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초등학생들의 측정으로서 분수에 대한 이해
: 공학도구를 활용한 기호적 중재

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Semiotic mediation through technology: The case of fraction reasoning

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초록

본 연구는 초등학생들이 공학도구를 활용하여 측정으로서의 분수의 과제를 해결하는 과정을 분석하고 해결전략의 변화 과정에 대해서 논의하였다. 초등학생 13명을 대상으로 과제 중심의 임상면담을 실시하였고, 특히 분수를 처음 학습한 3학년 학생들의 측정 문제 해결 전략을 심층분석하였다. 그 결과, 추측하기에서 반복적인 분할하기, 임의의 단위 사용에서 주어진 단위 사용과 같은 두 가지 프로파일이 발견되었다. 각 프로파일의 대표적인 사례를 바탕으로, 공학도구의 활용이 역동적인 단위 개념을 형성하는데 기여하고 또한 분수와 관련된 의미형성과정에 드래깅과 같은 수학적 조작 활동이 영향을 줄 수 있음을 알 수 있었다. 본 연구의 결과가 분수의 다양한 의미를 탐구하고 학습하는 후속 연구를 위한 밑거름이 되길 기대한다.

Abstract

This study investigates students' conceptions of fractions from a measurement approach while providing a technological environment designed to support students' understanding of the relationships between quantities and adjustable units. 13 third-graders participated in this study and they were involved in a series of measurement tasks through task-based interviews. The tasks were devised to investigate the relationship between units and quantity through manipulations. Screencasting videos were collected including verbal explanations and manipulations. Drawing upon the theory of semiotic mediation, students' constructed concepts during interviews were coded as mathematical words and visual mediators to identify conceptual profiles using a fine-grained analysis. Two students changed their strategies to solve the tasks were selected as a representative case of the two profiles: from guessing to recursive partitioning; from using random units to making a relation to the given unit. Dragging mathematical objects plays a critical role to mediate and formulate fraction understandings such as unitizing and partitioning. In addition, static and dynamic representations influence the development of unit concepts in measurement situations. The findings will contribute to the field's understanding of how students come to understand the concept of fraction as measure and the role of technology, which result in a theory-driven, empirically-tested set of tasks that can be used to introduce fractions as an alternative way.

* 주요어 : 측정으로서의 분수, 공학도구, 기호적 중재
* **Key words** : fraction as measure, dynamic technology, semiotic mediation
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Ⅰ. Introduction

The use of fractions is prevalent throughout everyday life (e.g., half of a gallon as the capacity of a container). The Common Core State Standards for Mathematics (CCSSM) suggested a longitudinal focus of fraction instruction from third grade to middle school (Common Core State Standards Initiative, 2010). National Mathematics Advisory Panel(2008) indicated that algebra is the gateway for later success in mathematics, but students often struggle with algebra due to poor proficiency with fraction concepts. One of the main reasons for this struggle is a dominant approach to introducing fractions via a part-whole conception (a/b means a parts out of b equal parts) which might limit children's understanding of fraction as quantity, measure, or amount (Behr, Harel, Post, & Lesh, 1992). This is not an issue only for the United States. Many countries emphasize this similar approach to teaching and learning fractions. For example, the current 2015 revised Korean national mathematics curriculum also emphasizes equal partitioning of a single object then identifies unit fractions to extend the number system in introducing fractions (Ministry of Education, 2015; Son, Hwang, & Yeo, 2020).

This could be exacerbated by limited opportunities to use various representations to visualize fraction quantities. To enhance a better understanding of fraction concepts, the CCSSM emphasizes various visual fraction models (Webel, Krupa, & McManus, 2016). In addition, the lack of opportunities to interpret fractions in other ways might constitute a negative influence on what students learn about advanced concepts and operations (Lee & Pang, 2014; Thompson & Saldanha, 2003). Thus, there has been a call for examining alternative approach with continuous models (e.g., tape diagram) rather than discrete ones (e.g., set models), and developing empirically-tested tasks to promote fundamental understanding of fraction concepts.

Due to the accelerating pace of social and technological change, digital technology has been increasingly integrated into the mathematics classroom, and this integration has piqued researchers' interest in understanding the role of technology in teaching, learning, and doing mathematics. Most studies show plenty of evidence that digital technology allows visualizing students' mathematical ideas (e.g., Moreno-Armella, Hegedus, & Kaput, 2008), to facilitate organizing and analyzing data (e.g., Konold, Harradine, & Kazak, 2007), and to support investigation in various content areas (e.g., Dick & Hollerbrands, 2011). In the domain of fractions, Steffe, Olive(2002) designed JavaBars to develop cognitive models of children's fractional knowledge. Students manipulate with visualized representations in potential operations using JavaBars. Hands-on manipulatives (e.g., fraction circles or Cuisenaire rods) would not be able to manipulate part of the partitioned whole leaving the whole intact, whereas this digital tool enables students to operationalize the part-whole relations (Olive & Labato, 2008).

Many researchers have discussed the importance of fraction learning from different approaches and suggest the promise of the use of digital technology to support the formation of students' conceptual understanding of fractions rather than to use the practice of procedural knowledge. However, there is relatively little research that has examined elementary students' understanding of fractions from a measurement approach (Davydov & Tsvetkovich, 1991), which examined the ratio between units and quantities. In addition, there is much less research that has investigated elementary students' understanding of fractions as measure within a technological environment (Simon, Placa, Avitzur, & Kara, 2018).

The current study explores students' understanding of fractions in terms of quantitative relationships between units and quantities and the role of digital technology to mediate their understanding. The research questions driving the study were: (1) How do children describe relationships between units and quantities and use a digital tool when solving the measurement tasks?; and (2) In what ways does a digital tool appear to mediate students' fraction understanding? As practitioners implement this measurement approach to fractions with technology in their field, it might be fundamental for stakeholders such as teachers, researchers, and curriculum developers to understand the nature of students' understanding of fraction as measure when such learning takes place within a technological environment. Beyond instructional potential, the study can build a bridge between educational access and equity and technology by providing high-quality learning opportunities for all students (Schoenfeld, 2002).

