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ON THE DEGENERATE MAXIMAL SURFACES IN $\mathbb{L}^4$

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Abstract. The purpose of this paper is to investigate various kinds of degeneracy of maximal surfaces in $\mathbb{L}^4$ in view of the generalized Gauss map.

1. Introduction

We adopt the notations in [5]. Denote by M a Riemannian surface, and define a maximal (spacelike) surface S in $\mathbb{L}^4$ by an embedding $X : M \longrightarrow \mathbb{L}^4$, where local coordinates ξ^1 , ξ^2 on M serve as isothermal parameters for the surface and $z = \xi^1 + i\xi^2$ as a complex coordinate on M. The Gauss map $\Phi(z) = (\phi_1(z), \phi_2(z), \phi_3(z), \phi_4(z))$ from M into $\mathbb{Q}^2_+$ is given in local complex coordinate on M as in [5]. We adopt terminologies about the causal character of a subspace of $\mathbb{C}P^3$ naturally according to the causal character of a subspace H of $\mathbb{C}^4_1$ under the natural projection $\pi : \mathbb{C}^4_1 \longrightarrow \mathbb{C}P^3$.

In this paper, we are concerned with maximal spacelike surfaces in $\mathbb{L}^4$ with degenerate Gauss map. In the classical case in the Euclidean space $\mathbb{R}^4$, there are several types of degeneracy of the Gauss map. We can think of similar types of degeneracy of maximal spacelike surfaces in $\mathbb{L}^4$. The main purpose of this paper is to investigate various kinds of the degeneracy of maximal spacelike surfaces in $\mathbb{L}^4$ in view of the generalized Gauss map.

2. On the Degenerate Maximal Surfaces

Definition 1. The maximal surface S *lies fully* in $\mathbb{L}^4$ if the image $X(M)$ does not lie in any proper affine subspace of $\mathbb{L}^4$, and *degenerate of the first kind* if its Gaussian image $\Phi(M)$ lies fully in a spacelike subspace of $\mathbb{C}P^3$, *degenerate of the second kind* if its Gaussian image $\Phi(M)$ lies fully in a timelike subspace of $\mathbb{C}P^3$, *degenerate of the third kind* if its Gaussian image $\Phi(M)$ lies fully in a null subspace of $\mathbb{C}P^3$, and is *k-degenerate* if k is the largest integer such that

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the image under Gauss map $\Phi(M)$ lies in a projective subspace of codimension k in $\mathbb{C}P^3$.

Remark 1. S is degenerate of the first kind if there exists a nonzero timelike vector $A = (a_1, \ldots, a_4)$ in $\mathbb{C}^4_1$ such that

$$\sum_{j=1}^{4} \epsilon_j a_j \phi_j \equiv 0 \ . \tag{1}$$

Furthermore, S is 2-degenerate of the first kind if we can find exactly 2-orthonormal vectors A_1 , A_2 in $\mathbb{C}^4_1$ for which such an equation holds, where A_1 is timelike.

Proposition 2.1. *Let S be a maximal surface in $\mathbb{L}^4$. Then*

(1) *S lies fully in $\mathbb{L}^4$ if and only if it is locally real part of a complex analytic curve lying fully in $\mathbb{C}^4_1$.*
(2) *S is 2-degenerate if and only if it does not lie in a plane and is the direct sum of lightlike line in $\mathbb{L}^2$ and a nonconstant complex or anti-analytic curve with respect to an orthonormal complex structure of $\mathbb{R}^2$.*
(3) *S is 3-degenerate if and only if it lies in a plane.*

Proof. (1) If $X : M \longrightarrow \mathbb{L}^4$ defines a maximal spacelike surface in local isothermal parameters in a simply-connected domain D on M, then the coordinate functions x^k's are harmonic on M. Hence there exists analytic functions f_k's on D such that $x_k = Ref_k$, $k = 1, 2, 3, 4$. Note that the metric on the analytic curve $\frac{1}{\sqrt{2}}(f_1, \ldots, f_4)$ induced from $\mathbb{C}^4_1$ coincides with the metric on the original surface. Now the assertion follows immediately from the local isometric version of a maximal surface $\frac{1}{\sqrt{2}}(f_1, \ldots, f_4)$.

