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APPROXIMATE SOLUTIONS OF SCHRÖDINGER EQUATION WITH A QUARTIC POTENTIAL

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Abstract. Recently we investigated a type of Hyers-Ulam stability of the Schrödinger equation with the symmetric parabolic wall potential that efficiently describes the quantum harmonic oscillations. In this paper we study a type of Hyers-Ulam stability of the Schrödinger equation when the potential barrier is a quartic wall in the solid crystal models.

1. Introduction

In 1940, Ulam [21] proposed a number of important unsolved problems at a conference at the University of Wisconsin. One of them was the following question about the stability of group homomorphisms:

> *Suppose G_1 is a group and G_2 a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, can we find $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for any $x, y \in G_1$, then there exists a group homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for any $x \in G_1$?*

The following year, Hyers [7] answered the Ulam's question for approximately additive functions with the assumption that G_1 and G_2 are Banach

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spaces. Indeed, he showed that each solution to the inequality $\|f(x+y)-f(x)-f(y)\| \le \varepsilon$ (for any x and y) can be approximated by an exact solution, that is, by an additive function. In this case, the Cauchy additive equation, $f(x+y)=f(x)+f(y)$, is said to have the Hyers-Ulam stability.

Meanwhile, Rassias [18], while trying not to strongly limit the Cauchy difference, tried to weaken the condition for the Cauchy difference as following:

$$\|f(x+y)-f(x)-f(y)\| \le \varepsilon\big(\|x\|^p + \|y\|^p\big),$$

where $p < 1$ is a fixed real number, and he proved the theorem of Hyers in this case. That is to say, he showed the Hyers-Ulam-Rassias stability (or generalized Hyers-Ulam stability) of the Cauchy additive functional equation. P. Găvruţa [6] expanded the theorem of Rassias, and since then, both have attracted the attention of many mathematicians (see [8, 11, 19]).

Again we assume that $I=(a,b)$ is an open interval with $-\infty \le a < b \le +\infty$ and n a fixed positive integer. Let us consider the linear differential equation of nth order

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) = g(x), \tag{1.1}$$

where $y : I \to \mathbb{C}$ is an n times continuously differentiable function, $a_0, \ldots, a_n : I \to \mathbb{C}$ are continuous coefficient functions, and $g : I \to \mathbb{C}$ is a continuous function.

We say that the differential equation (1.1) satisfies the Hyers-Ulam stability if the following statement is true for any $\varepsilon > 0$: For any n times continuously differentiable function $y : I \to \mathbb{C}$ which satisfies

$$\Big|a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) - g(x)\Big| \le \varepsilon$$

for all $x \in I$, there is a solution $y_0 : I \to \mathbb{C}$ to the differential equation (1.1) such that

$$|y(x) - y_0(x)| \le K(x,\varepsilon)$$

for each $x \in I$, where $K(x,\varepsilon)$ depends on x and ε and $\lim_{\varepsilon\to 0} K(x,\varepsilon) = 0$ no matter what the value of x is.

When the limit $\lim_{\varepsilon\to 0} K(x,\varepsilon)$ indeed depends on the value of x, it appears somewhat suitable for Hyers-Ulam-Rassias stability in a broad sense, but not in its strict sense. Therefore, the differential equation (1.1) may be said to have a type of Hyers-Ulam stability as there is no other appropriate official terminology yet. We refer to [8, 11, 19] for the more detailed definition of Hyers-Ulam stability.

To the best of our knowledge, Obłoza [15, 16] is the first mathematician who demonstrated Hyers-Ulam stability of the differential equations. Indeed, Obłoza perfectly showed the Hyers-Ulam stability of linear differential equations of the form

$$y'(x) + f(x)y(x) = g(x). \tag{1.2}$$

Since then, more mathematicians have dealt with this topic more broadly and in depth (see [1, 2, 4, 5, 9, 10, 13, 14, 17, 18, 20, 22]).

In a recent paper [12], the authors studied a type of Hyers-Ulam stability for a one-dimensional time independent Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x), \tag{1.3}$$

when the system has a symmetric parabolic wall potential that has a strong relationship with the quantum harmonic oscillations.

