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COLLECTIVE FIXED POINTS FOR GENERALIZED CONDENSING MAPS IN ABSTRACT CONVEX UNIFORM SPACES

Hoonjoo Kim

Department of Mathematics Education
Sehan University, YoungAm-Gun, Chunnam, 58447, Korea
e-mail: `hoonjoo@sehan.ac.kr`

Abstract. In this paper, we present a fixed point theorem for a family of generalized condensing multimaps which have ranges of the Zima-Hadžić type in Hausdorff KKM uniform spaces. It extends Himmelberg et al. type fixed point theorem. As applications, we obtain some new collective fixed point theorems for various type generalized condensing multimaps in abstract convex uniform spaces.

1. Introduction and Preliminaries

In [4], Himmelberg et al. introduced condensing multimaps defined on subsets of locally convex spaces and they obtained a fixed point theorem for a condensing multimap with convex values, closed graph, and bounded range. The concept of condensing multimaps was extended to generalized condensing multimaps by Petryshyn and Fitzpatrick [17]. Huang et al. [6] modified the definition of generalized condensing multimaps and got Himmelberg et al. type fixed point theorem on LG-spaces. Influenced by [1], Amini-Harandi et al. [2] presented a fixed point theorem for generalized condensing multimaps on abstract convex uniform spaces. The concept of abstract convex uniform

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spaces was introduced by Park [12] as a generalization of locally convex spaces, LG-spaces and other abstract locally convex structures.

The aim of this paper is to present new collective fixed point theorems for generalized condensing multimaps on abstract convex uniform spaces. We begin by extending the concepts of generalized condensing multimaps on locally convex topological vector spaces in [17] to product of abstract convex uniform spaces. In Section 3, we present a fixed point theorem for a family of generalized condensing multimaps which have ranges of the Zima-Hadžić type in Hausdorff KKM uniform spaces. The result extends Himmelberg et al. type fixed point theorem.

As applications, we obtain some new collective fixed point results for generalized condensing multimaps in $\mathfrak{KC}$ class (or KKM class) whose ranges are Φ-sets in the setting of abstract convex uniform spaces in Section 4. $\mathfrak{KC}$ class is equivalent to s-KKM class with surjective function s. So we reformulated the results in $\mathfrak{KC}$ class to those in s-KKM class. We show that generalized condensing multimaps in the 'better' admissible class defined on abstract convex uniform spaces have fixed point properties whenever their ranges are Klee approximable in Section 5.

A *multimap* (or simply, a *map*) $F : X \multimap Y$ is a function from a set X into the power set of Y; that is, a function with the *values* $F(x) \subset Y$ for $x \in X$ and the *fibers* $F^-(y) := \{x \in X \,|\, y \in F(x)\}$ for $y \in Y$. For $A \subset X$, let $F(A) := \bigcup\{F(x) \,|\, x \in A\}$. Throughout this paper, we assume that multimaps have nonempty values otherwise explicitly stated or obvious from the context. The closure operation and the interior operation of F are denoted by "$\overline{\quad}$" and Int, respectively.

Let $\langle X \rangle$ denote the set of all nonempty finite subsets of a set X.

The followings are due to Park [10, 12].

An *abstract convex space* $(X, D; \Gamma)$ consists of a topological space X, a nonempty set D, and a map $\Gamma : \langle D \rangle \multimap X$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D \rangle$. For any nonempty $D' \subset D$, the Γ-*convex hull* of D' is denoted and defined by $\mathrm{co}_\Gamma D' := \bigcup\{\Gamma_A \,|\, A \in \langle D' \rangle\} \subset X$.

When $D \subset X$ in $(X, D; \Gamma)$, the space is denoted by $(X \supset D; \Gamma)$ and in case $X = D$, let $(X; \Gamma) := (X, X; \Gamma)$. When $(X \supset D; \Gamma)$, a subset X' of X is said to be Γ-*convex* if $\mathrm{co}_\Gamma(X' \cap D) \subset X'$. This means that $(X', D'; \Gamma')$ itself is an abstract convex space where $D' := X' \cap D$ and $\Gamma' : \langle D' \rangle \multimap X'$ a map defined by $\Gamma'_A := \Gamma_A \subset X'$ for $A \in \langle D' \rangle$.

An *abstract convex uniform space* $(X, D; \Gamma; \mathcal{U})$ is an abstract convex space with a basis $\mathcal{U}$ of a uniform structure of X. $A \subset X$ and $U \in \mathcal{U}$, the set $U[A]$ is defined to be $\{y \in X : (x, y) \in U \text{ for some } x \in A\}$. An abstract convex uniform space $(X \supset D; \Gamma; \mathcal{U})$ is called an *$L\Gamma$-space* if D is dense in X and $U[C]$ is Γ-convex for each $U \in \mathcal{U}$ whenever $C \subset X$ is Γ-convex.

