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MAJORIZATION PROBLEMS FOR UNIFORMLY STARLIKE FUNCTIONS BASED ON RUSCHEWEYH $q-$DIFFERENTIAL OPERATOR RELATED WITH EXPONENTIAL FUNCTION

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Abstract. The main object of this present paper is to study some majorization problems for certain classes of analytic functions defined by means of $q-$calculus operator associated with exponential function.

1. Introduction

Let $\mathcal{A}$ be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

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which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. For given $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A}$, the Hadamard product of f and g is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z).$$

For two analytic functions $f, g \in \mathcal{A}$, we say that f is subordinate to g, denoted by $f \prec g$, if there exists a Schwarz function $\omega(z)$ which is analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ for all $z \in \mathbb{U}$, such that $f(z) = g(\omega(z))$ for $z \in \mathbb{U}$. Note that, if the function g is univalent in $\mathbb{U}$, due to Miller and Mocanu [6], we have

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

If f and g are analytic functions in $\mathbb{U}$, following MacGregor [5], we say that f is majorized by g in $\mathbb{U}$, that is $f(z) \ll g(z)$ $(z \in \mathbb{U})$ if there exists a function $\phi(z)$, analytic in $\mathbb{U}$, such that

$$|\phi(z)| < 1 \text{ and } f(z) = \phi(z) g(z) \quad (z \in \mathbb{U}).$$

It is of interest to note that the notion of majorization is closely related to the concept of quasi-subordination between analytic functions.

Now we recall here the notion of *q-operator* that is, *q-difference operator* that play vital role in the theory of hypergeometric series, quantum physics and in the operator theory. The application of *q-calculus* was initiated by Jackson [3], recently Kanas and Răducanu [4] have used the fractional *q-calculus operators* in investigations of certain classes of functions which are analytic in $\mathbb{U}$.

Let $0 < q < 1$. For any non-negative integer n, the *q-integer number n* is defined by

$$[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + \cdots + q^{n-1}, \qquad [0]_q = 0. \tag{1.2}$$

In general, we will denote

$$[x]_q = \frac{1 - q^x}{1 - q}$$

for a non-integer number x. Also the *q-number shifted factorial* is defined by

$$[n]_q! = [n]_q [n-1]_q ... [2]_q [1]_q, \qquad [0]_q! = 1. \tag{1.3}$$

Clearly,

$$\lim_{q \to 1^-} [n]_q = n \qquad \text{and} \qquad \lim_{q \to 1^-} [n]_q! = n!.$$

For $0 < q < 1$, the Jackson's *q-derivative operator* (or *q-difference operator*) of a function $f \in \mathcal{A}$ given by (1.1) defined as follows [3]:

$$\mathfrak{D}_q f(z) = \begin{cases} \dfrac{f(z) - f(qz)}{(1-q)z} & \text{for} \quad z \neq 0 \\ \\ f'(0) & \text{for} \quad z = 0 \end{cases}, \tag{1.4}$$

$\mathfrak{D}_q^0 f(z) = f(z)$, and $\mathfrak{D}_q^m f(z) = \mathfrak{D}_q(\mathfrak{D}_q^{m-1} f(z))$, $m \in \mathbb{N} = \{1, 2, \dots\}$. From (1.4), we have

$$\mathfrak{D}_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1} \qquad (z \in \mathbb{U}), \tag{1.5}$$

where $[n]_q$ is given by (1.2).

For a function $\psi(z) = z^n$, we obtain

$$\mathfrak{D}_q \psi(z) = \mathfrak{D}_q z^n = \frac{1-q^n}{1-q} z^{n-1} = [n]_q z^{n-1}$$

and

$$\lim_{q \to 1^-} \mathfrak{D}_q \psi(z) = \lim_{q \to 1^-} \left([n]_q z^{n-1}\right) = n z^{n-1} = \psi'(z),$$

where ψ' is the ordinary derivative.