Ⅱ. Theoretical background

1. Semiotic mediation through digital technology

Semiotic mediation is the process of meaning-making from signs (Bartolini Bush & Mariotti, 2008; Yeo, 2020). Vygotsky(1978) defined signs as symbolic tools (e.g., speech, number) in mental work, which was distinguished from material tools (e.g., computer, ruler) in the physical work. Both symbolic and material tools can mediate children's understanding. Suppose that a child is solving an addition problem. The child might use a snap cube to manipulate (material tool) or generate a number expression with mathematical symbols (symbolic tool). The understandings of addition operation and results can be mediated by how to use the tools they are using. If the child uses a snap cube, putting together two different colored cubes can be joining action between two quantities and the total amount of cubes refers to the result of the addition. If the child uses symbolic notations with numbers, the context of the problem can be signified through those numerical symbols.

Building on the theory of semiotic mediation, a recent study (Noh, Lee, & Moon, 2019) employed a reasoning task on the relationships between properties of parallelograms. They analyzed multimodal semiotic resources students produced and used in order to develop understandings of the spatial structures of diagrams. For example, students articulated their reasoning with consistent use of words (e.g., "here", "go") and gestures (e.g., moving a fingertip toward a specific direction). As a result, they identified semiotic potential which is used as a means in solving reasoning tasks and as a tool of semiotic mediation.

When it comes to the use of digital technology, specifically, the signs generated by technological tools are internalized and directed to other signs (e.g., words, drawings, and gestures) in the social activity. Bartolini Bush, Mariotti(2008) described semiotic mediation as a process of meaning-making through internalizing the signs that are produced from an external, interpersonal activity such as many drawings with Dynamic Geometry Software (DGS). Then, a goal-oriented activity such as dragging and tracing in DGS can be internalized to construct individual meanings. The use of dragging and tracing tools also facilitates the whole-class discussion to mediate the collaborative meaning-making process in the activities. Arzarello, Robutti(2008) termed the "semiotic bundle" as the collection of the signs and their reciprocal relationships. Therefore, the goal of teaching can be to generate mathematical signs through the meaning-making process of learners.

Mediation through dynamic mediators which are generated by digital technology invokes mathematical relationships and properties to students. Ng, Sinclair(2015) investigated how children's geometric conceptions of symmetry emerged and transformed through language, gestures, and the use of technology and what role of dynamic digital technology has in the learning of symmetry from the semiotic perspective. Children manipulated squares and/or asymmetry lines in "The Symmetry Machine" activity, dragging squares and observing what happens to the corresponding symmetry points on the interactive whiteboard. The actions to move squares as the pivot sign produced both the artifact signs (the movement of a particular square) and the mathematical signs (the square as a mathematical object with the line of symmetry) as the following statement: "it will move like opposite, like this one will move to the windows, and this one will move to the wall" (p. 432). In the process of semiotic mediation, visual representations, gestures, and oral, written languages are obviously crucial mediators to facilitate the whole-class discussions (Battista, 2008; Kaur, 2015). More importantly, the use of technology activates new types of potential mediators. The dragging tool on the Sketchpad was utilized to mediates children with the dynamic action of the line of symmetry and the image squares. This tool allowed the children to attain the main features of symmetry: the shape of one side is the same as the other; a component is in the same distance as the counterpart; pre-image and image make a pair.

2. Measurement approach to fractions

Researchers have explored alternative instructional interventions to develop a fundamental and conceptual understanding of fractions (Davydov & Tsvetkovich, 1991; Empson, 1999; Empson, Junk, Dominguez, & Turner, 2006; Morris, 2000). The measurement approach (Davydov & Tsvetkovich, 1991; Morris, 2000) emphasizes mathematical generalities between whole numbers and rational numbers. This approach is based on the concepts of measurement units and unit ratios between whole units and partitive units. Researchers have explored the measurement approach to developing a fundamental understanding of fractions as quantity, not merely an arrangement of whole numbers. Fraction as measure is one of the five core subconstructs of fraction concepts which include part-whole, quotient, ratio, and operator (Behr et al., 1992; Kieren, 1988). Traditionally, the concept of fraction as measure refers to using a subdivided number line with fractional length units (Lamon, 2012). For example, Saxe, Diakow, Gearhart(2013) developed a curriculum for teaching integers and fractions on the number line which is a continuous linear measurement model. They defined a unit as a distance between zero and 1 or its equivalence distance and subunits as equally partitioned distances of a unit. Based on the part-whole concept and unit-subunit relationships, they defined denominators as the number of subunits in a unit and numerators as counts of subunits. This approach was designed to emphasize noticing the distance from zero or a benchmark fraction was crucial in comparing and locating different fractions on the number line with continuous units rather than discrete ones. However, Davydov, Tsvetkovich(1991) designed an instructional approach around the idea of fraction as measure. This approach is not restricted to only fraction concepts, but penetrates the whole number system and rational numbers. Consider measuring length A with five iterations of the unit (a). This means A is five times the length of unit a ($A = 5a$ or $A/a = 5$), and so length A can be represented as whole number 5 in terms of unit a. If length B is measured with two units of a with a remainder, the remainder could be measured with a new smaller unit (b). This smaller unit is compared with the original unit ($b = \frac{1}{2}a$, then $B = a + a + b = 2a + b = 2a + \frac{1}{2}a = (2 + \frac{1}{2})a = 2\frac{1}{2}a = 5/2\ a$), then length B can be represented as fraction $2\frac{1}{2}$ in terms of unit a. Both whole numbers (the case of length A) and fractions (the case of length B) have the same structure of generating numbers as a result of the measurement. The incorporation of a measurement approach to fraction learning has been investigated sporadically in developing instructional materials (Dougherty, 2008), reflecting current approaches to introduce fractions (Kang & Ko, 2003), identifying learning trajectory through teaching experiments (Simon et al., 2018), implementing to classroom settings (Schmittau & Morris, 2004), and comparing international textbooks (Alajmi, 2012).

In this study, I devised a novel digital technology to explore children's thinking about fractions based on the measurement approach and investigated how children were engaged in a series of measurement tasks. The purpose of the devised tasks was to provide opportunities to explore the relationship between quantities and units. At the same time, I also hypothesized that children might construct different sense making of fractions concepts, depending on how much they manipulate the tool dynamically.