For an abstract convex uniform space $(X \supset D; \Gamma; \mathcal{U})$, a subset S of X is said to be *of the Zima type* or *of the Zima-Hadžić type* if $D \cap S$ is dense in S and for each $U \in \mathcal{U}$ there exists a $V \in \mathcal{U}$ such that for each $N \in \langle D \cap S \rangle$ and any Γ-convex subset A of S, we have

$$A \cap V[z] \neq \emptyset, \quad \forall z \in N \quad \Longrightarrow \quad A \cap U[x] \neq \emptyset, \quad \forall x \in \Gamma_N.$$

The following lemma is in Park [12];

Lemma 1.1. *Any $L\Gamma$-space is of the Zima type.*

Even if $(X \supset D; \Gamma; \mathcal{U})$ is of the Zima type, it is not guaranteed that every subset S of X is of the Zima type, because $D \cap S$ is not dense in S in general. But when $D = X$, every subset of X is of the Zima type. Preserving the property of the Zima type on subsets makes theory efficient, so from now on, we assume that $D = X$.

A *generalized convex space* or a *G-convex space* $(X; \Gamma)$ consists of a topological space X such that for each $A \in \langle X \rangle$ with the cardinality $|A| = n + 1$, there exist a subset Γ_A of X and a continuous map $\phi_A : \Delta_n \to \Gamma_A$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma_J$. Here, Δ_n is the standard n-simplex with vertices $\{e_0\}_{i=0}^n$, and Δ_J is the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \ldots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\} \subset A$, then $\Delta_J = \mathrm{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}$.

A subset S of X is called a *Γ-convex subset* of $(X; \Gamma)$ if for any $N \in \langle S \rangle$, we have $\Gamma_N \subset S$. For details on G-convex spaces, see [14, 15, 16]. A *G-convex uniform space* $(X; \Gamma; \mathcal{U})$ is a G-convex space with a basis $\mathcal{U}$ of a uniform structure of X. A G-convex uniform space $(X; \Gamma; \mathcal{U})$ is said to be an *LG-space* if the uniformity $\mathcal{U}$ has a base $\mathcal{B}$ such that for each $U \in \mathcal{B}$, $U[C]$ is Γ-convex for each $U \in \mathcal{U}$ whenever $C \subset X$ is Γ-convex. The examples of G-convex uniform space are given in [9].

Let $(X; \Gamma)$ be an abstract convex space and Z be a set. For a multimap $F : X \multimap Z$, if a multimap $G : X \multimap Z$ satisfies $F(\Gamma_A) \subset G(A)$ for all $A \in \langle X \rangle$, then G is called a *KKM map* with respect to F. A *KKM map* $G : X \multimap Z$ is a KKM map with respect to the identity map 1_X.

A multimap $F : X \multimap Z$ is called a *$\mathfrak{K}$-map* if, for a KKM map $G : X \multimap Z$ with respect to F, the family $\{G(x)\}_{x \in X}$ has the finite intersection property. The set $\mathfrak{K}(X, Z)$ is defined to be $\{F : X \multimap Z \,|\, F \text{ is a } \mathfrak{K}\text{-map}\}$. Similarly, a *$\mathfrak{KC}$-map* is defined for closed-valued maps G and a *$\mathfrak{KO}$-map* for open-valued maps G.

For an abstract convex space $(X;\Gamma)$, the *KKM principle* is the statement $1_X \in \mathfrak{KC}(X,X) \cap \mathfrak{KO}(X,X)$. An abstract convex space is called a *KKM space* if it satisfies the KKM principle. A *KKM uniform space* $(X;\Gamma;\mathcal{U})$ is a KKM space with a basis $\mathcal{U}$ of a uniform structure of X. Known examples of KKM spaces are given in [11, 13] and the references therein. Note that a generalized convex space is also a KKM space.

2. Generalized condensing maps

Let $(X;\Gamma)$ be an abstract convex space, $A \subset X$ and put

$$\Gamma\text{-co}A = \bigcap\{C \mid C \text{ is a } \Gamma\text{-convex subset of } X \text{ containing } A\}$$

and

$$\Gamma\text{-}\overline{\text{co}}A = \bigcap\{C \mid C \text{ is a closed } \Gamma\text{-convex subset of } X \text{ containing } A\}.$$

Note that Γ-co$A$ and Γ-$\overline{\text{co}}A$ are the smallest Γ-convex set and the smallest closed Γ-convex set containing A, respectively. Clearly, $\text{co}_\Gamma A \subset \Gamma\text{-co}A$.

Lemma 2.1. *Let $(X;\Gamma;\mathcal{U})$ be an abstract convex uniform space of the Zima type.*

(1) *If C is a Γ-convex subset of X, then its closure $\overline{C}$ is Γ-convex.*
(2) *If $A \subset X$, then $\Gamma\text{-}\overline{\text{co}}A = \overline{\Gamma\text{-co}A}$.*

Proof. (1) For each $U \in \mathcal{U}$, let $V \in \mathcal{U}$ be the one satisfying the definition of the Zima type. For any $N \in \langle\overline{C}\rangle$, $C \cap V[z] \neq \emptyset$ for all $z \in N$, that means $C \cap U[x] \neq \emptyset$ for all $x \in \Gamma_N$. Hence $\Gamma_N \subset \bigcap_{U\in\mathcal{U}} U[C] = \overline{C}$.