Let $t \in \mathbb{R}$ and $n \in \mathbb{N}$. The *q-generalized Pochhammer symbol* is defined by

$$[t; n]_q = [t]_q [t+1]_q [t+2]_q ... [t+n-1]_q \tag{1.6}$$

and for $t > 0$ the *q-gamma function* is defined by

$$\Gamma_q(t+1) = [t]_q \Gamma_q(t) \qquad \text{and} \qquad \Gamma_q(1) = 1. \tag{1.7}$$

Using the *q-difference operator*, Kannas and Raducanu [4] defined the Ruscheweyh *q-differential operator* as below: For $f \in \mathcal{A}$,

$$\mathcal{R}_q^\delta f(z) = f(z) * F_{q,\delta+1}(z) \qquad (\delta > -1,\ z \in \mathbb{U}), \tag{1.8}$$

where

$$F_{q,\delta+1}(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma_q(n+\delta)}{[n-1]_q! \, \Gamma_q(1+\delta)} z^n = z + \sum_{n=2}^{\infty} \frac{[\delta+1; n-1]_q}{[n-1]_q!} z^n. \tag{1.9}$$

Making use of (1.8) and (1.9), we have

$$\mathcal{R}_q^\delta f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma_q(n+\delta)}{[n-1]_q! \, \Gamma_q(1+\delta)} a_n z^n \qquad (z \in \mathbb{U}). \tag{1.10}$$

$$\begin{aligned}\mathcal{R}_q^0 f(z) &= f(z),\\ \mathcal{R}_q^1 f(z) &= z\mathfrak{D}_q f(z),\\ \mathcal{R}_q^m f(z) &= \frac{z\mathfrak{D}_q^m(z^{m-1}f(z))}{[m]_q!} \quad (m\in\mathbb{N}).\end{aligned}$$

Also we have

$$\mathfrak{D}_q(\mathcal{R}_q^\delta f(z)) = 1 + \sum_{n=2}^{\infty}\Theta_n(q,\delta)a_n z^{n-1}, \tag{1.11}$$

where

$$\Theta_n := \Theta_n(q,\delta) = \frac{[n]_q\,\Gamma_q(n+\delta)}{[n-1]_q!\,\Gamma_q(1+\delta)}. \tag{1.12}$$

It is easy to check that

$$z\mathfrak{D}_q(F_{q,\delta+1}(z)) = \left(1+\frac{[\delta]_q}{q^\delta}\right)F_{q,\delta+2}(z) - \frac{[\delta]_q}{q^\delta}F_{q,\delta+1}(z) \quad (z\in\mathbb{U}). \tag{1.13}$$

Making use of (1.8)-(1.13) and the properties of Hadamard product, we obtain the following equality

$$z\mathfrak{D}_q(\mathcal{R}_q^\delta f(z)) = \left(1+\frac{[\delta]_q}{q^\delta}\right)\mathcal{R}_q^{1+\delta}f(z) - \frac{[\delta]_q}{q^\delta}\mathcal{R}_q^\delta f(z) \quad (z\in\mathbb{U}). \tag{1.14}$$

From (1.10), we note that

$$\lim_{q\to 1^-} F_{q,\delta+1}(z) = \frac{z}{(1-z)^{\delta+1}}, \qquad \lim_{q\to 1^-}\mathcal{R}_q^\delta f(z) = f(z) * \frac{z}{(1-z)^{\delta+1}}.$$

Thus, when $q\to 1^-$ we can say that Ruscheweyh $q-$differential operator reduces to the differential operator defined by Ruscheweyh [9] and (1.14) gives the well-known recurrent formula for Ruscheweyh differential operator.