In this paper, we prove a type of Hyers-Ulam stability of the one-dimensional Schrödinger equation (1.3) with a symmetric quartic wall potential, where $\psi : \mathbb{R} \to \mathbb{C}$ is the wave function, V a symmetric quartic potential function, $\hbar$ the reduced Planck constant, m the mass of the particle, and E the energy of the particle with $E > 0$.

Finally, we want to mention that we write this paper based on the ideas and experiences of the papers [3, 12, 13, 17].

2. Preliminaries

The formula that expresses the solution to the first-order linear inhomogeneous differential equation (1.2) is as follows.

Lemma 2.1. *Suppose that $f, g : \mathbb{R} \to \mathbb{C}$ are continuous functions such that each of the integrals below exists. Every continuously differentiable function $y : \mathbb{R} \to \mathbb{C}$ is a solution to the first-order linear inhomogeneous differential equation* (1.2) *if and only if y can be expressed as*

$$y(x) = \exp\left(-\int_0^x f(w)dw\right)\left(y(c) + \int_0^x g(s)\exp\left(\int_0^s f(w)dw\right)ds\right),$$

where $y(c)$ is an arbitrary complex number.

By using Lemma 2.1, we may prove easily the generalized Hyers-Ulam stability (or the Hyers-Ulam-Rassias stability) of the first-order linear inhomogeneous differential equation (1.2). The Hyers-Ulam stability version of the following theorem has been proved by Obłoza long years ago, but its easy-to-use version was proved in [12, Lemma 2] by using a way that is a little different from that of Obłoza's, so we will omit the proof here. Interested readers may well refer to the paper. By $\Re(z)$ we mean the real part of a complex number z.

Lemma 2.2. *Suppose that $f, g : \mathbb{R} \to \mathbb{C}$ and $\varphi : \mathbb{R} \to [0, \infty)$ are continuous functions such that each of the integrals below exists. If a continuously differentiable function $y : \mathbb{R} \to \mathbb{C}$ satisfies the inequality*

$$\left| y'(x) + f(x)y(x) - g(x) \right| \le \varphi(x)$$

for all $x \in \mathbb{R}$, then there exists a continuously differentiable solution $y_0 : \mathbb{R} \to \mathbb{C}$ to the first-order linear inhomogeneous differential equation (1.2) *such that*

$$\begin{aligned}&|y(x) - y_0(x)| \\ &\le \exp\left(-\Re\left(\int_0^x f(w)dw \right) \right) \left| \int_0^x \varphi(s) \exp\left(\Re\left(\int_0^s f(w)dw \right) \right) ds \right|\end{aligned}$$

for all $x \in \mathbb{R}$.

3. A type of Hyers-Ulam stability

In this section, we assume that the potential $V : \mathbb{R} \to \mathbb{R}$ is a quartic function defined by

$$V(x) = \frac{\hbar^2 \alpha^2}{2m} x^4 + \frac{\hbar^2 \alpha}{m} x + E, \tag{3.1}$$

where α is a fixed real number and $E > 0$.

Since the inequality (3.3) below is affected by the value of x, the property observed in the following theorem is called a type of Hyers-Ulam stability.

Theorem 3.1. *Assume that $V : \mathbb{R} \to \mathbb{R}$ is a quartic potential function defined by* (3.1)*, where α is a fixed real number and E is a positive real number as the energy of the particle under consideration. For any $\varepsilon > 0$, if a twice continuously differentiable function $\psi : \mathbb{R} \to \mathbb{C}$ satisfies the inequality*

$$\left| -\frac{\hbar^2}{2m} \frac{d^2 \psi(x)}{dx^2} + V(x)\psi(x) - E\psi(x) \right| \le \varepsilon \tag{3.2}$$

for all $x \in \mathbb{R}$, then there exists a twice continuously differentiable exact solution $\psi_0 : \mathbb{R} \to \mathbb{C}$ to the one-dimensional Schrödinger equation (1.3) *such that*

$$\begin{aligned}&|\psi(x) - \psi_0(x)| \\ &\le \frac{2m}{\hbar^2} \varepsilon \exp\left(\frac{\alpha}{3} x^3 \right) \left| \int_0^x \exp\left(-\frac{2}{3} \alpha s^3 \right) \left| \int_0^s \exp\left(\frac{\alpha}{3} w^3 \right) dw \right| ds \right|\end{aligned} \tag{3.3}$$

for all $x \in \mathbb{R}$.