(2) Since Γ-$\overline{\text{co}}A$ is closed, we have $\overline{\Gamma\text{-co}A} \subset \overline{\Gamma\text{-}\overline{\text{co}}A} = \Gamma\text{-}\overline{\text{co}}A$. By (1), $\overline{\Gamma\text{-co}A}$ is a closed Γ-convex set containing A, so $\Gamma\text{-}\overline{\text{co}}A \subset \overline{\Gamma\text{-co}A}$. □

A subset S of a uniform space X is said to be *precompact* if, for any entourage V, there is an $N \in \langle X\rangle$ such that $S \subset V[N]$.

For a subset A of a uniform space X with a basis $\mathcal{U}$, a *measure of precompactness* of A is defined by

$$\Psi(A) = \{V \in \mathcal{U} \mid A \subset \overline{V[K]} \text{ for some precompact subset } K \text{ of } X\}.$$

Lemma 2.2. ([18, Proposition 1.1]) *Let X be a uniform space with a basis $\mathcal{U}$ and $A, B \subset X$. Then*

(1) *A is precompact iff $\Psi(A) = \mathcal{U}$;*
(2) *if $A \subset B$, then $\Psi(B) \subset \Psi(A)$;*
(3) *$\Psi(\overline{A}) = \Psi(A)$; and*
(4) *$\Psi(A \cup B) = \Psi(A) \cap \Psi(B)$.*

Now, we extend the concepts of generalized condensing maps on locally convex topological vector spaces in [17] to the product of abstract convex uniform spaces.

Let $\{(X_i;\Gamma_i)\}_{i\in I}$ be a family of abstract convex spaces, and $i \in I$ be fixed. Let

$$X = \prod_{i\in I} X_i, \quad X^i = \prod_{j\in I\setminus\{i\}} X_j,$$

and $x_i = \pi_i(x)$ denote the projection of x in X_i. For any $N \in \langle X\rangle$, let set $\Gamma_N = \prod_{i\in I}\Gamma_i(\pi_i(N))$ for each $i \in I$. Note that $(X;\Gamma)$ forms an abstract convex space. For details, see [10].

For each $N \in \langle X\rangle$, Γ-co$N$ is called a *polytope* in X. An abstract convex uniform space $(X;\Gamma;\mathcal{U})$ is called an *abstract convex uniform space with precompact polytopes* if each polytope in X is precompact.

Let $\{(X_i;\Gamma_i;\mathcal{U}_i)\}_{i\in I}$ be a family of abstract convex uniform spaces with precompact polytopes. For each $i \in I$, let Ψ_i be a measure of precompactness in X_i. A map $T_i : X \multimap X_i$ is called Ψ_i*-condensing* provided that $\Psi_i(\pi_i(A)) \subsetneq \Psi_i(T_i(A))$ for any subset A satisfying $\pi_i(A)$ is not a precompact subset of X_i in [8]. T_i is called *generalized condensing* if $A \subset X$, $T_i(A) \subset \pi_i(A)$ and $\pi_i(A)\setminus\Gamma_i$-$\overline{\text{co}}\, T_i(A)$ is precompact, then $\pi_i(A)$ is precompact.

When $I = \{1\}$, Ψ_i-condensing map reduces to the usual Ψ-condensing map $T : X \multimap X$; that is, $\Psi(A) \subsetneq \Psi(T(A))$ for any nonprecompact subset A of X. And generalized condensing map $T : X \multimap X$ becomes a generalized condensing map in [17] which is similar that of in [2] and [6]; that is, if $A \subset X$, $T(A) \subset A$ and $A\setminus\Gamma$-$\overline{\text{co}}\, T(A)$ is precompact, then A is precompact.

Note that every compact map is condensing. An LG-space is said to be an *locally* G*-convex space* in [6] if $U[\{x\}]$ is Γ-convex for each $x \in X$ and $U \in \mathcal{U}$ and if Γ-co$A$ is precompact whenever A is precompact. So a locally G-convex space is an $L\Gamma$-space with precompact polytopes.