3. Developing fraction concepts with digital technology

In the domain of fractions, several studies have used digital technology. Although hands-on manipulatives (e.g., fraction circles) are more likely to be available to children in the classrooms, digital technology distinctively provides a powerful and interactive environment for them to represent fractions in different models. For example, Hunting, Davis, Pearn(1996) conducted a teaching experience with young children by investigating fraction learning based on their whole-number knowledge. They provided an operator-like technology called Copycat, which instantiated the fraction as operator concept (e.g., to find 3/4 of something is the same as to divide by 4, then to multiply by 3). Children were encouraged to determine which fraction was appropriate to represent fraction as operator between numerical value of input and corresponding output. In addition, they determined what would be a possible input (or output) through the operation with selected fractions in the Copycat environment. This study showed that the technological environment could develop children's understanding of the relationships between parts (output) and wholes (input) of a discrete representation (counters). They reported that children's whole-number knowledge (e.g., multiples, divisibility) enabled them to develop fraction concepts in solving fraction comparison problems.

Suh, Moyer, Heo(2005) also conducted a teaching experiment in a fifth-grade classroom through virtual fraction manipulatives from the National Library of Virtual Manipulatives. Also, the researchers used one applet from the NCTM online resources. The researchers examined students' learning of equivalent fractions (e.g., $5/7 = 10/14 = 15/21$) through pictorial representations. Students could manipulate up- and down-arrow buttons to change the number of partitions of the whole. When using the arrow buttons, students could explore multiple representations very quickly and make connections to symbolic representations. Ultimately, this tool allowed fifth graders to identify a mathematical relationship related to factors and multiples in denominators and numerators. Although the previous studies showed the affordance of digital technology in learning and teaching fractions with discrete or area models, little studies have used continuous and length models to develop fractions concepts.

Ⅲ. Method

1. Participants

13 third-grade students were drawn from a slightly larger unpublished study that included three grade levels (third, fourth, and fifth) within a single pubic school district in the Midwest region of the United States (Fall 2019). In the school district, the Everyday Math textbook series were used, which was initially introduced by basic fraction ideas in the third-grade level and emphasizing the part-whole concept of fractions with fraction circles. In this study, I focused on third graders who might develop initial fraction concepts such as identifying fractions and unit fractions in their classrooms. These third graders also experienced linear measurement with the inch unit and already learned basic fractional terms (e.g., a half inch, a quarter inch) in the Everyday Math curriculum. In addition, the participants were expected to have informal knowledge of fractions drawn from everyday life related to money, time, and food (Mack, 2001; Yeo, 2019). I conducted one time task-based interview after regular school hours. During the interview, my role as a researcher was to elicit their strategies verbally and to facilitate them in solving the measurement tasks with manipulation of a digital tool, rather than to teach the concept of fractions. The duration of the interview time was from 45 to 64 minutes. This variation was stemmed from administrating the interviews. As some participants struggled with initial tasks, they were able to solve an additional subtask and/or skipped a challenging task.

2. Dynamic Ruler tasks

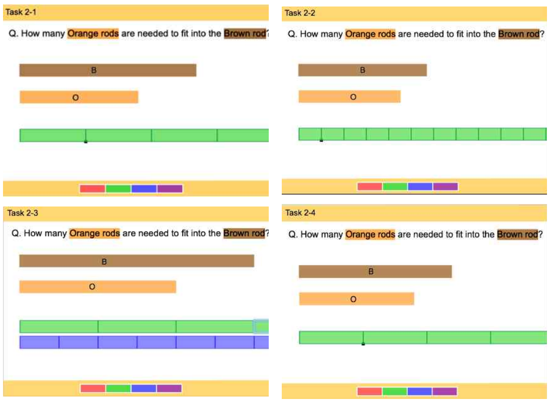
I developed the Dynamic Ruler tasks with New Cabri software (Laborde, 2016), which allows exploration of mathematical relationships and properties with dynamic actions. Like a ruler, this tool was designed to measure the length with adjustable units, which change simultaneously the unit size for the entire ruler. Although hands-on manipulatives (e.g., Cuisenaire rods) can be accessible in the classroom and familiar with children to explore numerical concept development, such dynamic features can be only implemented in a digital environment. The Dynamic Ruler tasks were stemmed from prior research on the teaching and learning of fractions from a measurement approach (Dougherty, 2008; Simon et al., 2018) and have been refined after the pilot studies. Although Simon and his colleagues (2018) employed the JavaBars tool to explore children's fraction as measure concept, I devised a new tool for two reasons. First, the Dynamic Ruler could be intuitively used by students by dragging. In JavaBars, students had to figure out the way how to manipulate with feature buttons. For example, if they would like to break down a piece with small parts, they have to choose the number of pieces then click the "PARTITION" button. However, in the Dynamic Ruler environment, students could change the size of units by dragging, which was easy and intuitive. Second, the Dynamic Ruler tasks had potential to develop flexible concept of units. In Simon and colleagues' (2018) study, the partial units were already pre-partitioned (e.g., 1/2, 1/3, 1/4). Therefore, students did not have any chance to think about other sizes of unit. However, the Dynamic Ruler provided the autonomy to choose partial unit sizes with continuous variations.

As a result, the four main task series (Table 1) were developed including the tutorial (exploring four features of the tool: moving a Dynamic Ruler to any place on the screen, adjusting the size of a unit, trimming the number of pieces, and selecting other different Dynamic Rulers), Task1 (using units to measure a length), Task 2 (using partial units to measure a leftover as unit fractions), and Task 3 (using partial units to measure a leftover with proper fractions). In addition, each task series consisted of four subtasks. The first subtask was an entry task and uses an initially longer adjustable unit of Dynamic Ruler (Figure 1). The second task was more difficult in terms of the fractional part and uses shorter adjustable units to encourage exploration with differently sized units compared to the first subtask. The third was an optional task for students who were struggling with previous tasks. It provided a pair of fixed unit-size rulers to ensure that students examine possibilities and consider the relationship between the two fixed rulers. The fourth subtask used the longer adjustable unit and presented a more difficult ratio such as 1/3.

[Table 1] Measurement task series

	sub1	sub2	sub3	sub4
Task 1	4:1	3:1	4:1	5:1
Task 2	3:2	5:4	3:2	4:3
Task 3	5:2	9:4	5:2	8:3

Note. The ratio was between the length quantity and the unit length. For example, the ratio in Task 2-1 was 3 : 2 = length quantity (brown rod) : orange rod (unit length) as shown in Figure 1.