(2) Suppose S is 2-degenerate. If S lies on a plane, it is clearly 3-degenerate. If S is 2-degenerate, its image $\hat{S}$ under the Gauss map lies in a complex line L, the intersection of two (non-degenerate or degenerate) hyperplanes of $\mathbb{C}P^3_+$. The line L must lie in $\mathbb{Q}^2$ or else intersect $\mathbb{Q}^2$ at isolated points. But in the latter case the Gauss map would be constant, and therefore S would be 3-degenerate. Thus $\hat{S} \subset L \subset \mathbb{Q}^2$. Observe that the complex line L lies in the tangent hyperplane to $\mathbb{Q}^2$ at any point, in other words, there is $A = (a_1, \ldots, a_4) \in \mathbb{Q}^2$ such that $\hat{S} \subset H : \sum_{k=1}^{4} \epsilon_k a_k z_k \equiv 0$. Denote $A = \alpha + i\beta$. Since $A \in \mathbb{Q}^2$, $< \alpha, \alpha > = < \beta, \beta > \geq 0$, $< \alpha, \beta > = 0$. Two cases may occur; A is spacelike or lightlike.

Case 1. A is spacelike.

ReA and ImA are two orthogonal spacelike vectors in $\mathbb{L}^4$. Let

$$\widetilde{e_3} = \frac{\alpha}{\sqrt{< \alpha, \alpha >}}, \widetilde{e_4} = \frac{\beta}{\sqrt{< \beta, \beta >}} \ .$$

Complete them to an orthonormal basis of $\mathbb{L}^4$. Then we will get

$$X = \widetilde{x_1}\widetilde{e_1} + \widetilde{x_2}\widetilde{e_2} + \widetilde{x_3}\widetilde{e_3} + \widetilde{x_4}\widetilde{e_4} \ ,$$

where

$$\begin{aligned} \widetilde{x_3} &= \frac{(\sum_k \epsilon_k \alpha_k x_k)}{\sqrt{<\alpha,\alpha>}}\,, \\ \widetilde{x_4} &= \frac{(\sum_k \epsilon_k \beta_k x_k)}{\sqrt{<\beta,\beta>}}\,. \end{aligned}$$

Therefore

$$\begin{aligned} \widetilde{\phi_3} &= \frac{\sum_k \epsilon_k \alpha_k \phi_k}{\sqrt{<\alpha,\alpha>}}\,, \\ \widetilde{\phi_4} &= \frac{\sum_k \epsilon_k \beta_k \phi_k}{\sqrt{<\beta,\beta>}}\,. \end{aligned}$$

Consequently,

$$\widetilde{\phi_3} + i\tilde{\phi_4} = \frac{\{\sum_k \epsilon_k(\alpha_k + i\beta_k)\phi_k\}}{\sqrt{<\alpha,\alpha>}} \equiv 0\,,$$

which implies $\widetilde{x_3} + i\widetilde{x_4}$ is analytic. Note that neither $\widetilde{\phi_3}$ nor $\widetilde{\phi_4}$ is identically zero, and, in turn, implies $\widetilde{x_3}+i\widetilde{x_4}$ is not constant. $\widetilde{\phi_1}^2 = \widetilde{\phi_2}^2$, since $\sum_k \epsilon_k {\phi_k}^2 \equiv \sum_k \epsilon_k \widetilde{\phi_k}^2 \equiv 0$. This implies $\widetilde{\phi_1} \equiv \widetilde{\phi_2}$ or $\widetilde{\phi_1} \equiv -\widetilde{\phi_2}$, where $\widetilde{\phi_1}$ is not identically zero. Otherwise S is 3-degenerate. Hence $(\widetilde{x_1}, \widetilde{x_2})$ defines a non-constant lightlike line in $\mathbb{L}^2$ and $(\widetilde{x_3}, \widetilde{x_4})$ defines a nonconstant analytic curve in $\mathbb{R}^2$ under the suitable orthogonal complex structure.