Proposition 2.3. *If* $(X;\Gamma;\mathcal{U})$ *is an* $L\Gamma$*-space with precompact polytopes, then every condensing map* $T : X \multimap X$ *is generalized condensing.*

Proof. First we show that if A is precompact subset of X, then Γ-co$A$ is precompact. For any $U \in \mathcal{U}$, we choose $V \in \mathcal{U}$ such that $V \circ V \subset U$. Since A is precompact, there exists an $N \in \langle X\rangle$ such that $A \subset V(N)$. Because $(X;\Gamma;\mathcal{U})$ is an $L\Gamma$-space, $V(\Gamma\text{-co}N)$ is a Γ-convex set containing $V(N)$. So $\Gamma\text{-co}A \subset \Gamma\text{-co}V(N) \subset V(\Gamma\text{-co}N)$. Since Γ-co$N$ is precompact, there exists an $M \in \langle X\rangle$ such that $\Gamma\text{-co}N \subset V(M)$. Therefore $\Gamma\text{-co}A \subset V(\Gamma\text{-co}N) \subset V(V(M)) \subset U(M)$.

Now we show that $\Psi(A) = \Psi(\Gamma\text{-co}A)$ for any $A \subset X$. By Lemma 2.2, $\Psi(A) \supset \Psi(\Gamma\text{-co}A)$. If $V \in \Psi(A)$, then $A \subset \overline{V(K)}$ for some precompact set K. Then $\Gamma\text{-co}A \subset \Gamma\text{-co}\overline{V(K)} \subset \overline{V(\Gamma\text{-co}K)}$, because $\overline{V(\Gamma\text{-co}K)}$ is a closed Γ-convex set containing $\overline{V(K)}$. Since Γ-co$K$ is precompact, $V \in \Psi(\Gamma\text{-co}A)$.

By Lemma 2.1 and Lemma 2.2, $\Psi(A) = \Psi(\Gamma\text{-co}A) = \Psi(\overline{\Gamma\text{-co}A}) = \Psi(\Gamma\text{-}\overline{\text{co}}\, A)$.

Suppose that there exists an $A \subset X$ such that $T(A) \subset A$ and $A \backslash \Gamma\text{-}\overline{\text{co}}\, T(A)$ is precompact, but A is not precompact. Since $A \subset \Gamma\text{-}\overline{\text{co}}\, T(A) \cup (A \backslash \Gamma\text{-}\overline{\text{co}}\, T(A))$,

$$\Psi(A) \supset \Psi(\Gamma\text{-}\overline{\text{co}}\, T(A)) \cap \Psi(A \backslash \Gamma\text{-}\overline{\text{co}}\, T(A)) = \Psi(\Gamma\text{-}\overline{\text{co}}\, T(A)) \cap \mathcal{U} = \Psi(T(A)).$$

Therefore T is not a condensing map. □

By Proposition 2.3, condensing maps are generalized condensing maps on locally convex spaces, LG-spaces with precompact polytopes and locally G-convex spaces.

3. Collective fixed points on KKM uniform spaces

From now on we assume that every abstract convex space is an abstract convex space with precompact polytopes.

The following proposition is a crucial tool for the proof of the main theorems.

Proposition 3.1. *Let $\{(X_i; \Gamma_i; \mathcal{U}_i)\}_{i \in I}$ be a family of abstract convex uniform spaces of the Zima type, $X = \prod_{i \in I} X_i$ and $T_i : X \multimap X_i$ be a map for each $i \in I$.*

(a) *If $Q = \prod_{i \in I} Q_i$ is a nonempty subset of X, then there is a closed Γ-convex subset $K = \prod_{i \in I} K_i$ of X such that $\Gamma_i\text{-}\overline{\text{co}}\,(T_i(K) \cup Q_i) = K_i$ for each $i \in I$.*

(b) *If X_i is a Hausdorff space and T_i is a generalized condensing closed multimap for each $i \in I$, then there exists a nonempty compact Γ-convex subset K of X, with $K = \prod_{i \in I} K_i$ such that $T_i(K) \subset K_i$ for each $i \in I$.*

Proof. (a) Let $\mathcal{A}$ be the family of all subsets A of X which satisfies the following conditions: $A = \prod_{i \in I} A_i$, where A_i is a closed Γ_i-convex subset of X_i such that $\Gamma_i\text{-}\overline{\text{co}}\,(T_i(A) \cup Q_i) \subset A_i$ for each $i \in I$. Since $X \in \mathcal{A}$, $\mathcal{A} \neq \emptyset$. Note that every $A \in \mathcal{A}$ is closed and Γ-convex.

Define a partial order by inverse inclusion, that is, for $A, B \in \mathcal{A}$, $A \leq B \Longleftrightarrow B \subset A$. Let $\mathcal{C}$ be any chain in $\mathcal{A}$. Put $M = \cap_{A \in \mathcal{C}} A$ and $M_i = \cap_{A \in \mathcal{C}} A_i$ for each $i \in I$, then $M = \prod_{i \in I} M_i$. For all $A \in \mathcal{C}$ and $i \in I$, each A_i is closed and Γ_i-convex, so is M_i. Moreover, $T_i(M) \cup Q_i \subset T_i(A) \cup Q_i$, hence $\Gamma_i\text{-}\overline{\text{co}}\,(T_i(M) \cup Q_i) \subset \Gamma_i\text{-}\overline{\text{co}}\,(T_i(A) \cup Q_i) \subset A_i$, and so $\Gamma_i\text{-}\overline{\text{co}}\,(T_i(M) \cup Q_i) \subset \cap_{A \in \mathcal{C}} A_i = M_i$. Thus $M \in \mathcal{A}$ and M is an upper bound of $\mathcal{C}$. By Zorn's lemma, $\mathcal{A}$ has a maximal element, say $K = \prod_{i \in I} K_i$.