Majorization problems for the class $\mathcal{S}^* = \mathcal{S}^*(0)$ had been investigated by MacGregor [5], further Altintas et al. [1] investigated a majorization problem for $\mathcal{S}(\gamma)$ the class of starlike functions of complex order γ $(\gamma\in\mathbb{C}\setminus\{0\})$, and Goyal and Goswami [2] generalized these results for the class of analytic functions involving fractional operator. Very lately, Tang and Deng [12] considered majorization properties for multivalent analytic functions related to the Srivastava-Khairnar-More operator and exponential function.

In this paper, using Ruscheweyh $q-$differential operator defined by (1.10) and motivated by recent works of [8], we define a new subclass of uniformly starlike functions associated with $q-$calculus operator, which are subordinate to exponential function, and investigate a majorization problem. Further we point out some special cases of our result.

Definition 1.1. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{RS}_q^\delta(\beta, e^z)$, if and only if

$$\left[\frac{z\mathfrak{D}_q(\mathcal{R}_q^\delta f(z))}{\mathcal{R}_q^\delta f(z)} - \beta\left|\frac{z\mathfrak{D}_q(\mathcal{R}_q^\delta f(z))}{\mathcal{R}_q^\delta f(z)} - 1\right|\right] \prec e^z, \tag{1.15}$$

where $\delta > -1$, $\beta > 0$ and $z \in \mathbb{U}$.

For $\beta = 0$ we have $\mathcal{RS}_q^\delta(\beta, e^z) \equiv \mathcal{RS}_q^\delta(e^z)$:

$$\frac{z\mathfrak{D}_q(\mathcal{R}_q^\delta f(z))}{\mathcal{R}_q^\delta f(z)} \prec e^z$$

where $\delta > -1$, $\beta > 0$ and $z \in \mathbb{U}$.

Further by taking $q \to 1^-$ and $\delta = 0$ we have $\mathcal{RS}_q^\delta(e^z) \equiv \mathcal{S}^*(e^z)$:

$$\frac{zf'(z)}{f(z)} \prec e^z \quad (z \in \mathbb{U}).$$

2. A majorization problem for the class $\mathcal{RS}_q^\delta(\beta, e^z)$

We state the following $q-$analogue of the result given by Nehari (cf. [7]) and Selvakumaran et al. [10].

Lemma 2.1. *If the function $\phi(z)$ is analytic and $|\phi(z)| < 1$ in $\mathbb{U}$, then*

$$|\mathfrak{D}_q\phi(z)| \leq \frac{1 - |\phi(z)|^2}{1 - |z|^2}. \tag{2.1}$$

Theorem 2.2. *Let the function $f \in \mathcal{A}$ and suppose that $g \in \mathcal{RS}_q^\delta(\beta, e^z)$. If $\mathcal{R}_q^\delta f(z)$ is majorized by $\mathcal{R}_q^\delta g$ in $\mathbb{U}$, then*

$$|\mathcal{R}_q^{\delta+1} f(z)| \leq |\mathcal{R}_q^{\delta+1} g(z)| \quad (|z| \leq r_1), \tag{2.2}$$

where $r_1 = r_1(\delta, \beta)$, is the smallest positive root of the equation

$$r^2 q^\delta e^r - r^2\{[\delta]_q - ([\delta]_q + q^\delta)\beta\} - q^\delta e^r - 2rq^\delta(1+\beta) + \{[\delta]_q - ([\delta]_q + q^\delta)\beta\} = 0, \tag{2.3}$$

where $[\delta]_q > ([\delta]_q + q^\delta)\beta + q^\delta e$ and $\beta \geq 0$.

