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FIXED POINTS AND COMMON FIXED POINTS THEOREMS IN CONE METRC-LIKE SPACES

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Abstract. In this paper, we introduce the new concept of a cone metric-like space and consider some fixed point theorems for generalized contractive mappings under suitable conditions in cone metric-like spaces. Our results generalize and unify the several main results of [1, 2, 9].

1. Introduction and Preliminaries

In 2007, Huang and Zhang [3] introduced a cone metric, as a generalization of a usual metric, and obtained fixed point theorems for some contractive mappings in cone metric spaces. Since then the fixed point theory for various mappings in a cone metric space has been rapidly developed and a lot of papers have appeared (see e.g., [4, 5, 6]). Later, Amini-Harandi [1] introduced a metric-like space, as a generalization of partial metric spaces, and considered some fixed point theorems for contractive mappings in metric-like spaces.

In 2002, Aamri and Moutawakil [7] introduced a property (E, A) for self mappings and obtained some fixed point theorems for such mappings under strict contractive conditions. Since the class of mappings satisfying property (E, A) contains the class of noncompatible mappings, the property (E, A) is very useful in the study of fixed point theorems of nonexpansive mappings(see [8]). Kim and Lee [2] introduced the property (C), which is a cone metric version of the usual metric property (E, A).

Inspired by the previous works, in this paper we introduce the concept of a cone metric-like, as a generalization of both cone metric and metric-like, and consider fixed point theorems for generalized contractive mappings in cone metric-like spaces. Our results generalize and unify the several main theorems of [1, 2, 9].

First of all, we recall some basic notions of a cone and a partial ordering.

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A nonempty subset P of a real Banach space E is called a cone if and only if
(P1) P is closed, $P \neq \{0\}$;
(P2) $a, b \in \mathbb{R}$ with $a, b \geq 0$, $x, y \in P \Rightarrow ax + by \in P$;
(P3) $x \in P$ and $-x \in P \Rightarrow x = 0$.

For a given cone $P \subset E$, we define a partial ordering '$\preceq$' with respect to P as follows; for $x, y \in E$, $x \preceq y$ if and only if $y - x \in P$. We shall note $x \ll y$ if and only if $y - x \in \text{int}P$, where $\text{int}P$ denotes the interior of P.

Now, we give the concept of a cone mteric-like space.

Definition 1. Let M be a nonempty set. Suppose that a mapping $d : M \times M \to (E, P)$ satisfies the following;
(d1) $d(x, y) = 0$ implies that $x = y$;
(d2) $d(x, y) = d(y, x)$ for all $x, y \in M$;
(d3) $d(x, y) \preceq d(x, z) + d(y, z)$ for all $x, y, z \in M$.
Then d is called a cone metric-like on M, and the set M with a cone metric-like d is called a cone metric-like space, denoted by (M, d).

If $E = \mathbb{R}$ and $P = \mathbb{R}_{\geq 0} := \{x | x \geq 0\}$, then a cone metric-like space (M, d) is a metric-like space in [1]. Therefore, every metric-like space can be regarded as a cone metric-like space.

Example 1.1. *Let $M = [0, 1]$, $E = \mathbb{R}^2$ be a Banach space with the standard norm, $P = \{(x, y) \in E; x, y \geq 0\}$ be a cone and let $d : M \times M \to E$ be a mapping of the form*

$$d(x, y) = \begin{cases} (0, 0), & x = y = 0 \\ (|x - y|, 1), & \textit{otherwise}. \end{cases}$$

Then the pair (M, d) is a cone metric-like space. However, since $d(1, 1) = (0, 1)$ and $d(1, 1) \neq (0, 0)$, (M, d) is not a cone metric space.

For the notion of convergence, the following definitions are considered in a cone metric-like space (M, d).

Definition 2. Let $\{x_n\}$ be a sequence in a cone metric-like space (M, d) and $x \in M$. If for every $c \in intP$, there is a natural number N such that for all $n > N$, $d(x_n, x) \ll c$, then we say that $\{x_n\}$ converges to x with respect to P and denote as $\lim_{n\to\infty} x_n = x$.

Definition 3. Let $\{x_n\}$ be a sequence in a cone metric-like space (M, d). If for every $c \in intP$, there is a natural number N such that for all $n, m > N$, $d(x_n, x_m) \ll c$, then we say that $\{x_n\}$ is a Cauchy sequence in (M, d).