We claim that $\Gamma_i\text{-}\overline{\text{co}}\,(T_i(K) \cup Q_i) = K_i$ for all $i \in I$. In fact, put $L_i = \Gamma_i\text{-}\overline{\text{co}}\,(T_i(K) \cup Q_i)$ and $L = \prod_{i \in I} L_i$, then $L_i \subset K_i$ and L_i is closed Γ_i-convex. Since $\Gamma_i\text{-}\overline{\text{co}}\,(T_i(L) \cup Q_i) \subset \Gamma_i\text{-}\overline{\text{co}}\,(T_i(K) \cup Q_i) = L_i$, we have $L \in \mathcal{A}$. By the maximality of K, we conclude that $K = L$, that is, $\Gamma_i\text{-}\overline{\text{co}}\,(T_i(K) \cup Q_i) = K_i$ for all $i \in I$.

(b) Choose $x_0 \in X$. Define a map $T : X \multimap X$ by $T(x) = \prod_{i \in I} T_i(x)$. Put $\Omega = \cup_{j \geq 0} T^j(x_0)$ where $T^0(x_0) = x_0$ and $T^{j+1}(x_0) = T \circ T^j(x_0)$. Then $T_i(\Omega) \subset \pi_i(\Omega)$ and $\pi_i(\Omega) \backslash T_i(\Omega) \subset \{\pi_i(x_0)\}$, so $\pi_i(\Omega) \backslash \Gamma_i\text{-}\overline{\mathrm{co}}\, T_i(\Omega)$ is precompact. As T_i is generalized condensing, $\pi_i(\Omega)$ is precompact, so $T_i(\Omega)$ is precompact. Therefore $\overline{T(\Omega)} = \prod_{i \in I} \overline{T_i(\Omega)}$ is compact.

For $i \in I$, define $G_i : \overline{T(\Omega)} \multimap \overline{T_i(\Omega)}$ by $G_i(x) = T_i(x) \cap \overline{T_i(\Omega)}$ for each $x \in \overline{T(\Omega)}$. Since T_i is closed and $\overline{T_i(\Omega)}$ is compact, $G_i(x) \neq \emptyset$ for all $x \in \overline{T(\Omega)}$, so G_i is well defined. Let $\mathcal{F}$ be a family of all subsets A of $\overline{T(\Omega)}$ which satisfies the following conditions: $A = \prod_{i \in I} A_i$, where A_i is a closed subset of $\overline{T_i(\Omega)}$ such that $G_i(A) \subset A_i$ for each $i \in I$.

Since $\overline{T(\Omega)} \in \mathcal{F}$, $\mathcal{F} \neq \emptyset$. Define a partial order $\leq$ on $\mathcal{F}$ by $A \leq B \iff B \subset A$, for any $A, B \in \mathcal{F}$. Let $\mathcal{C}$ be any chain in $\mathcal{F}$. Put $M = \cap_{A \in \mathcal{C}} A$ and $M_i = \cap_{A \in \mathcal{C}} A_i$ for $i \in I$. Then $M = \prod_{i \in I} M_i$ and M_i is a nonempty closed subset of $\overline{T_i(\Omega)}$ for all $i \in I$.

For all $x \in M$ and $A \in \mathcal{C}$, $G_i(x) \subset A_i$, so $G_i(x) \subset M_i$, that is, $G_i(M) \subset M_i$ for each $i \in I$. Thus M is an upper bound of $\mathcal{C}$, and so, by Zorn's Lemma, $\mathcal{F}$ has a maximal element, say $Q = \prod_{i \in I} Q_i$, then $G_i(Q) \subset Q_i$. Since T_i is closed, so is G_i, which in conjunction with the compactness of $\overline{T_i(\Omega)}$, shows that G_i is upper semicontinuous. Therefore $G_i(Q)$ is compact and closed.

Put $Y_i = G_i(Q)$ for each $i \in I$ and $Y = \prod_{i \in I} Y_i$. For $i \in I$, $G_i(Y) = G_i(\prod_{i \in I} G_i(Q)) \subset G_i(\prod_{i \in I} Q_i) = Y_i$, so the maximality of Q gives us that $Q = Y$. Thus $Q_i = G_i(Q) = T_i(Q) \cap \overline{T_i(\Omega)} \subset T_i(Q)$.