[Fig. 1] Task Series 2

3. Data sources and analysis

As a primary data source, task-based clinical interviews (Ginsburg, 1997) was conducted with individual students. Each interview was recorded by a screencasting tool (Quicktime Player) including their manipulations with a mouse and verbal explanations. All recorded videos were transcribed. This data allowed me to track how individual students interacted with the digital tool when they were engaged in the Dynamic Ruler tasks. Observation notes during the interview and reflection papers after the interview were also collected for triangulation.

I conducted multiple rounds of analysis. First, I analyzed each student's interview data to categorize their understanding of fractions in measurement with a mathematics education researcher from a holistic perspective: no partitioning (using an undecisive size of length units), pre-partitioning (counting the number of unit length and guessing the leftover with dragging), partitioning (counting the number of partitive unit lengths with dragging). We (I and the mathematics education researcher) coded the initial solution strategies from Task series 2 and 3 each other and came to reconcile any disagreement based on discussion until consensus. We did not include Task series 1 because the embedded mathematical concept was related to the whole number. Students' different ways of thinking with the tool were analyzed to identify conceptual profiles across multiple tasks (Mortimer & El-Hani, 2014). 5 students used the same types of strategies to solve different tasks during the interview, whereas 8 students had shifted the way of how to use the Dynamic Ruler across the tasks. To be specific, 2 students began with no partitioning but ended up with pre-partitioning and 6 students transformed from pre-partitioning to partitioning.

Second, I employed microgenetic analysis on the 13 students to capture the nature of students' understanding and the role of technology by tracing moment-by-moment change within a short span of time (Schoenfeld, Smith, & Arcavi, 1993). In other words, I focused on the two major aspects, mathematical words and visual mediators (Sfard, 2008). Mathematical words represent mathematical objects in discourse (e.g., fractions, measurement concepts) and visual mediators serve as instruments for semiotic mediation in students' learning (e.g., dragging to adjust unit size). We coded each sentence and a single action separately. This approach also provides qualitative changes by observing each case and by documenting fine-grained information.

Lastly, I investigated how students' conceptual profiles were mediated by the digital tool with synchronous and asynchronous analyses (Arzarello, 2006). In the synchronous phase, I examined the relationships between mathematical words and visual mediators at certain points from the microgenetic data tables. In the asynchronous phase, I looked for evidence of each student's conceptual profile as they use the Dynamic Rulers in successive moments. In this phase, my analysis focused on identifying and refining the mediated process that students expressed across the tasks. I met weekly with the other mathematics education researcher to discuss ongoing conjectures and reconcile differences in coding and interpretations of the data. These discussions were aimed at developing internal consistency and coherence with the data, and for consistency and coherence with preliminary research findings. Then, we finalized representative cases from each profile (Yin, 2014): pre-partitioning to partitioning (Sam[1]) and no partitioning to pre-partitioning (Mary). The two third-grade cases were chosen because they did show clear changes in their strategies in terms of the word use and the use of visual mediators in solving measurement tasks with the Dynamic Ruler across different tasks. In the case of Mary, she only represented two students out of 8 who changed the solution strategies. However, this was still significant because the focus of this study was how their fraction understandings had been changed through mediations of a digital tool, which might be more important than the group of students who used the same level of understanding of fractions in measurement across the tasks.

[1] All names are pseudonyms.

Ⅳ. Results

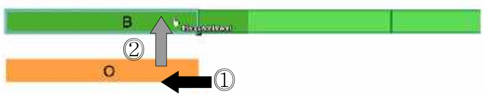
In this section, I describe the cases of Sam and Mary, who showed two distinct approaches: from guessing (pre-partitioning) to subsequent partitioning (partitioning), from random-sized unit (no partitioning) to the given size (pre-partitioning).

1. Sam: Guessing to subsequent partitioning

Sam, a third grader, was a representative case of children who changed his discourse across Tasks 2-2 and 2-4 from guessing the leftover part to partitioning it subsequently. Task 2-2 involved $\frac{1}{4}$ and Task 2-4 involved ⅓. Because children tend to experience difficulty to represent ⅓ with appropriate terms or symbols, Sam struggled a little bit in the latter task.

1) Guessing the leftover

In the next episode, Sam initially solved Task 2-2 by measuring the brown rod with the Green Dynamic Ruler, in which the pieces were the same size as the orange rod (Figure 2).



Note. The brown rod was in line with the Green Dynamic Ruler, which was why the letter B appears in horizontal alignment with the ruler. Sam first adjusted the size of the Green Dynamic Ruler like the orange rod, then moved up. The gray arrow shows the movement of the ruler and the black arrow shows the adjustment of the size.

[Fig. 2] Sam's strategies in Tasks 2-2

Sam (S): One and a quarter.

Interview (I): Can you tell me more? How did you get that? One and a quarter of what?

S: So, one and a quarter of orange.

I: Okay. So, can you tell me? How did you get that?

S: Wait, let me check. Yeah, it's a one and a quarter, I think.

I: How do know this is a quarter of the orange?

S: Because this is the half of orange rods and then there's two, two quarters in the one half. So, this is one and, and the quarter.

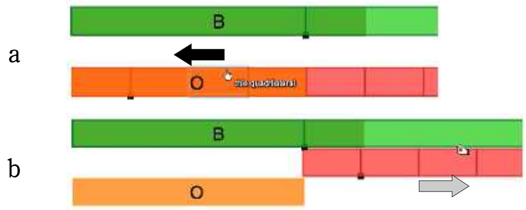
I: Okay. Can you use like a red ruler to show this is a quarter?

S: What's red?

I: Red ruler. So, do not delete that. Yeah, you can use this tool and then know this much is a quarter of the orange. I'm just curious about that relationship. How can you make me convinced that?

S: Like this. So, this is a half and then one quarter.

It was not surprising when Sam made the orange rod and a piece of the Green Dynamic Ruler the same size because he utilized the same type of the strategy in Task 2-1. He explained how he got the quarter size by halving the green piece and halving the half of the green piece. He was able to estimate the leftover size as a quarter ("one and a quarter"). When asked to verify the size of the leftover with the Red Dynamic Ruler, he created the length of one red piece by partitioning the orange rod into four pieces (Figure 3-a). He indicated that two red pieces were half of the orange rod, and one was a quarter. Then, Sam matched the leftover size and one red piece's size by placing the Red Dynamic Ruler below the green one (Figure 3-b).