Definition 4. If every Cauchy sequence in a cone metric-like space (M, d) is convergent, then (M, d) is called a complete cone metric-like space.

2. Fixed Point Theorems in Cone Metric-like Spaces

In this section, we establish fixed point theorems in two kinds of conditions satisfying the property (C) and the other. Firstly, we introduce the useful property (C) for checking the relationship of a sequence and its image converging to the same point.

Definition 5. Let M be a nonempty set with a cone metric-like $d : M \times M \to (E, P)$. A mapping $T : M \to M$ is said to satisfy the property (C) if there is a sequence $\{x_n\}$ in M such that

$$\lim_{n\to\infty} d(x_n, z) = 0 = \lim_{n\to\infty} d(Tx_n, z) \text{ for some } z \in M.$$

Definition 6. Let M be a nonempty set with a cone metric-like $d : M \times M \to (E, P)$. A mapping $T : M \to M$ is said to be (ψ, φ)-quasi weak contractive if for each $x, y \in M$,

$$\psi(d(Tx, Ty)) \preceq \psi(M_T(x, y)) - \varphi(M_T(x, y)),$$

where $\psi, \varphi : P \to P$ are mappings, provided that $M_T(x, y) := \max\{d(x, y),\ d(x, Tx),\ d(y, Ty),\ d(x, Ty),\ d(y, Tx),\ d(x, x),\ d(y, y)\}$.

Theorem 2.1. *Let M be a nonempty set with a cone metric-like $d : M \times M \to (E, P)$ and $T : M \to M$ a (ψ, φ)-quasi-weak contraction satisfying the property (C) with non-decreasing map ψ and non-increasing map φ satisfying $\varphi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.*

Proof. Let $\{x_n\}$ be a sequence in M satisfying

$$\lim_{n\to\infty} d(x_n, z) = 0 = \lim_{n\to\infty} d(Tx_n, z) \text{ for some } z \in M.$$

Then, we have

$$\begin{aligned}
\lim_{n\to\infty} M_T(z, x_n) &= \lim_{n\to\infty} \max\{d(z, x_n), d(z, Tz), d(x_n, Tx_n), d(z, Tx_n), d(x_n, Tz) \\
&\quad , d(z, z), d(x_n, x_n)\} \\
&\le \lim_{n\to\infty} \max\{d(z, x_n), d(z, Tz), d(x_n, z) + d(z, Tx_n), d(z, Tx_n), \\
&\quad d(x_n, z) + d(z, Tz), d(z, x_n) + d(x_n, z), d(z, x_n) + d(x_n, z)\} \\
&= d(z, Tz). \qquad (1)
\end{aligned}$$

Since T is a (ψ, φ)-quasi-weak contraction,

$$\psi(d(Tz, Tx_n)) \preceq \psi(M_T(z, x_n)) - \varphi(M_T(z, x_n)). \qquad (2)$$

On the other hand, $d(Tz, z) \preceq d(Tz, Tx_n) + d(Tx_n, z) = d(Tz, Tx_n)$. Since ψ is non-decreasing, we have

$$\psi(d(Tz, z)) \preceq \psi(d(Tz, Tx_n)) \text{ for } n \in \mathbb{N}. \qquad (3)$$

Letting $n \to \infty$, in the inequality (2) and (3), we obtain

$$\psi(d(Tz, z)) \preceq \lim_{n\to\infty} \psi(d(Tz, Tx_n)) \preceq \lim_{n\to\infty} (\psi(M_T(z, x_n)) - \varphi(M_T(z, x_n))).$$

Since ψ is non-decreasing and φ is non-increasing, from (1), we have

$$\psi(d(Tz,z)) \preceq \psi(d(z,Tz)) - \varphi(d(z,Tz))$$

Thus, $\varphi(d(z,Tz)) \preceq 0$ and the inequality implies that $\varphi(d(z,Tz)) = 0$ in a cone P. By the given property of φ, $d(Tz,z) = 0$. Since d is cone metric-like, $Tz = z$, that is z is a fixed point of T.