Proof. Considering the form of the potential function V given in (3.1), we define the following differential operators O_1 and O_2 by

$$\begin{aligned}(O_1 \psi)(x) &= \psi'(x) - \alpha x^2 \psi(x), \\ (O_2 \psi)(x) &= \psi'(x) + \alpha x^2 \psi(x)\end{aligned} \tag{3.4}$$

for any twice continuously differentiable function $\psi : \mathbb{R} \to \mathbb{C}$.

For every twice continuously differentiable function $\psi : \mathbb{R} \to \mathbb{C}$, it follows from (3.1) and (3.4) that

$$\begin{aligned}
-\frac{\hbar^2}{2m}((O_2 \circ O_1)\psi)(x) &= -\frac{\hbar^2}{2m}\left(\psi''(x) - \left(\alpha^2 x^4 + 2\alpha x\right)\psi(x)\right) \\
&= -\frac{\hbar^2}{2m}\psi''(x) + V(x)\psi(x) - E\psi(x)
\end{aligned}$$

for all $x \in \mathbb{R}$, which implies that

$$\left|-\frac{\hbar^2}{2m}\psi''(x) + V(x)\psi(x) - E\psi(x)\right| \le \varepsilon \text{ (for all } x \in \mathbb{R})$$

if and only if

$$|((O_2 \circ O_1)\psi)(x)| \le \frac{2m}{\hbar^2}\varepsilon \text{ (for all } x \in \mathbb{R})$$

which is again equivalent to

$$\left|\phi'(x) + \alpha x^2 \phi(x)\right| \le \frac{2m}{\hbar^2}\varepsilon \text{ (for all } x \in \mathbb{R}), \tag{3.5}$$

where $\phi(x) = (O_1\psi)(x)$.

Since $\phi(x) = \psi'(x) - \alpha x^2 \psi(x)$, by considering (3.2) and (3.5), we may apply Lemma 2.2 to the inequality (3.5), with the substitutions shown as follows: In (3.5), $\phi(x)$ is for $y(x)$, αx^2 is for $f(x)$, 0 is for $g(x)$, and $\frac{2m}{\hbar^2}\varepsilon$ is for $\varphi(x)$.

We conclude that there exists a continuously differentiable function $\phi_0 : \mathbb{R} \to \mathbb{C}$ satisfying

$$\phi_0'(x) + \alpha x^2 \phi_0(x) = 0 \tag{3.6}$$

and

$$\begin{aligned}
|\phi(x) - \phi_0(x)| &\le \frac{2m}{\hbar^2}\varepsilon \exp\left(-\int_0^x \alpha w^2 dw\right)\left|\int_0^x \exp\left(\int_0^s \alpha w^2 dw\right) ds\right| \\
&= \frac{2m}{\hbar^2}\varepsilon \left|\int_0^x \exp\left(-\frac{\alpha}{3}\left(x^3 - s^3\right)\right) ds\right|
\end{aligned} \tag{3.7}$$

for all $x \in \mathbb{R}$.

Because $\phi(x) = \psi'(x) - \alpha x^2 \psi(x)$, it follows from (3.7) that

$$\left|\psi'(x) - \alpha x^2 \psi(x) - \phi_0(x)\right| \le \frac{2m}{\hbar^2}\varepsilon \left|\int_0^x \exp\left(-\frac{\alpha}{3}\left(x^3 - s^3\right)\right) ds\right| \tag{3.8}$$

for all $x \in \mathbb{R}$. We apply again Lemma 2.2 to the inequality (3.8) with the substitutions presented as follows: In (3.8), $\psi(x)$ is for $y(x)$, $-\alpha x^2$ is for $f(x)$, $\phi_0(x)$ is for $g(x)$, and $\frac{2m}{\hbar^2}\varepsilon \left|\int_0^x \exp\left(-\frac{\alpha}{3}\left(x^3 - s^3\right)\right) ds\right|$ is for $\varphi(x)$.