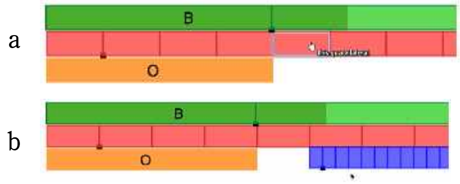


Note. The brown rod was in line with the Green Dynamic Ruler, which was why the letter B appears in horizontal alignment with the ruler. In 3-a, Sam first adjusted the size of the Red Dynamic Ruler to partition the orange rod into four pieces. The black arrow shows the adjustment of the size. In 3-b, he measured the leftover with one piece of the Red Dynamic Ruler. The gray arrow shows the movement of the ruler.

[Fig. 3] Sam's method of checking the leftover length with an additional dynamic ruler

2) Subsequent partitioning

Even though he estimated the leftover part with the Green Dynamic Ruler, Sam demonstrated his further understanding of how to measure the leftover that equals a partial unit size by using the Red Dynamic Ruler. This partial unit concept arose from the size of the leftover. Because he already knew the size of the leftover was a quarter by estimating the size of the remaining part with double-halving (partitioning a half into two parts), he seemed to decide to directly partition the orange rod into four quarters. Such a partitioning idea was evidenced even more during the following episode in Task 2-4.



Note. The brown rod was in line with the Green Dynamic Ruler, which was why the letter B appears in horizontal alignment with the ruler. In 4-a, Sam used the one fourth unit with the Red Dynamic Ruler to measure the leftover part. Then, in 4-b, he measured the leftover of the leftover with the one sixteenth piece of the Blue Dynamic Ruler.

[Fig. 4] Sam's subsequent partitioning

S: Wait, what is...I need to check. I need to check the quarter. Hmm. Did I even measure it right? Yes, I did. What is the...Oh wait, I think I know. One quarter and one and the half of quarter...One half...I mean one full quarter and one and a quarter of a quarter.

I: One and a quarter of a quarter. Okay. Why do you think this is...

S: Wait, I want to check if that's right. What is the quarter of a quarter?

I: Yeah. I am curious about it.

S: How long is it? Oh my. Nope. Wait, is that right? No. Oh, I think I know. Four quarters and a quarter of a quarter and the quarter.

I: How many quarters are there?

S: Look, look. So, four quarters equal one orange rod. And then one quarter. Plus, I just did four quarters of the quarters. And then I just look at it, and then it kind of looks like there's a quarter in a quarter of a quarter.

I: Interesting. Why did you use the third ruler?

S: This? To check how many quarters there are, I mean...to check how long the quarter is in the quarter.

Sam changed the size of the Green Dynamic Ruler so that one unit on the ruler was the same size as the orange rod, as he did in previous tasks (2-1 and 2-2). Then, he pulled out the Red Dynamic Ruler to measure the leftover part (Figure 4-a). While he usually used additional Dynamic Rulers when I asked him to use those for probing, this was the first time he used multiple Dynamic Rulers of his own volition to figure out the leftover length more precisely. Sam used fourths to measure the leftover like in previous tasks, but it was not enough to measure a new leftover. Therefore, he decided to iterate the same strategy to measure the difference between the leftover and one red piece by using the Blue Dynamic Ruler (Figure 4-b). He partitioned a fourth of the orange rod into four pieces and generated a unit on the Blue Dynamic Ruler that was 1/16 the size of a unit on the Green Dynamic Ruler. Finally, his solution for the task was the sum of three different unit sizes (a green piece: one whole, a red piece: one fourth, a blue piece: 1/16). The expected solution was 4/3 (1.3333), but his answer was close enough given the limitations of the Dynamic Ruler environment ($1 + \frac{1}{4} + 1/16 = 21/16 = 1.3125$).

When Sam first noticed the leftover between two rods, he used the Red Dynamic Ruler in Task 2-4. Rather than guessing the new leftover part, he subsequently used the Blue Dynamic Ruler to measure the new leftover between the brown rod and five red pieces. He drew three different unit sizes with a full understanding of the relationship: green equals the whole, red equals a quarter, and blue equals a quarter of a quarter. And this reciprocal relationship was based on the understanding of the whole and fractional units ("four quarters is equal to one orange rod").

3) Mediation process

When solving Task 2-4, Sam initially constructed the Green Dynamic Ruler with pieces that were the same size as the given unit. Then, he brought up the Red Dynamic Ruler and the Blue Dynamic Ruler to figure out the leftover size. Matching one green segment and four red segments, he found that the leftover size must be at least one quarter of the original unit. Subsequently, he adjusted the Blue Dynamic Ruler to partition one red piece into four blue pieces. In this task, Sam manipulated three rulers and adjusted the size of the rulers differently. Finally, he figured out the relationship between one green segment (1), one red segment ($\frac{1}{4}$), and one blue segment (1/16) with recursive measuring the leftovers, "Four quarters and a quarter of a quarter and the quarter".

What would influence such changing his discourse between two tasks? First, Sam seemed to realize that any size of length could be partitioned with the Dynamic Ruler. As checking the length of the leftover in Task 2-2, he used the recursive measurement strategy in Task 2-4. When dragging the Dynamic Ruler to change the size of unit length, it was possible to partition the leftover into equal-sized parts. Even though the leftover was a different length, Sam used dragging to make a much smaller size of unit length. Second, because Sam was a third grader, he had been more familiar with everyday fraction terms, such as a half and a quarter, rather than a third or a fourth as formal expressions in school mathematics. After the interview, he mentioned where his simple fraction expressions came from, “So, I saw basically money is one. Um, money is math too. And I used quarters, whole, and stuff like that, which is math too”. This analogy might be why he only partitioned units into two or four parts as a repeated halving strategy. In Task 2-4, he volunteered to articulate his thinking process through partitioning the given unit into a specific number of parts (“Wait, what is? I need to check. I need to check the quarter. Hmm? Did I even measure it right?”). He measured the leftover rather than estimating it, even though the size of the leftover was not familiar to him (one third), which was a key difference between guessing and “checking”. In a series of actions, he seemed to use his knowledge of partitioning money to the measuring situation. Like money unit (e.g., dollar), one length unit could be partitioned into two parts ($\frac{1}{2}$) and half of the unit also could be partitioned into two parts ($\frac{1}{4}$). This might even provide a chance for children to enhance their strategies concerning a composite unit as unit relation, which refers to a unit of a unit (“a quarter of a quarter”). Such an idea would not come up with simple estimating.