To prove its uniqueness, suppose that T has two distinct fixed points y and z in M. Then

$$\begin{aligned} M_T(y,z) &= \max\{d(y,z), d(y,Ty), d(z,Tz), d(y,Tz), d(z,Ty), d(y,y), d(z,z)\} \\ &= \max\{d(y,z), d(y,y), d(z,z), d(y,z), d(z,y), d(y,y), d(z,z)\} \\ &= \max\{d(y,z), d(y,y), d(z,z)\}. \end{aligned}$$

From the inequality $d(y,y) \preceq d(y,Ty) + d(y,Ty) = 0$, we have $d(y,y) = d(z,z) = 0$. Therefore, $M_T(y,z) = d(y,z)$. Since T is a (ψ,φ)-quasi-weak contraction, we have

$$\psi(d(y,z)) = \psi(d(Ty,Tz)) \preceq \psi(M_T(y,z)) - \varphi(M_T(y,z)) = \psi(d(y,z)) - \varphi(d(y,z))$$

Thus $\varphi(d(y,z)) \preceq 0$ which implies that $\varphi(d(y,z)) = 0$ and $d(y,z) = 0$. Since d is cone metric-like, $y = z$ and thus T has a unique fixed point. □

Example 2.2. *Let $M = \{0,1,2\}$, $E = \mathbb{R}^2$ be a Banach space with the standard norm and $P = \{(x,y) \in E; x,y \geq 0\}$ be a cone. If we define a mapping $d : M \times M \to E$ as follows;*

$$\begin{aligned} &d(0,0) = (0,0), \quad d(1,1) = (0,1), \quad d(2,2) = (2,1), \\ &d(0,1) = (1,1), \quad d(1,0) = (2,1), \quad d(0,2) = (1,1), \\ &d(2,0) = (2,1), \quad d(1,2) = (3,1), \quad d(2,1) = (3,1). \end{aligned}$$

then d is a cone metric-like on M. Let $T : M \to M$ be a mapping defined by

$$T0 = 0, \qquad T1 = 0, \qquad T2 = 1.$$

Then, we can easily see that T satisfies the property (C). If $\psi, \varphi : P \to P$ are the mappings defined by $\psi((x,y)) = (x^2, y - \frac{y^2}{2})$ and $\varphi((x,y)) = (\frac{x+y}{2}, 0)$ for each $(x,y) \in P$, then ψ and φ satisfy the condition of Theorem 2.1. Through the following calculation;

$$\begin{aligned} &M_T(0,0) = (0,0), \quad M_T(1,1) = (2,1), \quad M_T(2,2) = (3,1), \\ &M_T(0,1) = (2,1), \quad M_T(1,0) = (2,1), \quad M_T(0,2) = (3,1), \\ &M_T(2,0) = (3,1), \quad M_T(1,2) = (3,1), \quad M_T(2,1) = (3,1), \end{aligned}$$

we can induce that T is a (ψ,φ)-quasi-weak contraction. Therefore, T has a unique fixed point.

If (M,d) is a cone metic space and $M_T(x,y) \preceq d(x,y)$, then we have the following theorem as a corollary of Theorem 2.1. And, Theorem 2.1 makes it possible to omit the continuity of the following theorem.

Theorem 2.3. [2] *Let M be a nonempty set with a cone metric $d : M \times M \to (E, P)$ and $T : M \to M$ a generalized (ψ, φ)-weak contractive mapping satisfying the property (C) and for each $x, y \in M$,*

$$\psi(d(Tx, Ty)) \preceq \psi(d(x, y)) - \varphi(d(x, y)),$$

where $\psi, \varphi : P \to P$ are continuous mappings with $\varphi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

Now, we consider a fixed point theorem in complete cone metric-like spaces without the property (C).

Theorem 2.4. *Let (M, d) be a complete cone metric-like space and $T : M \to M$ be a mapping satisfying*

$$d(Tx, Ty) \preceq \alpha(d(x, y))d(x, y) - \beta(d(x, y))w \tag{4}$$

for each $w \in intP$, $x, y \in M$ with $x \neq y$, where $\alpha : P \to [0, 1)$ is non-increasing and $\beta : P \to [0, 1)$ with $\beta(0) = 0$. Then T has a unique fixed point.

