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MATHEMATICAL MODELLING FOR THE AXIALLY MOVING MEMBRANE WITH INTERNAL TIME DELAY

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Abstract. In [1], we studied the PDE system with time-varing delay. Time delay occurs due to loosening in a high-speed moving axially directed membrane (string, belt, or plate) at production. Our purpose in this work derives a mathematical model with internal time delay. First, we consider the physical phenomenon of axially moving membrane with respect to kinetic energy, potential energy and work done. By the energy conservation law in physics, we get the second order nonlinear PDE system with internal time delay.

1. Introduction

Many PDE systems induced the mathematical modeling on the mass production process are studied on sense of mathematical analysis (well-posedness,non-existence, energy decay rates and so on) (See [1, 2, 3, 4, 6]). In this work, we deal with a roller to roller part of the mechanical process. If the roller's shaft moves, then the system may have unstable energy around its boundaries. The energy is the work done WD. We try to check the PDE problem induced by mathematical modelling for axially moving membrane (in effect, string).

In particular, we focus on the time variable relating to the damping term. It is related to the energy generated from the boundary inward when the boundary moves. The energy generated at this time is non-conservative work done WD_{nc}. In case of the work done defined by the non conservative forces $f(x,t)$ in internal of domain and $f_c(t)$ at the boundary $x = l(t)$. In the free boundary, the axially moving string may slip. Therefore, it is necessary to define a different time. It is necessary to express the time differently from normal time t. I will express it in τ here. An also, work done on the outward direction at the right boundary that is free is WD_{rb}. Therefore, $WD = WD_{nc} + WD_{rb}$ on the near the right boundary. Note that work done on the fixed boundary $x = 0$ is zero [See FIGURE 1].

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Our work focus on WD_{nc} for the time delay relating $s(x,t-\tau)$ effected weight $\varpi(x)$ and moving speed ν (See [8]). Our purpose in this work derives a mathematical model with internal time delay. It is meaningful for analytical approaches of the PDE system concerning the time-varying delay in [1], physically.

This work is organized into some processes. First, we check some needed physical variables. Using the variables, we consider the kinetic energy, potential energy, and work done in great detail. Next, we consider the variation for energy. Using the energy conservation and the variation lemma, Hamilton's principle, integration by parts, and so on, we deduce the initial-boundary problem for nonlinear PDE, including time delay meaningfully.

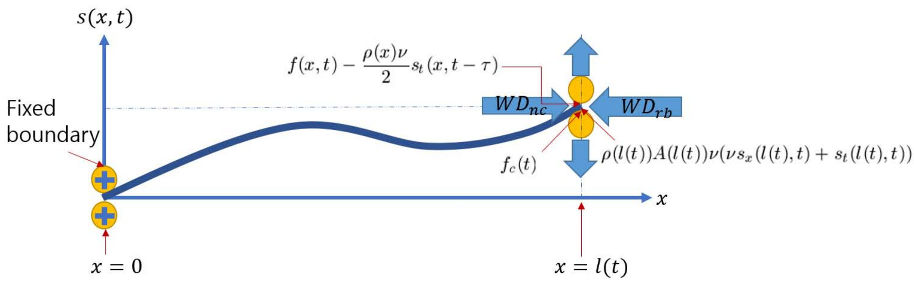

FIGURE 1. WD on the sense of physical situations.

2. Mathematical modelling

The string on the mass production process is steered axially through two ends, which are spaced apart by a distance of $l(t)$. The membrane has two variables. s is the displacement of string under the variables. The first variable is the spatial part x. The range of x is $0 \le x \le l(t)$. Indeed, we consider suitable time t of $l(t)$ is fixed. In case t is free, we can consider it as future work. It is a part of the mechanical process. More delicately, the part is roller to roller. In the case of the first boundary $x = 0$, physically, the roller's shaft is fixed. In the other case $l(t)$, the roller's shaft is mechanically free. The next variable is time t Over time, the string moves in a high-speed moving axial direction. At this time, a slip phenomenon occurs in the inner area with the right border $l(t)$. A variable related to the time that shows this phenomenon well is τ. f, f_c and ν change their symbols to ψ, ψ_c and $\overline{v}$, respectively. Some variables and constants that will be used in this paper are as follows:

$$\begin{cases} \overline{v} > 0 & : \text{ moving speed for axial direction;} \\ s_t + \overline{v}s_x & : \text{ transversal velocity of the string moving;} \\ s & : \text{ displacement of the string moving;} \\ (\cdot)_t = \partial(\cdot)/\partial t & : \text{ the partial derivative for time;} \\ (\cdot)_x = \partial(\cdot)/\partial x & : \text{ the partial derivative for time for domain value;} \\ C(x) & : \text{ the area of cross-section;} \\ \varpi(x) & : \text{ mass per unit(weight);} \\ Y & : \text{ Young's elastic modulus;} \\ \sigma(x,t) & : \text{ tensile stress;} \\ \zeta(x,t) & : \text{ strain;} \\ \iota_0 & : \text{ initial tension of string.} \end{cases}$$