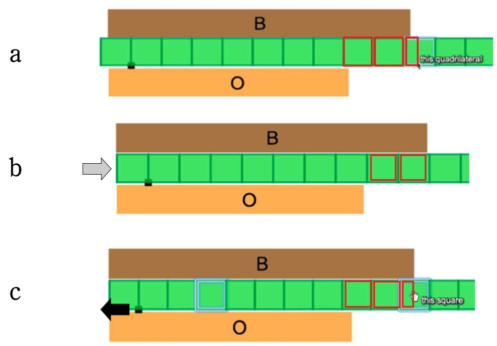
2. Mary: a random-sized unit to the given unit

Children often used the default (or random) size unit without consideration of the unit itself and showed limited understanding of measurement that assigns units along with the object. Even though students must decide the size of one unit to coordinate with a given length, such children might had little understanding of what would be the appropriate unit size to measure. However, in the Dynamic Ruler tasks, students had the opportunity to adjust the size of Dynamic Ruler’s pieces to match the given unit length and to count the number of pieces because the ruler could be coordinated automatically when deciding an appropriate size of pieces. Therefore, children’s understanding of unitizing could be mediated by matching the size of Dynamic Rulers with the given unit length.

Mary, a third grader, was selected as a representative case of moving from using random-sized units to making a relation to the given unit when solving Tasks 2-2 and 3-1.

1) Using random-sized units

In the following episode, Mary was working on Task 2-2 (as shown in Figure 5) and explained her thinking to get the number of orange rods to fit into the brown rod,



Note. The red boxes denote how she counted the numbers. The gray arrow shows the movement of the ruler and the black arrow shows the adjustment of the size.

[Fig. 5] Mary’s use of the Green Dynamic Ruler to measure the brown rod in Task 2-2

Mary (M): I just need to drag.

Interviewer (I): You can do that.

M: One, two, and a half of this, kinda. So, that's three and a half.

I: You said a half of this? How do you know that is a half?

M: Because you do this. [Mary physically demonstrated by moving the mouse cursor up and down within the 11th piece of the Green Dynamic Ruler, which is the piece that is intersected by the right edge of the brown rod in Figure 5-a.]

I: Okay. Um, if you, you have a two and a half more. Does it really fit?

M: [Mary aligned the Green Dynamic Ruler so that two of its units equaled the leftover length of the brown rod, as shown in Figure 5-b] One, two. It's two orange rods. It just shrank.

I: If you need two orange rods more, then I would think, like, there is a— one is here, and if you have two more, I think it'll be off of the screen. How do you think that?

M: Then they would have to back it up.

I: Then, still, do you think two orange rods? You need more?

M: [Mary manipulates a smaller unit size of the Dynamic Ruler.] I am good with smaller. One, two, and a half.

I: Okay. Two and a half you need. Interesting.

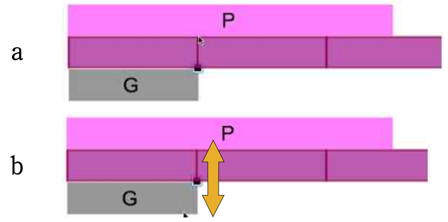
Mary consistently used random units to figure out the measures across the earlier tasks of her interview. In Task 2-2, Mary started dragging to change the unit size of the Green Dynamic Ruler and made a smaller size than the default size of the Dynamic Ruler. She placed the Green Dynamic Ruler between two rods, but the left side was not aligned well. Then, she counted the number of green pieces from the right edge of the orange rod to the right edge of the brown one (Figure 5-a). Counting the last green piece, she represented the part of it that was to the left of the brown rod's edge as "a half of this, kinda." When asked to validate her initial solution, she slightly moved the Dynamic Ruler to the right and found that she needed "two orange rods" more to fit into the brown rod (Figure 5-b). After I checked her unit size relevant to the numeral value ("If you have two more, I think it'll be off of screen"), she tried to change the size of the Dynamic Ruler and made its units a smaller size (Figure 5-c). Last, she counted the number of green pieces, "One, two, and a half."

In this episode, she seemed to only attempt to measure the difference between two rods with the Green Dynamic Ruler rather than to measure the brown rod with the orange rod. That is, she was looking for how many more green pieces were needed to cover the brown rod's leftover length after using the orange rod to measure a portion of the brown rod's length. In her first solution, she expressed $3\frac{1}{2}$ because she described the last piece as $1\frac{1}{2}$ instead of a half. Here, she used the green piece as her unit size to measure the leftover length. However, she seemed to feel that it was necessary to represent the leftover size as a fractional term for the first time during the interview. She related a leftover of the green piece with one green unit size and estimated this as a half. However, this leftover part was caused by a misalignment between the rods and the Green Dynamic Ruler. She realigned the Dynamic Ruler and figured out the difference was two but she referred to the green pieces as the orange rods ("two orange rods"). Mary tried to use a different unit size in her last attempt, but her answer was similar ("two and a half").

Mary was able to change the size of the Dynamic Ruler and counted the number of units that equaled the leftover length. This actually worked well to measure the difference between two quantities. However, she did not consider the relationship between the Green Dynamic Ruler and the given unit length (orange rod). Her fragile unitizing concept was illustrated by her answer of “two orange rods” when she counted the number of green pieces. She interchanged two different unit sizes (the green piece and the orange rod).

2) Making a relation to the given unit

Mary showed a more sophisticated understanding of unitizing by matching the size of the unit length with the Dynamic Ruler in Task 3-1. In the following dialogue, it was evident that she adjusted the size of the Purple Dynamic Ruler to make its units exactly the same size as the gray rod and related the partial leftover length of the pink rod to the gray rod (Figure 6).



Note. The yellow arrow shows the vertical movement for Mary to indicate the alignment between the first piece of the Purple Dynamic Ruler and the gray rod.

[Fig. 6] Mary’s use of the Purple Dynamic Ruler to match the gray rod

M: One, half. So, one and a half.

I: Can you tell me more?

M: Well, if you cut this in half, there would be just one and a half.

I: Um, interesting. Where did you get the half?

M: Well, if you just, like, give this with this...like lining up...then, that’s how I got the half. But if you just line it up with this another half.

I: Can you show me the half again. Okay. Um, why did you make the line up?

M: Like this? Because if you do it as the shape as this, then you’ll see how much is in between.