Proof. Let $x \in M$ and $x_n = T^n x$ for $n \in \mathbb{N}$. Consider

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \preceq \alpha(d(x_{n-1}, x_n))d(x_{n-1}, x_n) - \beta(d(x_{n-1}, x_n))w \\ &\preceq d(x_{n-1}, x_n) = d(Tx_{n-2}, Tx_{n-1}) \\ &\preceq \alpha(d(x_{n-2}, x_{n-1}))d(x_{n-2}, x_{n-1}) - \beta(d(x_{n-2}, x_{n-1}))w \\ &\preceq d(x_{n-2}, x_{n-1}) \preceq \cdots \preceq d(x_1, x_2), \end{aligned}$$

hence $\{d(x_n, x_{n+1)}\}$ is non-increasing. On the other hand, from (4), we have

$$\begin{aligned} &d(x_2, x_{n+1}) = d(Tx_1, Tx_n) \preceq \alpha(d(x_1, x_n))d(x_1, x_n) - \beta(d(x_1, x_n))w \\ &\preceq \alpha(d(x_1, x_n))\{d(x_1, x_2) + d(x_2, x_{n+1}) + d(x_{n+1}, x_n)\}, \end{aligned}$$

which implies that

$$(1 - \alpha(d(x_1, x_n))d(x_2, x_{n+1}) \preceq d(x_1, x_2) + d(x_{n+1}, x_n) \preceq 2d(x_1, x_2).$$

Since $\{d(x_n, x_{n+1)}\}$ is non-increasing, we get

$$(1 - \alpha(d(x_1, x_n))d(x_1, x_n) \preceq 4d(x_1, x_2),$$

which implies that

$$d(x_1, x_n) \preceq \frac{4d(x_1, x_2)}{1 - \alpha(d(x_1, x_n))}.$$

Since α is non-increasing, we have

$$d(x_1, x_n) \preceq \frac{4d(x_1, x_2)}{1 - \alpha(t)} \tag{5}$$

for some $t \in P$. Hence, $\{x_n\}$ is bounded.

If $d(x_k, x_{k+p}) \succeq c$ for $k = 1, \cdots, n-1$ and $c \in intP$, by the non-increasing property of α, we have $d(x_k, x_{k+p}) \preceq \alpha(c)w$ for $w \in intP$. From (5), we get

$$\begin{aligned} d(x_n, x_{n+p}) &\preceq d(x_1, x_p) \prod_{k=1}^{n-1} \alpha(d(x_k, x_{k+p})) \\ &\preceq \frac{4d(x_1, x_2)}{1-\alpha(t)} \{\alpha(c)\}^n w^n \to 0 \text{ as } n \to \infty. \end{aligned}$$

So, there exists $N \in \mathbb{N}$, independently of p, such that $d(x_N, x_{N+p}) \preceq \varepsilon$ for $p \in \mathbb{N}$, which proves that $\{x_n\}$ is a Cauchy sequence, hence we have $\lim_{m,n\to\infty} d(T^n x, T^m x) = 0$. From the completeness of (M, d), there exists $x_0 \in M$ such that

$$\lim_{n\to\infty} d(T^n x, x_0) = d(x_0, x_0) = \lim_{m,n\to\infty} d(T^n x, T^m x) = 0. \tag{6}$$

From (4), we get

$$\begin{aligned} d(T^n x, Tx_0) &\preceq \alpha(d(T^{n-1}x, x_0))d(T^{n-1}x, x_0) - \beta(d(T^{n-1}x, x_0))w \\ &\preceq \alpha(d(T^{n-1}x, x_0))d(T^{n-1}x, x_0). \end{aligned} \tag{7}$$

By (6) and (7), we have $\lim_{n\to\infty} d(T^n x, Tx_0) = 0$. Thus,

$$\lim_{n\to\infty} d(T^n x, Tx_0) = d(Tx_0, Tx_0) = \lim_{m,n\to\infty} d(T^n x, T^m x) = 0.$$

From the inequality (d3) of a cone metric-like d, we have

$$d(x_0, Tx_0) \preceq d(T^n x, x_0) + d(T^n x, Tx_0) \to 0 \text{ as } n \to \infty$$

and so $x_0 = Tx_0$, that is x_0 is a fixed point of T.

To prove the uniqueness, suppose that T has two distinct fixed points y and z in M. Then, from (4),

$$d(y, z) = d(Ty, Tz) \preceq \alpha(d(y, z))d(y, z) - \beta(d(y, z))w \preceq \alpha(d(y, z))d(y, z),$$

which implies that

$$(1 - \alpha(d(y, z)))d(y, z) \preceq 0.$$

Thus, $d(y, z) = 0$ which implies the unique existence of fixed point of T. □

By putting $\beta \equiv 0$, then the following theorem in [1] is a corollary of Theorem 2.4.