We also define certain energies and vibrations physically. K is kinetic energy. P is potential energy. δWD_{nc} is the variation of non-conservative work done. δWD_{rb} is the variation of work done at the right boundary. More specifically, all of them are given by

$$K = \frac{1}{2}\int_0^{l(t)} \varpi(x)C(x)[\overline{v}^2 + (\overline{v}s_x + s_t)^2]dx, \tag{1}$$

$$P = \int_0^{l(t)} \left[\left(\iota_0 + \frac{YC(x)}{4}\int_0^1 \left(\frac{\partial s}{\partial x}\right)^2 dx\right) + \frac{1}{2}C(x)\sigma(x,t)\right]\zeta(x,t)dx, \tag{2}$$

$$\delta WD_{nc} = \int_0^{l(t)} \left[\psi(x,t) - \frac{\varpi(x)\overline{v}}{2}s_t(x,t-\tau)\right]\delta s(x,t)dx + \psi_c(t)\delta s(l(t),t) \tag{3}$$

$$\delta WD_{rb} = \varpi(l(t))C(l(t))\overline{v}(\overline{v}s_x(l(t),t) + s_t(l(t),t))\delta s(l(t),t). \tag{4}$$

Now,we calculate the potential energy P in more detail. Let Z and $\zeta(x,t) = \frac{dy-dx}{dx}$(,where $|dx| = 1$) be the string's tension and the strain under physical situations, respectively [See FIGURE 2].

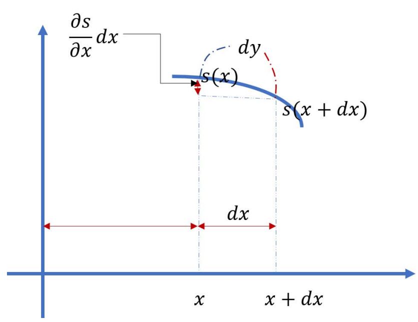

FIGURE 2. Strain on the sense of physical situations.

Then, we have

$$
\begin{aligned}
P &= \int_0^{l(t)} Z\zeta(x,t)dx \\
&= \int_0^{l(t)} \left[\sigma(\varepsilon(x,t)) + \frac{\sigma'(\varepsilon(x,t))}{2}\int_0^{l(t)} \left(\frac{\partial s}{\partial x}\right)^2 dx\right]\zeta(x,t)dx.
\end{aligned}
$$

Next we set $\sigma(\varepsilon(x,t)) = \iota_0 + \frac{YC(x)}{2}\varepsilon(x,t)$ with ι_0. Therefore the potential energy U changes

$$
P = \int_0^{l(t)} \left[\iota_0 + \frac{YC(x)}{2}\varepsilon(x,t) + \frac{YC(x)}{4}\int_0^{l(t)} \left(\frac{\partial s}{\partial x}\right)^2 dx\right]\zeta(x,t)dx.
$$

Because of $\sigma(\varepsilon(x,t))$ which is defined by $\sigma(x,t)$, we can apply $\sigma(\varepsilon(x,t)) = Y\varepsilon(x,t)$. So, we finally get

$$
P = \int_0^{l(t)} \left[\left(\iota_0 + \frac{YC(x)}{4}\int_0^{1} \left(\frac{\partial s}{\partial x}\right)^2 dx\right) + \frac{1}{2}C(x)\sigma(x,t)\right]\zeta(x,t)dx.
$$

By the Taylor's theorem, we have the strain

$$
\begin{aligned}
\zeta(x,t) &= \frac{dy-dx}{dx} = \frac{1}{2}\left(\frac{\partial s}{\partial x}\right)^2 - \frac{1}{8}\left(\frac{\partial s}{\partial x}\right)^4 + \cdots + \frac{-\frac{1}{4}\cdots\left(\frac{1}{2}-n+1\right)}{n!}\left(\frac{\partial s}{\partial x}\right)^n + \cdots \\
&\approx \frac{1}{2}\left(\frac{\partial s}{\partial x}\right)^2 \\
&\ll 1
\end{aligned}
$$

So, we approximately get

$$
P = \frac{1}{2}\int_0^{l(t)} \left[\left(\iota_0 + \frac{YC(x)}{4}\int_0^{l(t)} \left(\frac{\partial s}{\partial x}\right)^2 dx\right) + \frac{1}{2}C(x)\sigma(x,t)\right]\left(\frac{\partial s}{\partial x}\right)^2 dx.
$$