Case 2. A is lightlike.

Then both α and β are lightlike, and therefore they are linearly dependent. Therefore A can be considered as a real lightlike vector. By Proposition 2.6 [6], $\hat{S}$ lies in the null hyperplane $z_1 = z_2$ under a suitable orthogonal coordinate change in $\mathbb{L}^4$. Hence $\phi_1 = \phi_2$, $\phi_1 \neq 0$, and ${\phi_3}^2 + {\phi_4}^2 = 0$. Therefore (x_1, x_2) defines a lightlike line in $\mathbb{L}^2$ and (x_3, x_4) defines a nonconstant holomorphic or anti-holomorphic map with respect to an orthonormal complex structure.

The converse is trivial since the hypothesis imply ${\phi_1}^2 \equiv {\phi_2}^2$, ${\phi_3}^2 + {\phi_4}^2 \equiv 0$ and $\phi_1 \neq 0$, $\phi_3 \neq 0$. Since it does not lie in a plane, $\phi_1 \equiv c\phi_3$ for no $c \in \mathbb{C}$ and therefore S should be 2-generate.

(3) S is 3-degenerate if and only if $\hat{S}$ is constant in $\mathbb{Q}^2_+ \subset \mathbb{C}P^3_+$. In fact, only the first kind of degeneration is possible, since $\hat{S}$ is in $\mathbb{Q}^2_+$. In turn, $\hat{S}$ is constant if and only if S has the same tangent space everywhere, which is equivalent to the statement that S lies on a plane in $\mathbb{L}^4$.

□

Theorem 2.2. *Let M be a Riemann surface, F a non-constant meromorphic function on M, h a (non-constant) harmornic function on M, and c a complex constant. Suppose they satisfy the following:*

(1) $|c| < 1$*;*
(2) *the analytic differential ω defined on M in terms of a local parameter $z = \xi^1 + i\xi^2$ by*

$$\omega = \left(\frac{\partial h}{\partial \xi^1} - i \frac{\partial h}{\partial \xi^2} \right) \tag{2}$$

has zeros coinciding in position and order with zeros and poles of F;

(3) *if c is not real, then h has single-valued harmonic conjugate on M;*

(4) *if we denote*

$$d = c^2 - 1 \tag{3}$$

then for every closed curve C on M,

$$\int_C \frac{d}{F}\omega = -\overline{\int_C F\omega}. \tag{4}$$

Then the surface $X : M \longrightarrow \mathbb{L}^4$ defined by

$$X = Re \int \left(-c, 1, \frac{1}{2}(\frac{d}{F} + F), \frac{i}{2}(\frac{d}{F} - F) \right) \omega \tag{5}$$

is a 1-degenerate maximal surface of the first kind. Here the integral is taken from a fixed point to a variable point M along an arbitrary path.

Conversely, to a 1-degenerate maximal surface $\mathbb{S}$ of the first kind in $\mathbb{L}^4$, we may assign a quadruple $\{M, F, h, c\}$ which satisfies the hypotheses. The surface $\mathbb{S}$ is given, up to congruence, by (5).