Proof. Since $g \in \mathcal{RS}_q^\delta(\beta, e^z)$, we find from (1.15) that

$$\left[\left(\frac{z\mathfrak{D}_q(\mathcal{R}_q^\delta g(z))}{\mathcal{R}_q^\delta g(z)}\right) - \beta\left|\frac{z\mathfrak{D}_q(\mathcal{R}_q^\delta g(z))}{\mathcal{R}_q^\delta g(z)} - 1\right|\right] = e^{w(z)}, \tag{2.4}$$

where $w(z) = c_1 z + c_2 z^2 + c_3 z^3 + \cdots$ is analytic in $\mathbb{U}$, with $w(0) = 0$ and $|w(z)| \leq |z|$ for all $z \in \mathbb{U}$. Letting

$$\varpi = \frac{z\mathfrak{D}_q(\mathcal{R}_q^\delta g(z))}{\mathcal{R}_q^\delta g(z)}$$

in (2.4), we have

$$\varpi - \beta|\varpi - 1| = e^{w(z)}$$

which implies

$$\varpi = \frac{e^{w(z)} - \beta\ e^{-i\varphi}}{1 - \beta\ e^{-i\varphi}}.$$

This is, from (2.4), we get

$$\frac{z\mathfrak{D}_q(\mathcal{R}_q^\delta g(z))}{\mathcal{R}_q^\delta g(z)} = \frac{e^{w(z)} - \beta e^{-i\varphi}}{1 - \beta\ e^{-i\varphi}}. \tag{2.5}$$

Now, by applying the relation (1.14) in (2.5), we get

$$\frac{\mathcal{R}_q^{\delta+1} g(z)}{\mathcal{R}_q^\delta g(z)} = \frac{[\delta]_q - ([\delta]_q + q^\delta)\beta e^{-i\varphi} + q^\delta e^{w(z)}}{([\delta]_q + q^\delta)(1 - \beta\ e^{-i\varphi})} \tag{2.6}$$

which yields that

$$\left|\mathcal{R}_q^\delta g(z)\right| \leq \frac{([\delta]_q + q^\delta)\ (1+\beta)}{[\delta]_q - ([\delta]_q + q^\delta)\beta - q^\delta e^{|z|}} \left|\mathcal{R}_q^{\delta+1} g(z)\right|. \tag{2.7}$$

Since $\mathcal{R}_q^\delta f$ is majorized by $\mathcal{R}_q^\delta g(z)$ in $\mathbb{U}$, we have

$$\mathcal{R}_q^\delta f(z) = \phi(z)\mathcal{R}_q^\delta g(z).$$

By applying $q-$differentiation with respect to z, we get

$$z\mathfrak{D}_q(\mathcal{R}_q^\delta f(z) = z\mathfrak{D}_q(\phi(z))\mathcal{R}_q^\delta g(z) + z\phi(z)\mathfrak{D}_q(\mathcal{R}_q^\delta g(z)). \tag{2.8}$$

Noting the fact that Schwarz function $\phi(z)$ satisfies the $q-$analogue of the result given by Nehari (cf. [7]) proved in Lemma 2.1,

$$|\mathfrak{D}_q\phi(z)| \leq \frac{1 - |\phi(z)|^2}{1 - |z|^2} \tag{2.9}$$

and using (1.14), (2.7) and (2.9) in (2.8), we have

$$|\mathcal{R}_q^{\delta+1} f(z)| \leq \left(|\phi(z)| + \left(\frac{1 - |\phi(z)|^2}{1 - |z|^2}\right) \frac{|z| q^\delta (1+\beta)}{[\delta]_q - ([\delta]_q + q^\delta)\beta - q^\delta e^{|z|}}\right) |\mathcal{R}_q^{\delta+1} g(z)|.$$

Setting $|z| = r$ and $|\phi(z)| = \rho$ $(0 \leq \rho \leq 1)$, the above inequality leads us to the inequality

$$|\mathcal{R}_q^{\delta+1} f(z)| \leq \left(\rho + \left(\frac{1 - \rho^2}{1 - r^2}\right) \frac{r q^\delta (1+\beta)}{[\delta]_q - ([\delta]_q + q^\delta)\beta - q^\delta e^r}\right) |\mathcal{R}_q^{\delta+1} g(z)|. \tag{2.10}$$