From Lemma 2.2, there exists a continuously differentiable function $\psi_0 : \mathbb{R} \to \mathbb{C}$ satisfying

$$\psi_0'(x) - \alpha x^2 \psi_0(x) = \phi_0(x) \tag{3.9}$$

and

$$|\psi(x) - \psi_0(x)| \leq \frac{2m}{\hbar^2} \varepsilon \exp\left(\frac{\alpha}{3} x^3\right) \left| \int_0^x \exp\left(-\frac{2}{3}\alpha s^3\right) \left| \int_0^s \exp\left(\frac{\alpha}{3} w^3\right) dw \right| ds \right|$$

for all $x \in \mathbb{R}$.

Further, by Lemma 2.1 and (3.9), ψ_0 has the form

$$\psi_0(x) = \exp\left(\frac{\alpha}{3} x^3\right) \left(\psi_0(c) + \int_0^x \phi_0(s) \exp\left(-\frac{\alpha}{3} s^3\right) ds\right),$$

where $\psi_0(c)$ is an arbitrary complex number. Because ϕ_0 is continuously differentiable, we see that ψ_0 is twice continuously differentiable.

Finally, using (3.6) and (3.9), it is easily shown that $\psi_0 : \mathbb{R} \to \mathbb{C}$ is a solution to the one-dimensional Schrödinger equation (1.3). □

For any real number $\alpha < 0$, it holds that

$$\begin{aligned}\int_0^s \exp\left(\frac{\alpha}{3} w^3\right) dw &= \frac{1}{3\left(-\frac{\alpha}{3}\right)^{1/3}} \gamma\left(\frac{1}{3}, -\frac{\alpha}{3} s^3\right) \\ &= \frac{1}{3^{2/3}(-\alpha)^{1/3}} \int_0^{-\frac{\alpha}{3}s^3} e^{-w} w^{-2/3} dw,\end{aligned}$$

where the incomplete gamma function $\gamma(\beta, x)$ is defined by

$$\gamma(\beta, x) = \int_0^x e^{-t} t^{\beta-1} dt.$$

Hence, if the parameter α of the potential function (3.1) is negative, then the inequality (3.3) becomes

$$|\psi(x) - \psi_0(x)| \leq \frac{2m}{\sqrt[3]{-9\alpha}\hbar^2} \varepsilon \exp\left(\frac{\alpha}{3} x^3\right) \left| \int_0^x \exp\left(-\frac{2}{3}\alpha s^3\right) \left| \int_0^{-\frac{\alpha}{3}s^3} \frac{dw}{\sqrt[3]{w^2 e^w}} \right| ds \right|$$

for all $x \in \mathbb{R}$.

4. Discussion

The Schrödinger equation is based on the postulates of quantum mechanics, and the perturbation theory of Schrödinger equation may be very useful for the case when it is very hard to find the exact solution for some potentials. It is also possible to use the one-dimensional Schrödinger equation to examine the state of particles reflected on the rectangular potential wall that is somewhat related to the subject matter of this paper.

Because the distance between the perturbed solution ψ and the exact solution ψ_0 of the one-dimensional time independent Schrödinger equation (1.3) is influenced strongly by x, we know that in Theorem 3.1 we have failed to demonstrate the exact Hyers-Ulam stability of the Schrödinger equation when the relevant potential function is quartic and $E > 0$. So instead of saying that we have proved the Hyers-Ulam stability in this paper, we are saying that we have proved a type of Hyers-Ulam stability.

We think Lemma 2.2 should be greatly improved to demonstrate the Hyers-Ulam stability of the Schrödinger equation (1.3) with quartic potential. And we believe that it will take a considerable amount of time to achieve this, so we would like to leave this improvement as a task to be done next.

5. Conclusions

Applying the operator method, we have verified that the one-dimensional time independent Schrödinger equation has a type of Hyers-Ulam stability if the associated potential function is given by the form (3.1). The main result of the present paper did not address the more general case of the Schrödinger equation when the associated potential function is in the form:

$$V(x) = Ax^4 + Bx^3 + Cx^2 + Dx + E,$$

where the parameters A, B, C, D, and E meet the minimum necessary conditions. Nevertheless, we think it is worthwhile that we considered the Schrödinger equation with a quartic potential function and investigated a type of Hyers-Ulam stability of it.

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