Proof. Suppose a quadruple $\{M, F, h, c\}$ satisfies the hypotheses i) - iv). Then for the holomophic 1-form ω, $Re \int_C \omega = 0$ for any closed smooth curve in M. iv) guarantees $Re \int \frac{1}{2}(\frac{d}{F} + F)\omega = Re \int \frac{i}{2}(\frac{d}{F} - F)\omega = 0$ for any closed curve in M. Hence (5) gives us a well-defined map $X : M \to \mathbb{L}^4$. Since M is locally simply-connected, x_k is a real part a holomorphic map, $\frac{\partial x_k}{\partial \xi^1} - i \frac{\partial x_k}{\partial \xi^2} = \phi_k$ is also holomorphic and ϕ_k is the ontegrand in (5). Directly from (5),

$$-{\phi_1}^2 + {\phi_2}^2 + {\phi_3}^2 + {\phi_4}^2 = (1 - c^2 + d){\phi_2}^2 = 0\,,$$

and

$$\sum_{k=1}^{4} \epsilon_k |\phi_k|^2 > 0\,.$$

Hence (5) defines a maximal surface in $\mathbb{L}^4$. Since $\phi_1 = -c\phi_2$, S is degenerate. We have to show S is exactly 1-degenerate of the first kind. Suppose $\sum \epsilon_k a_k \phi_k \equiv 0$, where $\phi_1 = -c\phi_2$, ${\phi_3}^2 + {\phi_4}^2 = d{\phi_2}^2$, $d = c^2 - 1$, and $|c| < 1$. Here ϕ_2 is not identically zero since h is nonconstant. $\sum \epsilon_k a_k \phi_k \equiv 0$ means that

$$2(a_1 c + a_2)F + d(a_3 + ia_4) + F^2(a_3 - ia_4) \equiv 0\,.$$

If $a_3 - ia_4 \neq 0$, then $F \equiv$ *constant*, a contradiction. Hence $a_3 = ia_4$. Also $a_1 + a_2 = 0$, i.e. $a_2 = -a_1 c$. Consequently, $\sum \epsilon_k a_k \phi_k = 0$ implies

$$\begin{aligned} -a_1\phi_1 - a_1 c\phi_2 + a_3\phi_3 + a_4\phi_4 &= -a_1(\phi_1 + c\phi_2) + a_4(i\phi_3 + \phi_4) \\ &= a_4(i\phi_3 + \phi_4) = 0\,. \end{aligned}$$

If $a_4 \neq 0$, then $i\phi_3 + \phi_4 = 0$ which implies $d = c^2 - 1 = 0$, a contradiction to the fact that $c \neq \pm 1$. Hence $a_3 = a_4 = 0$. The vector $(a_1, -ca_1, 0, 0)$ is clearly timelike since $|c| < 1$ and therefore there is only one timelike vector (up to complex multiple) which satisfies the linear equation $\sum \epsilon_k a_k \phi_k \equiv 0$.

Suppose S is a 1-degenerate maximal surface of the first kind. Then there is an orthonormal basis of $\mathbb{L}^4$ with respect to which the Gauss map of S satisfies

$$\begin{aligned} \phi_1 &= c\phi_2, \\ {\phi_3}^2 &+ {\phi_4}^2 = d\phi^2, \\ d &= c^2 - 1 \end{aligned}$$

with respect to a local isothermal parameter $z = \xi^1 + i\xi^2$ on S. Since S does not lie in a plane in $\mathbb{L}^4$, ϕ_1 is not identically zero. Thus the function $F = \frac{\phi_3 + i\phi_4}{\phi_2}$ is meromorphic in S. F cannot be identically zero. For it would imply ${\phi_3}^2 + {\phi_4}^2 \equiv 0$, $-{\phi_1}^2 + {\phi_2}^2 \equiv 0$, and also imply that S cannot be 1-degenerate, which is contrary to the assumption. Similarly $c \neq \pm 1$, since $c = \pm 1$ implies $\phi_1 = \pm\phi_2$ and ${\phi_3}^2 + {\phi_4}^2 \equiv 0$. From ${\phi_3}^2 + {\phi_4}^2 = (\phi_3 + i\phi_4)(\phi_3 - i\phi_4) = d\phi^2$, we have $\phi_3 - i\phi_4 = \frac{d}{F}\phi_2$, and $\phi_3 + i\phi_4 = F\phi_2$. Hence we find

$$\begin{aligned} \phi_3 &= \tfrac{1}{2}\left(\tfrac{d}{F} + F\right)\phi_2 \, , \\ \phi_4 &= \tfrac{i}{2}\left(\tfrac{d}{F} - F\right)\phi_2 \, . \end{aligned}$$