That is,

$$|\mathcal{R}_q^{\delta+1}f(z)| \leq \Theta_1(r,\rho)|\mathcal{R}_q^{\delta+1}g(z)|,$$

where the function $\Theta_1(r,\rho)$ is given by

$$\Theta_1(r,\rho) = \rho + \frac{r(1-\rho^2)q^\delta(1+\beta)}{(1-r^2)\{[\delta]_q - ([\delta]_q + q^\delta)\beta - q^\delta e^r\}}.$$

In order to determine the bound of $\Theta_1(r,\rho)$, we have to choose

$$\begin{aligned} r_1 &= \max\{r \in [0,1) : \Theta_1(r,\rho) \leq 1,\ \rho \in [0,1]\} \\ &= \max\{r \in [0,1) : \Theta_2(r,\rho) \geq 0,\ \rho \in [0,1]\}, \end{aligned}$$

where

$$\Theta_2(r,\rho) = (1-r^2)\{[\delta]_q - ([\delta]_q + q^\delta)\beta - q^\delta e^r\} - r(1+\rho)q^\delta(1+\beta).$$

Obviously, for $\rho = 1$, the function $\Theta_2(r,\rho)$ takes its minimum value, namely

$$\min\{\Theta_2(r,\rho) : \rho \in [0,1]\} = \Theta_2(r,1) = \Theta_2(r),$$

where

$$\Theta_2(r) = (1-r^2)\{[\delta]_q - ([\delta]_q + q^\delta)\beta - q^\delta e^r\} - 2rq^\delta\ (1+\beta).$$

Furthermore, if $\Theta_2(0) = [\delta]_q > ([\delta]_q + q^\delta)\beta + q^\delta\ e$ and $\Theta_2(1) = -2q^\delta(1+\beta) < 0$, then there exists r_1 such that $\Theta_2(r) \geq 0$ for all $r \in [0,r_1]$, where $r_1 = r_1(\delta,\beta)$, the smallest positive root of the equation (2.3). This completes the proof. □

Putting $\beta = 0$ and $\rho = 1$ in Theorem 2.2, we have the following corollary:

Corollary 2.3. *Let the function $f \in \mathcal{A}$ and suppose that $g \in \mathcal{RS}_q^\delta(e^z)$. If $\mathcal{R}_q^\delta f$ is majorized by $\mathcal{R}_q^\delta g$ in $\mathbb{U}$, then*

$$|\mathcal{R}_q^{\delta+1}f(z)| \leq |\mathcal{R}_q^{\delta+1}g(z)|,\ \ |z| \leq r_2, \tag{2.11}$$

where $r_2 = r_2(\delta)$, is the smallest positive root of the equation

$$r^2q^\delta e^r - r^2[\delta]_q - q^\delta e^r - 2rq^\delta + [\delta]_q = 0. \tag{2.12}$$

For $\beta = 0,\ q \to 1^-$ and $\delta = 0$, Corollary 2.3 reduces to the following result:

Corollary 2.4. *Let the function $f \in \mathcal{A}$ be analytic and univalent in the open unit disk $\mathbb{U}$ and suppose that $g \in \mathcal{S}^*(e^z)$. If f is majorized by g in $\mathbb{U}$, then*

$$|f'(z)| \leq |g'(z)|\ \ (|z| \leq r_3),$$

where r_3 is the smallest positive root of $r^2e^r - 2r - e^r = 0$.

3. A majorization problem for the class $\mathcal{R}(\mu,\tau)$

Due to Alitintas et al. [1], we recall the definition of the function class $\mathcal{R}(\mu,\tau)$, the class of functions h of the form

$$h(z) = 1 - \sum_{n=1}^{\infty} c_n z^n \quad (c_n \geq 0\ ; z \in \mathbb{U}), \tag{3.1}$$

which are analytic in $\mathbb{U}$ and satisfy the inequality

$$|h(z) + \mu z h'(z) - 1| < |\tau| \quad (\tau \in \mathbb{C} \setminus \{0\}; \Re(\mu) \geq 0).$$

Further we recall the following lemmas, which will be required in our investigation of the majorization problem for the class $\mathcal{R}(\mu,\tau)$.