I: Okay, I think you used this given size at first, but at this time, you changed the size. Is there any reason for that? Why did you change at this time? You just used the original size, right, usually? But at this time, you changed the size. Why did you change that?

M: Because if you do it as the length like this, then you’ll see how much there is.

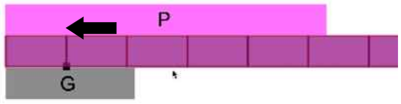
In the beginning, Mary did not change the size of the Purple Dynamic Ruler but placed the Dynamic Ruler between the two rods. Then she adjusted the unit size of the Dynamic Ruler so that it was the same size as the gray rod (Figure 6-a). Like the previous task, she found the difference between the two rods with then number of purple pieces (“one and a half”). When asked the origin of the half, she moved the mouse cursor vertically to show the alignment of the Purple Dynamic Ruler with the gray rod (Figure 6-b). She explained the importance of lining up (“Because if you do it as the shape as this, then you’ll see how much is in between”) and adjusting the size of Dynamic Ruler related to the given unit length (“Because if you do it as the length like this, then you’ll see how much there is”).

In this episode, Mary showed her better understanding of the task by utilizing the gray rod as the unit of measure. Previously, she counted the number of pieces and estimated partial values with a fractional term. However, in this task, she made a unit relation between the length of the Dynamic Ruler’s units and the gray rod. This approach signaled a significant change across her interview since she usually decided to use the default (or random) sizes of the Dynamic Rulers as units. However, in Task 3-1, she was able to unitize one piece of the Purple

Dynamic Ruler to determine the length of the specified rod (pink rod) by matching the size of the given unit (gray rod) to the size of the ruler's pieces. Then, she employed the purple pieces to measure the difference and found "one and a half." She valued the alignment between the Purple Dynamic Ruler and the gray rod and regarded the adjusted purple pieces as a copy of the gray rod. In other words, she seemed to perceive that counting the number of the purple pieces was the same as counting the gray rod's unit length. More surprisingly, Mary explained that half the size of the gray rod was also able to measure the pink rod in the following exchange:

I: Can you use a different size?
M: Yeah, like the half, like through the G.
I: Okay. So, can you tell me more? How can you use this size?
M: Oh, you can just count it, and they will be more like more halves.
I: More halves?
M: Yeah. Because look. One, two, three, three, and a half.
I: Okay. So, do you, how do you know? Oh, this one, the purple is a half of gray?
M: Um, because if you do that G, then it's half. I just learned that by myself. Well, I'm just gonna break a half of this.

Mary seemed to think about the letter G on the gray rod as the midpoint of the rod, so she considered a new adjusted size of the Purple Dynamic Ruler's pieces as a half (Figure 7). Then she counted the number of purple pieces, "three and a half."Although this was not a correct answer for the task since the letter G was not the actual midpoint of the gray rod, she showed her clear intention to make a new unit size (purple piece) related to the referent unit (gray rod).



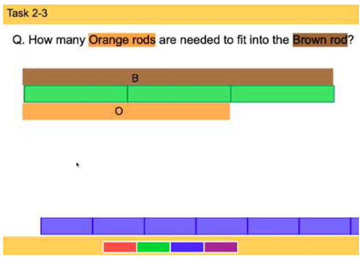
Note. Mary moved up the Dynamic Ruler at first, then she adjusted the size of it. The black arrow shows the adjustment of the size.

[Fig. 7] Mary's Use of a Partial Unit to Measure the Pink Rod

3) Mediation process

Unitizing is the mental process of assigning a unit to measure a given quantity (Lamon, 1996). Across the interview, between Tasks 2-2 and 3-1, I observed Mary's progress in unitizing by dragging the Dynamic Ruler so that it aligned with both the given unit and the given length to be measured. The goal of the tasks was not to count the number of any size pieces but to measure the linear quantity with a relevant unit size, such as the same size (whole number) or a partial size (rational number). In Task 2-2, Mary only focused on the difference between the rods and the number of pieces in the Dynamic Ruler. While the relationships among the rods and Dynamic Ruler might have been significant, Mary struggled to identify a correct unit to measure length and did not make a rational connection between the given unit (the orange rod in Figure 5) and the pieces that could assist in measurement (the pieces of the Green Dynamic Ruler). However, in Task 3-1, she aligned the gray rod and the Purple Dynamic Ruler (Figure 6) to make the ruler's units match the length of the gray rod. Furthermore, she used the letter G on the gray rod as a perceived midpoint and created pieces of the Purple Dynamic Ruler that were half as long as the gray rod (Figure 7). In so doing, Mary determined the size of one unit, then coordinated the unit to the quantity.

What might have stimulated her reasoning to use a specific unit size for the Dynamic Ruler in Task 3-1 instead of the default (or undecisive) size she relied on Tasks 2-1 and 2-2? One way to explain this mediation might be to consider the prepartitioned Dynamic Ruler that Mary used in Task 2-3, which was a task focused on using prepartitioned rulers (Figure 8). Since she was struggling to understand the goal of Task 2-1 and 2-2, I decided to ask Mary to solve the optional Task 2-3 prior to Task 3-1. In the previous tasks (2-1 and 2-2), she ignored or paid little attention to the given unit length (orange rod in Figure 6). Instead, she counted the number of pieces in the Green Dynamic Ruler. However, in Task 2-3, the Dynamic Rulers were already partitioned evenly in relation to the orange rod. One piece of the Green Dynamic Ruler was one half of the orange rod, and one piece of the Blue Dynamic Ruler was one fourth of the orange rod. These fixed sizes of the Dynamic Rulers might have lowered Mary's cognitive load and helped her to see the goal of the task (to relate the given unit to the given quantity) and possibly to see how units that come from partitions of the given unit can be used to quantify the relationship between the two rods. In other words, the prepartitioned nature of the rulers in Task 2-3 may have encouraged children such as Mary to focus more on unit relationships as the key idea of the tasks.



[Fig. 8] Mary's Solution in Task 2-3

In the following exchange, I asked Mary to explain her understanding of the relationship between the green pieces and the orange rod.

I: [I pointed to one piece of the Green Ruler in Figure 6.] Then how much [is a green piece] of the orange? Does it make things clear for you? So, this [orange rod] is a one. How much is this [green piece]?

M: One and a half.

I: One and a half? What about this?

M: Full one.

I: Full one. What about this?

M: Another one.

I: Okay. And if you fit them into the brown rod, how many orange rods do you need total?