Now the Gauss map takes the form

$$\Phi = \phi\left(-c, 1, \frac{1}{2}(\frac{d}{F} + F), \frac{i}{2}(\frac{d}{F} - F)\right) .$$

If F were constant, the Gauss map would be constant, but that is not the case. Put $h = x_2$. Then $\omega = \phi_2 dz$, and (5) follows from the fact that $X = Re \int \Phi dz$. Since X is single-valued on S, for any closed curve in S,

$$Re \int \frac{1}{2}(\frac{d}{F} + F)\omega = Re \int \frac{i}{2}(\frac{d}{F} - F)\omega = 0 \, ,$$

from which we can get

$$\int \frac{d}{F}\omega = -\overline{\int F\omega} \, .$$

Similarly, $Re \int \omega = Re \int -c\omega = 0$ implies $\int \phi_2 dz = 0$ for any closed curve in S if c is not real. It follows that the harmonic conjugate of h, say

$$\int \frac{\partial h}{\partial \xi^1} d\xi^1 - \frac{\partial h}{\partial \xi^2} d\xi^2$$

is single-valued in S. □

Theorem 2.3. (1) *Let M be a Riemann surface, F a non-constant meromorphic function on M, h a (non-constant) harmornic function on M, and c a complex constant. Suppose they satisfy the following:*

(a) $|c| < 1$, $c \neq \pm i$;

(b) *the analytic differential* ω *defined on* M *in terms of a local parameter* $z = \xi^1 + i\xi^2$ *by*

$$\omega = \left(\frac{\partial h}{\partial \xi^1} - i\frac{\partial h}{\partial \xi^2}\right) dz \tag{6}$$

has zeros coinciding in position and order with zeros and poles of F*;*

(c) *if* c *is not real, then* h *has single-valued harmonic conjugate on* M*;*

(d) *if*

$$d = c^2 + 1, \tag{7}$$

then

$$Re \int_C F\omega = Re \int_C \frac{d}{F}\omega = 0 \tag{8}$$

for every closed curve C *on* M*,*

Then the surface $X : M \longrightarrow \mathbb{L}^4$ *defined by*

$$X = Re \int \left(\frac{1}{2}(F - \frac{d}{F}), \frac{1}{2}(F + \frac{d}{F}), 1, c\right) \omega \tag{9}$$

is a 1-degenerate maximal surface of the second kind and its Gaussian image lies in a hyperplane which is SO(1,3)-equivalent to $cz_3 - z_4 = 0$*,* $|c| \leq 1$*,* $c \neq \pm i$*. Here the integral is taken from a fixed point to a variable point* M *along an arbitrary path.*

Conversely, let S *be a 1-degenerate maximal surface of the second kind which lies in a hyperplane that is SO(1,3)-equivalent to* $cz_3 - z_4 = 0$*,* $|c| \leq 1$*,* $c \neq \pm i$*. To such an* S*, we may assign a quadruple* $\{M, F, h, c\}$ *which satisfies the hypotheses. The surface* $\mathbb{S}$ *is actually given, up to congruence, by (9).*

(2) *Let* M *be a Riemann surface,* F *a non-constant meromorphic function on* M*,* h *a (non-constant) harmornic function on* M*, and* c *a complex constant. Suppose they satisfy the following:*

(a) $|c| < 1$*;*

(b) *the analytic differential* ω *defined on* M *in terms of a local parameter* $z = \xi^1 + i\xi^2$ *by*

$$\omega = \left(\frac{\partial h}{\partial \xi^1} - i\frac{\partial h}{\partial \xi^2}\right) dz \tag{10}$$

has zeros coinciding in position and order with zeros and poles of F*;*

(c) *if* c *is not real, then* h *has a single-valued harmonic conjugate on* M*;*

(d) *if*

$$d = -c^2 + 1, \tag{11}$$

then

$$-\overline{\int_C F\omega} = \int_C \frac{d}{F}\omega \tag{12}$$

for every closed curve C on M.