Lemma 3.1. ([1]) *If the function h defined by* (3.1) *is in the class $\mathcal{R}(\mu,\tau)$, then*

$$\sum_{n=1}^{\infty} c_n \leq \frac{|\tau|}{1+\Re(\mu)}. \tag{3.2}$$

Lemma 3.2. ([1]) *If the function h defined by is in the class $\mathcal{R}(\mu,\tau)$, then*

$$1 - \frac{|\tau|}{1+\Re(\mu)}|z| \leq |h(z)| \leq 1 + \frac{|\tau|}{1+\Re(\mu)}|z| \quad (z \in \mathbb{U}). \tag{3.3}$$

Theorem 3.3. *Let the function f and g be analytic in $\mathbb{U}$ and suppose that the function g is normalized and also satisfies the following inclusion property:*

$$\left(\frac{z\mathfrak{D}_q(\mathcal{R}_q^\delta g(z))}{\mathcal{R}_q^\delta g(z)}\right) \in \mathcal{R}(\mu,\tau).$$

If $\mathcal{R}_q^\delta f$ is majorized by $\mathcal{R}_q^\delta g$ in $\mathbb{U}$, then

$$|\mathcal{R}_q^{\delta+1} f(z)| \leq |\mathcal{R}_q^{\delta+1} g(z)| \quad (|z| \leq r_4), \tag{3.4}$$

where $r_4 = r_4(\tau,\mu,\delta)$ is the root of the cubic equation

$$\begin{aligned} &q^\delta|\tau|r^3 - \{(q^\delta - [\delta]_q)(1+\Re(\mu)) - 2|\tau|\}r^2 \\ &- [2(1+\Re(\mu)) + q^\delta|\tau|]r + (q^\delta - [\delta]_q)[1+\Re(\mu)] = 0 \end{aligned} \tag{3.5}$$

which lies in the interval $(0,1)$ *and* $(q^\delta - [\delta]_q)(1+\Re(\mu)) > 0$.

Proof. For an appropriately normalized analytic function g satisfying the inclusion property, we find from the assertion of Lemma 3.2 that

$$\left|\frac{z\mathfrak{D}_q(\mathcal{R}_q^\delta g(z))}{\mathcal{R}_q^\delta g(z)}\right| \geq 1 - \frac{|\tau|}{1+\Re(\mu)} r \quad (|z| = r,\ 0 < r < 1) \tag{3.6}$$

or, equivalently, that

$$|\mathcal{R}_q^\delta g(z)| \le \frac{(q^\delta+[\delta]_q)(1+\Re(\mu)-|\tau|r)}{(q^\delta-[\delta]_q)(1+\Re(\mu))-q^\delta|\tau|r}|(\mathcal{R}_q^{\delta+1}g(z))| \quad (|z|=r,\ 0<r<1). \tag{3.7}$$

Since

$$\mathcal{R}_q^\delta f(z) \ll \mathcal{R}_q^\delta g(z) \quad (z\in\mathbb{U}),$$

there exists an analytic function ϕ such that

$$\mathcal{R}_q^\delta f(z) = \phi(z)\mathcal{R}_q^\delta g(z) \ \ \textit{and} \quad |\phi(z)|<1.$$

By applying $q-$differentiation with respect to z, we get

$$z\mathfrak{D}_q(\mathcal{R}_q^\delta f(z) = z\mathfrak{D}_q(\phi(z))\mathcal{R}_q^\delta g(z) + \phi(z) z\mathfrak{D}_q(\mathcal{R}_q^\delta g(z)). \tag{3.8}$$