Let $K_i = \Gamma_i\text{-}\overline{\mathrm{co}}\,(T_i(K) \cup Q_i)$ for each $i \in I$ and $K = \prod_{i \in I} K_i$ in (a), then $Q_i \subset T_i(Q)$ and $Q \subset K$ imply that $Q_i \subset T_i(K)$. Hence

$$K_i = \Gamma_i\text{-}\overline{\mathrm{co}}\,(T_i(K) \cup Q_i) = \Gamma_i\text{-}\overline{\mathrm{co}}\,(T_i(K)).$$

Since $T_i(K) \subset K_i$ and T_i is generalized condensing, K_i is compact and so K is compact. $\square$

Notes. 1. The proof of Proposition 3.1 is motivated by Lemma 3.4 and Lemma 3.5 in [6]. Lemma 3.4 and Lemma 3.5 are on a locally G-convex space $(X; \Gamma)$ with $I = \{1\}$.

2. If $\{(X_i \supset D_i; \Gamma_i; \mathcal{U}_i)\}_{i \in I}$ be a family of $L\Gamma$-spaces and T_i is a condensing multimap for each $i \in I$, the same conclusion in (b) is obtained without assuming closedness of T_i [8].

From now on, all topological spaces are assumed to be Hausdorff. The following fixed point theorem is in [12];

Proposition 3.2. *Let $(X; \Gamma; \mathcal{U})$ be a KKM uniform space and $T : X \multimap X$ be a compact upper semicontinuous map with closed Γ-convex values. If $T(X)$ is of the Zima type, then T has a fixed point.*

Let $\{(X_i;\Gamma_i;\mathcal{U}_i)\}_{i\in I}$ be a family of abstract convex uniform spaces, $X = \prod_{i\in I} X_i$ and $\mathcal{U}$ be the base of a uniform structure of X generated by $\{\mathcal{U}_i\}_{i\in I}$. Clearly $(X;\Gamma;\mathcal{U})$ is an abstract convex uniform space.

Lemma 3.3. *Let $\{(X_i;\Gamma_i;\mathcal{U}_i)\}_{i\in I}$ be a family of abstract convex uniform spaces and $X = \prod_{i\in I} X_i$. If $(X;\Gamma;\mathcal{U})$ is of the Zima type, then $(X_i;\Gamma_i;\mathcal{U}_i)$ is of the Zima type for each $i \in I$.*

Proof. For $i \in I$, and $U_i \in \mathcal{U}_i$, let

$$U = \{(x, y) \in X \times X \mid (\pi_i(x), \pi_i(y)) \in U_i\}$$

and $V \in \mathcal{U}$ be the one satisfying the definition of the Zima type. Let $N_i \in \langle X_i\rangle$ and A_i be a Γ_i-convex subset of X_i such that $A_i \cap p_i(V)[z_i] \neq \emptyset$ for all $z_i \in N_i$ where $p_i(V)$ denotes the projection of V in $X_i \times X_i$. Choose $y \in V$ and $N \in \langle X\rangle$ such that $\pi_i(N) = N_i$ and $\pi_j(N) = \pi_j(y)$ if $j \neq i$. Put $A = A_i \times X^i$, then A is a Γ-convex subset of X and $A \cap V[z] \neq \emptyset$ for all $z \in N$, which implies $A \cap U[z] \neq \emptyset$ for all $x \in \Gamma_N = \prod_{j\in I} \Gamma_j(\pi_j(N))$. Therefore $A_i \cap U_i[z] \neq \emptyset$ for all $x_i \in \Gamma_i(N_i)$. □

The following generalization of a Himmelberg et al. [4] fixed point theorem, is a main result of this paper;

Theorem 3.4. *Let $\{(X_i;\Gamma_i;\mathcal{U}_i)\}_{i\in I}$ be a family of abstract convex uniform spaces, $X = \prod_{i\in I} X_i$, $(X;\Gamma;\mathcal{U})$ be a KKM space of the Zima type and $T_i : X \multimap X_i$ be a generalized condensing closed multimap with Γ_i-convex values for each $i \in I$. Then there exists an $x \in X$ such that $x \in \prod_{i\in I} T_i(x)$.*

Proof. By Lemma 3.3, each $(X_i;\Gamma_i;\mathcal{U}_i)$ is of the Zima type. By Proposition 3.1, there exists a compact Γ-convex subset $K = \prod_{i\in I} K_i$ of X such that $T_i(K) \subset K_i$ for each $i \in I$. Define $T : X \multimap X$ by $T(x) = \prod_{i\in I} T_i(x)$ for all $x \in X$, then $T(K) \subset K$. Since $T|_K$ is compact and closed, $T|_K$ is an upper semicontinuous map with closed Γ-convex values and $T(K)$ is of the Zima type. By Proposition 3.2, $T|_K$ has a fixed point. □

Corollary 3.5. *Let $(X;\Gamma;\mathcal{U})$ be a KKM uniform space of the Zima type. If $T : X \multimap X$ is a generalized condensing closed multimap with Γ-convex values, then T has a fixed point.*