Then the surface $X : M \longrightarrow \mathbb{L}^4$ defined by

$$X = Re \int \left(1, -c, \frac{1}{2}(F + \frac{d}{F}), \frac{i}{2}(\frac{d}{F} - F), \right) \omega \tag{13}$$

is a 1-degenerate maximal surface of the second kind and its Gaussian image lies in a hyperplane which is SO(1,3)-equivalent to $cz_1 + z_2 = 0$, $|c| < 1$. Here the integral is taken from a fixed point to a variable point M along an arbitrary path.

Conversely, let S be a 1-degenerate maximal surface of the second kind which lies in a hyperplane that is SO(1,3)-equivalent to $cz_1 + z_2 = 0$, $|c| < 1$. To such an S, we may assign a quadruple $\{M, F, h, c\}$ which satisfies the hypotheses. The surface S is actually given, up to congruence, by (13).

(3) *Let M be a Riemann surface, F a non-constant meromorphic function on M, g and h a (non-constant) harmonic function on M. Suppose they satisfy the following:*

(a) *the analytic differential λ defined on M in terms of a local parameter $z = \xi^1 + i\xi^2$ by*

$$\lambda = \left(\frac{\partial g}{\partial \xi^1} - i\frac{\partial g}{\partial \xi^2}\right) dz = \psi dz \tag{14}$$

has zeros coinciding in position and order with zeros of F;

(b) *the analytic differential μ defined on M in terms of a local parameter $z = \xi^1 + i\xi^2$ by*

$$\mu = \left(\frac{\partial h}{\partial \xi^1} - i\frac{\partial h}{\partial \xi^2}\right) dz = \Psi dz \tag{15}$$

has zeros coinciding in position and order with poles of F;

(c)

$$\begin{aligned} &F\Psi + \tfrac{\psi}{F} = -\sqrt{2}i\Psi\,, \\ &|F\Psi|^2 - 2Re(\psi\Psi) + |\tfrac{\psi}{F}|^2 > 0\,; \end{aligned} \tag{16}$$

(d) *for every closed curve C on M,*

$$\int_C F\mu = -\overline{\int_C \frac{\lambda}{F}}\,. \tag{17}$$

Then the surface $X : M \longrightarrow \mathbb{L}^4$ defined by

$$X = \frac{1}{2}Re \int (1, -1, F, -iF)\,\mu + \frac{1}{2}Re \int \left(1, 1, \frac{1}{F}, \frac{i}{F}\right) \lambda \tag{18}$$

is a 1-degenerate maximal surface of the second kind and its Gaussian image lies in a hyperplane which is SO(1,3)-equivalent to $-z_1 + z_2 + \sqrt{2}z^3 = 0$. Here the integral is taken from a fixed point to a variable point M along an arbitrary path.

Conversely, let S be a 1-degenerate maximal surface of the second kind which lies in a hyperplane that is SO(1,3)-equivalent to $-z_1+z_2+\sqrt{2}z^3=0$. To such an S, we may assign a quadruple $\{M,F,g,h\}$ which satisfies the hypotheses. The surface S is actually given, up to congruence, by (18).

Proof. (1) Put $F=\frac{\phi_1+\phi_2}{\phi_3}$.

(2) Put $F=\frac{\phi_3+i\phi_4}{\phi_1}$.