From now on, we start calculating the variation between kinetic and potential energy. By using the Gâteatux derivative, we get variations of K and P like as:

$$
\begin{aligned}
\delta K(s;\phi) &= \lim_{\varepsilon\to 0}\frac{K(s+\varepsilon\phi)-K(s)}{\varepsilon} \\
&= \int_0^{l(t)} \varpi(x)C(x)\left[(s_t+\overline{v}s_x)\phi_t + \overline{v}(s_t+\overline{v}s_x)\phi_x\right]dx,
\end{aligned}
\tag{5}
$$

$$
\begin{aligned}
\delta P(s;\phi) &= \lim_{\varepsilon\to 0}\frac{P(s+\varepsilon\phi)-P(s)}{\varepsilon} \\
&= \int_0^{1} \left[\left(\iota_0 + \frac{YC(x)}{4}\int_0^{l(t)} \left(\frac{\partial s}{\partial x}\right)^2 dx\right) + \frac{1}{2}C(x)\sigma(x,t)\right] s_x\phi_x dx,
\end{aligned}
\tag{6}
$$

where ϕ is the C^1 function which depends on x and t.

Apply for the Hamilton's Principle, (3)-(4) and (5)-(6) are as follows:

$$\int_{t_0}^{t_1} (\delta K - \delta P + \delta WD_{nc} - \delta WD_{rb}) dt = 0, \ for\ all\ t \in [t_0, t_1] \tag{7}$$

Accordingly, (7) can be replaced with

$$\begin{aligned}
&\int_{t_0}^{t_l} \int_0^{l(t)} \varpi(x) C(x) \left[(s_t + \overline{v} s_x)\phi_t + \overline{v}(s_t + \overline{v} s_x)\phi_x \right] dxdt \\
&+ \int_{t_0}^{t_l} \int_0^{l(t)} \left[\iota_0 + \frac{YC(x)}{4} \int_0^{l(t)} \left(\frac{\partial s}{\partial x}\right)^2 dx + \frac{1}{2} C(x)\sigma(x,t) \right] s_x \phi_x dxdt \\
&+ \int_{t_0}^{t_l} \int_0^{l(t)} \left[\psi(x,t) - \frac{\varpi(x)\overline{v}}{2} s_t(x, t-\tau) \right] \phi dxdt \\
&+ \int_{t_0}^{t_l} \left[F_c(t)\phi(l(t),t) - \varpi(l(t)) C(l(t)) \overline{v}(\overline{v} s_x(l(t),t) + s_t(l(t),t))\phi(l(t),t) \right] dt = 0
\end{aligned} \tag{8}$$

By using the integration by parts, we deduce

$$\begin{aligned}
&\int_{t_0}^{t_1} \Bigg[\varpi(l(t)) C(l(t)) \overline{v}(\overline{v} s_x(l(t),t) + s_t(l(t),t)) - \left(\iota_0 + \frac{YC(l(t))}{4} \int_0^{l(t)} \left(\frac{\partial s}{\partial x}\right)^2 dx \right) s_x(l(t),t) \\
&\quad + F_c(t) - \frac{1}{2} C(l(t)) \sigma(l(t),t) s_x(l(t),t) - \varpi(l(t)) C(l(t)) \overline{v}(\overline{v} s_x(l(t),t) + s_t(l(t),t)) \Bigg] \phi(l(t),t) dt \\
&- \int_{t_0}^{t_1} \Bigg[\varpi(0) C(0) \overline{v}(\overline{v} s_x(0,t) + s_t(0,t)) - \left(\iota_0 + \frac{YC(0)}{4} \int_0^{l(t)} \left(\frac{\partial s}{\partial x}\right)^2 dx \right) s_x(0,t) \\
&\quad - \frac{1}{2} C(0) \sigma(0,t) s_x(0,t) \Bigg] \phi(0,t) dt \\
&\quad + \int_0^{l(t)} \varpi(x) C(x) (\overline{v} s_x(x,t_1) + s_t(x,t_1)) \phi(x,t_1) dx \\
&\quad + \int_0^{l(t)} \varpi(x) C(x) (\overline{v} s_x(x,t_0) + s_t(x,t_0)) \phi(x,t_0) dx \\
&- \int_{t_0}^{t_1} \int_0^{l(t)} \Bigg[s_{tt} - \left(\frac{\iota_0}{\varpi(x) C(x)} - \overline{v}^2 + \frac{Y}{4\varpi(x)} \int_0^{l(t)} \left(\frac{\partial s}{\partial x}\right)^2 dx \right) s_{xx} \\
&\quad + \left[\left(\frac{\varpi_x(x)}{\varpi(x)} + \frac{C_x(x)}{C(x)} \right) \overline{v}^2 - \frac{YC_x(x)}{4\varpi(x) C(x)} \int_0^{l(t)} \left(\frac{\partial s}{\partial x}\right)^2 dx \right] s_x \\
&\quad + \left(\frac{\varpi_x(x)}{\varpi(x)} + \frac{C_x(x)}{C(x)} \right) \overline{v} s_t + 2\overline{v} s_{xt} \\
&\quad - \frac{1}{2\varpi(x) C(x)} \frac{\partial}{\partial x} \left[C(x) \sigma(x,t) \frac{\partial s}{\partial x} \right] - \left[\psi(x,t) - \frac{\varpi(x)\overline{v}}{2} s_t(x, t-\tau) \right] \Bigg] dxdt = 0.
\end{aligned}$$