It is shown that if $\{(X_i;\Gamma_i;\mathcal{U}_i)\}_{i\in I}$ is a family of $L\Gamma$-spaces, then $X = \prod_{i\in I} X_i$ is an $L\Gamma$-space, so the following corollary holds;

Corollary 3.6. *Let $\{(X_i;\Gamma_i;\mathcal{U}_i)\}_{i\in I}$ be a family of $L\Gamma$-spaces, $X = \prod_{i\in I} X_i$, $(X;\Gamma)$ be a KKM space and $T_i : X \multimap X_i$ be a generalized condensing closed multimap with Γ_i-convex values for each $i \in I$. Then there exists an $x \in X$ such that $x \in \prod_{i\in I} T_i(x)$.*

Corollary 3.7. *Let $(X;\Gamma;\mathcal{U})$ be an LG-space and $T : X \multimap X$ be a generalized condensing closed multimap with Γ-convex values. Then T has a fixed point.*

Corollary 3.7 relaxes the conditions of Theorem 3.7 in [6].

4. Fixed point theorems for $\mathfrak{KC}$ maps

For a given abstract convex space $(X;\Gamma)$ and a topological space Y, a map $H : Y \multimap X$ is called a Φ-*map* if there exists a map $G : Y \multimap X$ such that

(i) for each $y \in Y$, $\mathrm{co}_\Gamma G(y) \subset H(y)$;
(ii) $Y = \bigcup\{\ \mathrm{Int} G^-(x) \mid x \in X\}$.

In an abstract convex uniform space $(X;\Gamma;\mathcal{U})$, a subset S of X is called a Φ-*set* if, for any entourage $U \in \mathcal{U}$, there exists a Φ-map $H : S \multimap X$ such that the graph of H is contained in U.

The following propositions are in [10], [12];

Proposition 4.1. *Let $(X;\Gamma)$ be an abstract convex space, C be a Γ-convex subset of X and Z be a set. If $T \in \mathfrak{K}(X,Z)$, then $T|_C \in \mathfrak{K}(C,Z)$.*

Proposition 4.2. *Let $(X;\Gamma;\mathcal{U})$ be an abstract convex uniform space, and $T \in \mathfrak{KC}(X,X)$ be a compact closed map. If $\overline{T(X)}$ is a Φ-set, then T has a fixed point.*

If every singleton of $(X;\Gamma;\mathcal{U})$ is Γ-convex, then any subset of the Zima type in X is a Φ-set [10]. Therefore the following theorem holds;

Theorem 4.3. *Let $\{(X_i;\Gamma_i;\mathcal{U}_i)\}_{i\in I}$ be a family of abstract convex uniform spaces such that every singleton is Γ_i-convex, and $X = \prod_{i\in I} X_i$ be of the Zima type. If $T_i : X \multimap X_i$ is a generalized condensing closed multimap for each $i \in I$ and $\prod_{i\in I} T_i \in \mathfrak{KC}(X,X)$, then there exists an $x \in X$ such that $x \in \prod_{i\in I} T_i(x)$.*

Proof. By Proposition 3.1, there exists a compact Γ-convex subset K of X, with $K = \prod_{i\in I} K_i$ such that $T_i(K) \subset K_i$ for each $i \in I$. Put $T = \prod_{i\in I} T_i$, then $T|_K$ is compact closed. Since every singleton is Γ_i-convex for each $i \in I$, so is $(X;\Gamma;\mathcal{U})$. Therefore $\overline{T(K)}$ is a Φ-set. By Proposition 4.1 and Proposition 4.2, $T|_K \in \mathfrak{KC}(K,K)$ and $T|_K$ has a fixed point. □

Since any $L\Gamma$-space is of the Zima type, the following theorem holds;

Theorem 4.4. *Let $\{(X_i;\Gamma_i;\mathcal{U}_i)\}_{i\in I}$ be a family of $L\Gamma$-spaces, $X = \prod_{i\in I} X_i$ and $T_i : X \multimap X_i$ be a generalized condensing closed multimap for each $i \in I$. If $\prod_{i\in I} T_i \in \mathfrak{KC}(X,X)$ and $\overline{\prod_{i\in I} T_i(X)}$ is a Φ-set, then there exists an $x \in X$ such that $x \in \prod_{i\in I} T_i(x)$.*

From Theorem 4.4, we obtain the following corollary in [2];

Corollary 4.5. *Let* $(X;\Gamma;\mathcal{U})$ *be an* $L\Gamma$*-space, and* $T : X \multimap X$ *be a generalized condensing closed multimap. If* $T \in \mathfrak{KC}(X,X)$ *and* $\overline{T(X)}$ *is a* Φ*-set, then* T *has a fixed point.*