M: One.

Prior to this exchange, I questioned Mary about the relationship between the Dynamic Ruler and the given unit length (orange rod), but Mary ignored the meaning of the orange rod as the unit to measure. Even though she mentioned some fractional terms, they were not related to the orange rod but to the pieces of the Green Dynamic Ruler. However, in this exchange, it was significant that she tried to find meaning in the green piece's relation to the orange rod ("One and a half?" "Full one."). Concerning the first green piece, she seemed to recognize the length of one green piece was half of the orange rod. This reasoning contrasted with how she described the third green piece without considering the orange rod ("Another one."). Because Task 2-3 was designed for children who had struggled to use the Dynamic Ruler tool for unitizing, they might have had less cognitive demand in using the tool itself during Task 2-3. Thus, they could focus more on interpreting the relationship between the given unit (of the rod that was specified) and the prepartitioned units (of the Dynamic Ruler). Therefore, Mary may have used the opportunity to take the meaning that she gleaned from Task 2-3—that one green piece equals half of the orange rod—and employ it on Task 3-1. Eventually, this contrasting experience between static (Task 2-3) and dynamic (Tasks 2-2 or 3-1) features of the tasks might elicit to decide the size of units of the Dynamic Ruler more meaningfully and intentionally.

V. Discussion and Conclusion

I examined children's understanding of fraction concepts from a measurement approach as the children developed such conceptions within digital technological environment. Based on the analysis of each representative case, I identified distinct changes in children's discourse and strategies across tasks series. Sam used the Dynamic Ruler by matching the given unit length in Task 2-2. However, he implemented the dragging feature more actively to check the leftover size in Task 2-4. Similarly, Mary constructed a random size of units of the Dynamic Ruler but she matched the unit length to quantify the longer rod. I also unpacked how their understandings were mediated through the use of devised a digital tool and measurement tasks. Based on the findings regarding the mediated understandings of fractions, this section discusses several implications related to teaching and learning fractions through digital technology.

First, students can develop and progress their understanding of fraction as measure in solving the measurement tasks. 8 third-grade students out of 13 had changed the way to solve the measurement tasks. To be specific, 6 students shifted from pre-partitioning to partitioning. This shift includes qualitative changes in their understanding of fractions. When students keep exploring the relationships between quantities, it would be affordable to deepen their unit concept. For example, Sam described a half as one piece out of four pieces by comparing the bottom rod with the leftover part visually in Task 2-2. This might be still related to the part-whole concept rather than fraction as measure. However, he showed clear evidence to prove his rigorous understanding of unit relations (units of units) in Task 2-4. Sam constructed a partitive unit with the Red Dynamic Ruler ("a quarter") and eventually found the ratio between the original unit and a piece of the Blue Dynamic Ruler ("a quarter of a quarter"). Because the core idea of fraction as measure was to find a ratio between units, this understanding was crucial. As Sam used a digital tool to explore unit relation dynamically, younger students are required to be given more opportunities to use a variety of unit size to quantify fractional values.

Second, dragging action plays a critical role in the mediation of children's mathematical understandings. In the case of Sam, he checked the leftover with a partial unit size by dragging to formulate the meaning of the partial length quantity in Task 2-2, then he made a much smaller size of units through equal partitioning of the additional leftovers in Task 2-4. In the latter task, Sam seems to use dragging to test his assumptions that any length could be partitioned into equal parts through the use of the Dynamic Ruler tool (Hollebrands, 2007). This was contrasting with Mary's initial use of Dynamic Ruler. She used dragging randomly to make any small size of unit and there was no relation between the given unit and the generated unit, which was a critical concept for measurement approach to fractions. In Task 3-1, she changed the way to use dragging as formulating meaning of fractional quantity. The way to use dragging as a visual mediator (Sfard, 2008) contributes to developing the meanings of fractions in terms of units and the relations. In the theory of semiotic mediation, the dragging feature of the tool was considered as a pivot sign to mediate sense-making to students. This was aligned with the findings from a recent study about the dragging activities of a dynamic software (Yang & Shin, 2014). They found 'random dragging' and 'guided dragging' enabled to help the students formulate mathematical conjectures in the mathematical reasoning process; 'dragging test' could be exploited to confirm the conjectures.

Third, students have an opportunity to compare static and dynamic representations and to interpret the difference between them actively with various subtasks. In this study, I asked students how they found the length with prepartitioned units as one of the subtasks. Similar tasks were also used by Simon and colleagues (2018), who found students could come to understand that a quantity was measured by iterating the unit, a partial unit, or a combination of both as an initial concept of fractions with prepartitioned units. However, more general measurement situations have another facet that evaluates the amount of length with undecided units (i.e., the quantity is given, and its unit is not). When students adjust the unit size of the Dynamic Ruler, they might have an opportunity to measure a quantity and a unit with various expressions. For instance, Mary measured a difference between two rods with partial units by dragging the Dynamic Ruler. She came up with the result of 3/2 because she measured the leftover between the given quantity and unit as the new partial unit ($\frac{1}{2}$). This would suggest that some of the tasks used in this study may help students deepen fraction conceptions as they use the digital tool, especially when contrasted with prior studies that found students lacked this understanding with static representations (Dougherty, 2008).

One might argue that physical manipulatives or direct modeling (e.g., drawings) would be productive to diagnose children's misunderstanding of fractions such as leaving gaps between iterations or selecting unrelated unit sizes. However, in the microworld of Dynamic Ruler, students can be engaged intensively in the relationships between adjustable units and quantities without potential distractions. This would be more important opportunities for them to formulate the concepts of fractions.

Although this study investigated a limited sample of elementary students, there is little study done to introduce fractions from measurement approach and even with digital technology. In this study, I tested and refined theories of how students come to mediated understandings of fraction concepts with the use of digital technology. I also produced a theory-driven, empirically tested set of tasks accompanied by illustrations of students' problem-solving in a dynamic digital environment. The findings will help bridge the gap between theory and practice and analyze alternative methods for using fractions to teach number and math fundamentals, as well as advanced concepts such as algebra and geometry. This study aimed to use the fraction as measure concept instead of the part-whole concept to enhance students' knowledge. As a result, my findings offer one way to expand students' stereotype conceptions of fractions. By developing a robust understanding of fractions, this study's learning approach has the potential to increase students' critical thinking skills both within and beyond mathematics.

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