Theorem 2.5. *Let (M, d) be a complete metric-like space and $T : M \to M$ be a mapping satisfying*

$$d(Tx, Ty) \leq \alpha(d(x, y))d(x, y)$$

for each $x, y \in M$ with $x \neq y$ with $\alpha : [0, \infty) \to [0, 1)$ is non-increasing. Then T has a unique fixed point.

3. Common Fixed Point Theorems in Cone Metric-like Spaces

Definition 7. Two mappings $S, T : M \to M$ are weakly compatible if $STx = TSx$ whenever $Sx = Tx$.

Definition 8. Let M be a nonempty set with a cone metric-like $d : M \times M \to (E, P)$. Two mappings $S, T : M \to M$ are said to satisfy the property (C) if there is a sequence $\{x_n\}$ in M such that

$$\lim_{n\to\infty} d(Sx_n, z) = 0 = \lim_{n\to\infty} d(Tx_n, z) \text{ for some } z \in M.$$

Theorem 3.1. *Let M be a nonempty set with a cone metric-like $d : M \times M \to (E, P)$ and $S, T : M \to M$ be mappings satisfying the property (C), S be onto, and for each $x, y \in M$,*

$$\psi(d(Tx, Ty)) \preceq \psi(d(Sx, Sy)) - \varphi(d(Sx, Sy))$$

where ψ is non-decreasing and φ is non-increasing self-mappings on P. Then S and T have a coincidence point in M. Moreover, if S and T are weakly compatible, then S and T have a unique common fixed point.

Proof. Let $\{x_n\}$ be a sequence in M satisfying

$$\lim_{n\to\infty} d(Sx_n, z) = 0 = \lim_{n\to\infty} d(Tx_n, z) \text{ for some } z \in M.$$

Take $a \in M$ such that $z = Sa$, then

$$\lim_{n\to\infty} d(Sx_n, Sa) = 0 = \lim_{n\to\infty} d(Tx_n, Sa) \text{ for some } z \in M.$$

Since

$$\psi(d(Ta, Tx_n)) \preceq \psi(d(Sa, Sx_n)) - \varphi(d(Sa, Sx_n)),$$

we have

$$\lim_{n\to\infty} \psi(d(Ta, Tx_n)) \preceq \lim_{n\to\infty} (\psi(d(Sa, Sx_n)) - \varphi((Sa, Sx_n))),$$

which implies that

$$\psi(d(Ta, Sa)) \preceq \psi(d(Sa, Sa)) - \varphi(d(Sa, Sa)).$$

Thus, $d(Ta, Sa) = 0$.

Now, we show that $z = Ta$ is a common fixed point of S and T. Since S and T are weakly compatible, we have

$$\psi(d(Ta, TTa)) \preceq \psi(d(Sa, STa)) - \varphi(d(Sa, STa)) = \psi(d(Ta, TTa)) - \varphi(d(Ta, TTa)),$$

which implies that $Ta = TTa$. Hence $TTa = STa = Ta = z$. To prove the uniqueness, suppose that S and T have two distinct fixed points $y = Sy = Ty$ and $z = Sz = Tz$ in M, then

$$\psi(d(Tz, Ty)) \preceq \psi(d(Sz, Sy)) - \varphi(d(Sz, Sy)) = \psi(d(Tz, Ty)) - \varphi(d(Tz, Ty)).$$

Hence, $\varphi(d(Tz, Ty)) = 0$. □

By putting $\psi(t) = t$ and $\varphi(t) = 0$ in Theorem 3.1, we have the following common fixed point theorem.

Theorem 3.2. [2] *Let M be a nonempty set with a cone metric $d : M \times M \to (E, P)$ and $S, T : M \to M$ be mappings satisfying the property (C), S be onto, and for each $x, y \in M$,*

$$d(Tx, Ty) \preceq d(Sx, Sy).$$

Then S and T have a coincidence point in M. Moreover, if S and T are weakly compatible, then they have a unique common fixed point.

The following theorem in [9] is a corollary of Theorem 3.1.

Theorem 3.3. *Let (X, d) be a cone metric space, and P a normal cone with normal constant K. Suppose mappings $S, T : M \to M$ satisfy*

$$d(Tx, Ty) \preceq kd(Sx, Sy), \quad \text{for all } x, y \in M,$$

where $k \in [0, 1)$ is a constant. If the range of S contains the range of T and $S(M)$ is a complete subspace of M, then T and S have a unique point of coincidence in M. Moreover, if S and T are weakly compatible, then they have a unique common fixed point.

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