Let X be a nonempty set, $(Y;\Gamma)$ be an abstract convex space and Z be a topological space. If $S : X \multimap Y$, $T : Y \multimap Z$ and $F : X \multimap Z$ are three multimaps satisfying

$$T(\mathrm{co}_\Gamma S(A)) \subset F(A) \quad \text{for all} \quad A \in \langle X\rangle,$$

then F is called an *S-KKM map* with respect to T. If for any S-KKM map F with respect to T, the family $\{\overline{F(x)}\}_{x\in X}$ has the finite intersection property, then T is said to have the *S-KKM property*. The class *S-KKM*(X,Y,Z) is defined to be the set $\{T : Y \multimap Z \,|\, T$ has the S-KKM property$\}$. If S is the identity map 1_X, then S-KKM$(X,X,Z) = \mathfrak{KC}(X,Z)$. It was shown that $\mathfrak{KC}(Y,Z) \subset S$-KKM$(X,Y,Z)$ in [3].

The following proposition shows the relation between S-KKM maps and $\mathfrak{KC}$-maps more specifically;

Proposition 4.6. *Let* X *be a nonempty set,* $(Y;\Gamma)$ *be an abstract convex space and* Z *be a topological space. For any surjective function* $s : X \to Y$, $T \in \mathfrak{KC}(Y,Z)$ *if and only if* $T \in s$*-KKM*(X,Y,Z).

The proof of Proposition 4.6 is a modification of Proposition 2.4 [7] for a convex space. By Proposition 4.6, Theorem 4.3 is reformulated as follows;

Theorem 4.7. *Let* Z *be a nonempty set,* $\{(X_i;\Gamma_i;\mathcal{U}_i)\}_{i\in I}$ *be a family of abstract convex uniform spaces such that every singleton is* Γ_i*-convex, and* $X = \prod_{i\in I} X_i$ *be of the Zima type. If* $T_i : X \multimap X_i$ *is a generalized condensing closed multimap for each* $i \in I$ *and* $\prod_{i\in I} T_i \in s$*-KKM*(Z,X,X)*, then there exists an* $x \in X$ *such that* $x \in \prod_{i\in I} T_i(x)$.

Theorem 4.7 generalizes and deletes some extra conditions in Theorem 2.7 in [5].

5. Fixed point theorems for the class $\mathfrak{B}$ of multimaps

Now, we follow the definitions in [12]. Let $(E;\Gamma)$ be an abstract convex space, X be a nonempty subset of E, and Y be a topological space. *The better admissible class* $\mathfrak{B}$ of maps from X into Y is defined as follows:

$F \in \mathfrak{B}(X,Y) \iff F : X \multimap Y$ is a map such that, for any $\Gamma_N \subset X$, where $N \in \langle X\rangle$ with the cardinality $|N| = n+1$, and for any continuous function

$p : F(\Gamma_N) \to \Delta_n$, there exists a continuous function $\phi_N : \Delta_n \to \Gamma_N$ such that the composition $p \circ F|_{\Gamma_N} \circ \phi_N : \Delta_n \to \Delta_n$ has a fixed point.

Let $(E; \Gamma; \mathcal{U})$ be an abstract convex uniform space. A subset K of E is said to be *Klee approximable* if, for each entourage $U \in \mathcal{U}$, there exists a continuous function $h : K \to E$ satisfying

(1) $(x, h(x)) \in \mathcal{U}$ for all $x \in K$;
(2) $h(K) \subset \Gamma_N$ for some $N \in \langle K \rangle$;
(3) there exist continuous functions $p : K \to \Delta_n$ and $\phi_N : \Delta_n \to \Gamma_N$ with $|N| = n + 1$ such that $h = \phi_N \circ p$.

For a G-convex uniform space $(X; \Gamma; \mathcal{U})$, every nonempty compact Φ-subset K of X is Klee approximable.

The following proposition is a fixed point theorem for the class $\mathfrak{B}$ of multimaps in [12];

Proposition 5.1. *Let $(X; \Gamma; \mathcal{U})$ be an abstract convex uniform space and $T \in \mathfrak{B}(X, X)$ be a closed map such that $T(X)$ is compact Klee approximable. Then T has a fixed point.*

Theorem 5.2. *Let $\{(X_i; \Gamma_i; \mathcal{U}_i)\}_{i \in I}$ be a family of abstract convex uniform spaces of the Zima type, $X = \prod_{i \in I} X_i$, and $T_i : X \multimap X_i$ be a generalized condensing closed multimap for each $i \in I$. Suppose that $T := \prod_{i \in I} T_i \in \mathfrak{B}(X, X)$ and $T(C)$ is Klee approximable for each compact Γ-convex subset C of X. Then T has a fixed point.*

Proof. By Proposition 3.1, there exists a compact Γ-convex subset K of X such that $T(K) \subset K$. Then $T|_K$ is closed and $T(K)$ is compact Klee approximable. $T \in \mathfrak{B}(X, X)$ implies $T|_K \in \mathfrak{B}(K, K)$. By Proposition 5.1, $T|_K$ has a fixed point. □

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