Thus in view of (3.7) and using (1.14), just as in the proof of Theorem 2.2, we have

$$|\mathfrak{D}_q(\phi(z)| \le \frac{1-|\phi(z)|^2}{1-|z|^2} \ \ (z\in\mathbb{U})$$

and

$$|\mathfrak{D}_q(\mathcal{R}_q^\delta f(z)| \le \left(|\phi(z)| + \tfrac{1-|\phi(z)|^2}{1-|z|^2}\cdot\tfrac{(1+\Re(\mu)-|\tau|r)|z|}{(q^\delta-[\delta]_q)(1+\Re(\mu))-q^\delta|\tau|r}\right)|\mathfrak{D}_q(\mathcal{R}_q^\delta g(z))|,$$

$|z|=r,\ 0<r<1$. That is,

$$\begin{aligned}|\mathcal{R}_q^{\delta+1}f(z)| \le & \left(|\phi(z)| + \frac{1-|\phi(z)|^2}{1-r^2}\cdot\frac{(1+\Re(\mu)-|\tau|r)r}{(q^\delta-[\delta]_q)(1+\Re(\mu))-q^\delta|\tau|r}\right) \\ & \times \mathcal{R}_q^{\delta+1}g(z)| \\ =& \Lambda_1(\rho,r)|\mathcal{R}_q^{\delta+1}g(z)|,\end{aligned} \tag{3.9}$$

where $|z|=r,\ 0<r<1$. We set $|\phi(z)|=\rho$ and the function $\Lambda_1(\rho,r)$ defined by

$$\Lambda_1(\rho,r) = \rho + \frac{1-\rho^2}{1-r^2}\cdot\frac{(1+\Re(\mu)-|\tau|r)r}{(q^\delta-[\delta]_q)(1+\Re(\mu))-q^\delta|\tau|r}. \tag{3.10}$$

In order to determine the bound of $\Lambda(\rho,r)$, we have to choose

$$\begin{aligned} r_1 &= \max\{r\in[0,1) : \Lambda_1(\rho,r)\le 1,\ \rho\in[0,1]\} \\ &= \max\{r\in[0,1) : \Lambda_2(\rho,r)\ge 0,\ \rho\in[0,1]\},\end{aligned}$$

where, for $0\le\rho\le 1$.

$$\Lambda_2(r,\rho) = (1-r^2)\{(q^\delta-[\delta]_q)(1+\Re(\mu))-q^\delta|\tau|r\} - r(1+\rho)(1+\Re(\mu)-|\tau|r).$$

Obviously, for $\rho=1$, the function $\Lambda_2(r,\rho)$ takes its minimum value, namely

$$\min\{\Lambda_2(r,\rho):\rho\in[0,1]\} = \Lambda_2(r,1) = \Lambda_2(r),$$

where

$$\Lambda_2(r) = (1-r^2)\{(q^\delta - [\delta]_q)(1+\Re(\mu)) - q^\delta|\tau|r\} - 2r(1+\Re(\mu)-|\tau|r).$$

Furthermore, if $\Lambda_2(0) = (q^\delta - [\delta]_q)(1+\Re(\mu)) > 0$ and $\Lambda_2(1) = -2(1+\Re(\mu) - |\tau|) < 0$, then there exists r_4 such that $\Lambda_2(r) \geq 0$ for all $r \in [0, r_4]$, where $r_4 = r_4(\tau, \mu, \delta)$, the smallest positive root of the equation (3.5) which completes the proof of Theorem 3.3. □

Remark 3.4. Specializing the parameters δ, β in (1.15) one can define the various other interesting subclasses of $\mathcal{RS}_q^\delta(\beta, e^z)$, involving $q-$calculus operator and one can easily derive the result as in Theorem 2.2. Further as mentioned in[11] we can define new subclasses $\mathcal{RS}_q^\delta(\beta, 1+\sin z)$, $\mathcal{RS}_q^\delta(\beta, \cos z)$, and $\mathcal{RS}_q^\delta(\beta, z+\sqrt{1+z^2})$, and investigate a majorization problem for these classes.

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