Apply for the variational lemma, finally we get the following mathematical modelling considering internal time delay

$$\begin{aligned}
s_{tt}- & \left[\frac{1}{\varpi(x)C(x)}(\iota_0 + C(x)\sigma(x,t)) - \overline{v}^2 + \frac{Y}{4\varpi(x)}\int_0^{l(t)}\left(\frac{\partial s}{\partial x}\right)^2 dx\right] s_{xx} \\
& + \left[\left(\frac{\varpi_x(x)}{\varpi(x)} + \frac{C_x(x)}{C(x)}\right)\overline{v}^2\right. \\
& \left. - \frac{1}{2\varpi(x)C(x)}\frac{\partial}{\partial x}\left(C(x)\sigma(x,t)\right) - \frac{YC_x(x)}{4\varpi(x)C(x)}\int_0^{l(t)}\left(\frac{\partial s}{\partial x}\right)^2 dx\right] s_x \qquad (9) \\
& + \left(\frac{\varpi_x(x)}{\varpi(x)} + \frac{C_x(x)}{C(x)}\right)\overline{v}s_t + \frac{\varpi(x)\overline{v}}{2}s_t(x,t-\tau) + 2\overline{v}s_{xt} = \psi(x,t) \\
& \text{in } (0,l(t)) \times (0,T),
\end{aligned}$$

the boundary conditions on $(0,T)$, as

$$\begin{aligned}
\psi_c(t) - & \left[\left(\iota_0 + \frac{YC(l(t))}{4}\int_0^{l(t)}\left(\frac{\partial s}{\partial x}\right)^2 dx\right) + C(l(t))\sigma(l(t),t)\right] \qquad (10) \\
& \times s_x(l(t),t) = 0
\end{aligned}$$

$$s_x(0,t) = s_t(0,t) = 0 \qquad (11)$$

and the initial conditions as

$$s(x,0) = s_0(x) \text{ and } s_t(x,0) = s_l(x) \qquad \text{in } (0,l(t)). \qquad (12)$$

3. Conclusion

Under the suitable initial condition relating (12), the generalization (considering coefficient of s_{xx}) for the equation (9) is given a name to the Kirchhoff based type equation (See [5]). In the boundaries $x = l(t)$ and $\psi_c(t)$, mathematically we regard them as Dirichlet boundary, Neumann boundary and so on. Especially, the boundary relating (10) can be unstable because of WD. So, we may need consisting to the boundary feedback control $-g(u_t)$. we can get the generalized PDE system which is set $K(x,t,\|\nabla u(t)\|^2)$ by

$$\frac{1}{\varpi(x)C(x)}(\iota_0 + C(x)\sigma(x,t)) - \nu^2 + \frac{Y}{4\varpi(x)}\int_0^{l(t)}\left(\frac{\partial s}{\partial x}\right)^2 dx,$$

and the rest of terms $\left(except\left(\frac{\varpi_x(x)}{\varpi(x)} + \frac{C_x(x)}{C(x)}\right)\overline{v}s_t + \frac{\varpi(x)\nu}{2}s_t(x,t-\tau)\right)$ in (9) to be the nonlinear function by

$$N(x,t,s_x,s_{xt}),$$

which is concerned about s_x, s_{xt} and $\psi(x,t)$. If $\left(\frac{\varpi_x(x)}{\varpi(x)} + \frac{C_x(x)}{C(x)}\right)\overline{v}s_t + \frac{\varpi(x)\nu}{2}s_t(x,t-\tau)$ is generalized to $\mu_1(x)s_t + \mu_2(x)s_t(x,t-\tau)$, it is meaningful to establish the relationship between $\mu_1(x)$ and $\mu_2(x)$ mathematically.

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