(3) Since g and h are harmonic functions on M, $Re\int_C \lambda = Re\int_C \mu = 0$ for any closed (smooth) curve in M. Also iv) guarantees

$$Re\int_C \left(F\mu+\frac{\lambda}{F}\right) = Re\int_C i\left(-F\mu+\frac{\lambda}{F}\right)=0\,.$$

Hence (18) gives us a well-defined map $X : M \to \mathbb{L}^4$. Since $g = Re\int \psi dz$, $h = Re\int \Psi dz$, we obtain $x_1=\frac{1}{2}(g+h)$, and $x_2=\frac{1}{2}(g-h)$. Since M is locally simply-connected and all of the integrands in (18) are holomorphic on M, all x_k's are harmonic and hence $\phi_k = \frac{\partial x_k}{\partial \xi^1} - i\frac{\partial x_k}{\partial \xi^2}$ are holomorphic for $1\le k\le 4$, and ϕ_k is the integrand in (18). Directly from (18),

$$\begin{aligned} - \quad & {\phi_1}^2+{\phi_2}^2+{\phi_3}^2+{\phi_4}^2 \\ &= \tfrac{1}{4}\{-(\psi+\Psi)^2+(\psi-\Psi)^2+(F\Psi+\tfrac{\psi}{F})^2-(-F\Psi+\tfrac{\psi}{F})^2\} \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} - \quad & |\phi_1|^2+|\phi_2|^2+|\phi_3|^2+|\phi_4|^2 \\ &= \tfrac{1}{2}\left(|F\Psi|^2-(2Re(\psi\Psi)+|\tfrac{\psi}{F}|^2\right) > 0\,. \end{aligned}$$

Hence $X : M \longrightarrow \mathbb{L}^4$ defines a maximal surface in $\mathbb{L}^4$. Now we prove that it is exactly 1-degenerate of the second kind. Suppose $\sum \epsilon_k a_k \phi_k \equiv 0$. Since $F\Psi+\frac{\psi}{F}=-\sqrt{2}i\Psi$, it follows that $\psi=(F^2-\sqrt{2}iF)\Psi$ except isolated points. Note that neither Ψ nor ψ is identically zero. Let

$$\begin{aligned} \alpha &= -a_1-a_2-\sqrt{2}ia_3+\sqrt{2}a_4\,, \\ \beta &= \sqrt{2}ia_1-\sqrt{2}ia_2-2ia_4\,, \\ \gamma &= -a_1+a_2\,. \end{aligned}$$

Then $0\equiv 2\sum\epsilon_k a_k\phi_k \equiv \alpha\Psi+\beta F\Psi+\gamma F^2\Psi$. Since Ψ is not identically zero, we have the quadratic equation of F with the form $\alpha+\beta F+\gamma F^2\equiv 0$, where α, β, γ are defined as above. Since F is not constant, all the coefficients are zeros, otherwise F would be constant in terms of a_1, a_2, a_3, a_4. Hence we obtain

$$a_1=a_2=a, a_4=0, a_3=\frac{i}{\sqrt{2}}(a_1+a_2)=\sqrt{2}ia\,.$$

There is only one, up to a linear factor, spacelike vector $(1,1,\sqrt{2}i,0)$ which satisfies the equation (1). Hence S is a 1-degenerate maximal surface of the second kind.

We will prove the converse. Hypotheses guarantees the existence of an orthonormal basis of $\mathbb{L}^4$ with respect to which the Gauss nap of S satisfies $\phi_3 = \frac{i}{\sqrt{2}}(\phi_2 - \phi_1)$ with respect to a local isothermal parameter $z = \xi^1 + i\xi^2$ on M. If $\phi_2 - \phi_1 \equiv 0$, then $\phi_3 \equiv \phi_4 \equiv 0$. This would imply that the Gauss map is constant. In the similar way we can show $\phi_1 + \phi_2$ does not vanish everywhere. Thus the function

$$F = \frac{\phi_3 + i\phi_4}{\phi_1 - \phi_2} = \frac{\phi_1 + \phi_2}{\phi_3 - i\phi_4}$$

is meromorphic on M. F does not vanish everywhere, otherwise $0 \equiv {\phi_3}^2 + {\phi_4}^2 \equiv {\phi_1}^2 + {\phi_2}^2$ would imply either $\phi_1 + \phi_2 \equiv 0$ or $\phi_1 - \phi_2 \equiv 0$. If F is constant, then the Gauss map would be constant, that is, S could not be 1-degenerate. Consider the map $X : M \longrightarrow \mathbb{L}^4$ defines a maximal surface S. Define $g = x_1 + x_2$, $h = x_1 - x_2$ so that both become harmonic maps on M. Note that neither of them is constant, because none of $\phi_1 - \phi_2$ and $\phi_1 + \phi_2$ vanish everywhere. Define $\psi = \frac{\partial g}{\partial \xi^1} - i\frac{\partial g}{\partial \xi^2}$ and $\Psi = \frac{\partial h}{\partial \xi^1} - i\frac{\partial h}{\partial \xi^2}$. Then

$$\psi = \phi_1 + \phi_2 \ , \ \Psi = \phi_1 - \phi_2 \ . \tag{19}$$

According to the definition of F,

$$\phi_3 + i\phi_4 = F\Psi \ , \ \phi_3 - i\phi_4 = \frac{\psi}{F} \ . \tag{20}$$

Note here $F\Psi$ and $\frac{\psi}{F}$ are holomorphic on M, and therefore the hypotheses (a) and (b) are satisfied. From (19) and (20) we obtain

$$\begin{array}{rcl} \phi_1 & = & \frac{1}{2}(\psi + \Psi), \\ \phi_2 & = & \frac{1}{2}(\psi - \Psi), \\ \phi_3 & = & \frac{1}{2}(F\Psi + \frac{\psi}{F}), \\ \phi_1 & = & \frac{i}{2}(\frac{\psi}{F} - F\Psi). \end{array} \tag{21}$$

Direct computation shows (c) is also satisfied. Since $X = Re\int \Phi dz$, (18) follows easily up to the choice of a fixed point and a path to a variable point from it, in other words, up to a congruence in $\mathbb{L}^4$. For any closed curve C on M, $Re\int_C \left(F\mu + \frac{\lambda}{F}\right) \equiv 0$, that is $\int_C F\mu + \overline{\int_C \frac{\lambda}{F}} \equiv 0$ and therefore (d) is satisfied.

□

Theorem 2.4. *Let M be a Riemann surface, F a non-constant meromorphic function on M, h a (non-constant) harmornic function on M. Suppose they satisfy the following:*

(1) *the analytic differential* ω *defined on* M *in terms of a local parameter* $z = \xi^1 + i\xi^2$ *by*

$$\omega = \left(\frac{\partial h}{\partial \xi^1} - i\frac{\partial h}{\partial \xi^2}\right) dz \tag{22}$$

has zeros coinciding in position and order with zeros and poles of F*;*

(2) h *has a single-valued harmonic conjugate on* M*;*

(3) *for every closed curve* C *on* M*,*

$$\int_C F\omega = 2\overline{\int_C \frac{\omega}{F}} \tag{23}$$

Then the surface $X : M \longrightarrow \mathbb{L}^4$ *defined by*

$$X = Re\int \left(i, 1, \frac{1}{2}(F - \frac{2}{F}), -\frac{i}{2}(F + \frac{2}{F})\right)\omega \tag{24}$$

is a 1-degenerate maximal surface of the third kind. Here the integral is taken from a fixed point to a variable point M *along an arbitrary path.*

Conversely, to a 1-degenerate maximal surface S *of the third kind in* $\mathbb{L}^4$*, we may assign a triple* $\{M, F, h\}$ *which satisfies the hypotheses. The surface* S *is actually given, up to congruence, by (24).*

Proof. Put $F = \frac{\phi_3 + i\phi_4}{\phi_2}$ and $x_2 = h$. □

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