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GREEN FUNCTIONS FOR FLOW VELOCITY OF STATIONARY STOKES SYSTEMS

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Abstract. We establish existence and uniqueness of Green functions for flow velocity of stationary Stokes systems, under a continuity assumption of weak solutions to the system, in a bounded domain such that the divergence equation is solvable there. We also obtain pointwise bounds of the Green functions.

1. Introduction

This paper is a continuation of [6], in which the authors studied Green functions for the flow velocity of stationary Stokes systems with variable coefficients in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d \geq 3$. In [6], they established the existence of the Green function satisfying pointwise bound away from the boundary of the domain under the regularity assumption that weak solutions of the system are Hölder continuous in the interior of the domain. They also obtained global pointwise bound for the Green function under the additional assumption that weak solutions of Dirichlet problems are locally bounded up to the boundary. Such type of regularity assumptions were introduced in [12, 13] to deal with fundamental solutions and Green functions for elliptic systems with irregular coefficients. We refer the reader to [4, 5, 7, 2] and the references therein for some work in this direction. We also refer to [10, 3] for Green functions in two dimensional domains without any regularity assumptions.

In this paper, we extend the results in [6] to domains such that the divergence equation is solvable there. This solvability assumption is sufficiently general to allow the domain Ω to be, for example, a John domain. Hence, the class of domains we consider includes Lipschitz domains, Reifengerg flat domains, and Semmes-Kenig-Toro (SKT) domains. Note that, in establishing the existence of the Green function, we use a weaker condition that weak solutions of the system are continuous in the interior of the domain. For further details, see Theorem 2.6.

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As an application of our results combined with $W^{1,q}$-estimates for the Stokes system established in [8], we have that if the coefficients of the Stokes system are merely measurable in one direction, which may differ depending on the local coordinates, and have small mean oscillations in the other directions (variably partially BMO) and the domain is a John domain, then the Green function exists and satisfies the pointwise bound away from the boundary of the domain. Moreover, the Green function satisfies the pointwise bound globally if the domain is Reifenberg flat. Note that Stokes systems with variably partially BMO coefficients can be used to describe the motion of inhomogeneous fluids with density dependent viscosity and two fluids with interfacial boundaries; see [8, 9] and the references therein.

The remainder of the paper is organized as follows. In Section 2, we state our main results along with some definitions and assumptions. Section 3 is devoted to the construction of approximated Green functions. In Section 4, we provide the proofs of the main theorems.

2. Main results

Throughout the paper, we denoted by Ω a bounded domain in the Euclidean space $\mathbb{R}^d$, where $d \geq 3$. For any $x \in \Omega$ and $R > 0$, we write $\Omega_R(x) = \Omega \cap B_R(x)$, where $B_R(x)$ is a usual Euclidean ball of radius R centered at x. For $q \in [1, \infty]$, we define

$$\tilde{L}^q(\Omega) = \{u \in L^q(\Omega) : (u)_\Omega = 0\},$$

where $L^q(\Omega)$ is the set of all measurable functions on Ω that are qth integrable and $(u)_\Omega$ is the average of u over Ω, i.e.,

$$(u)_\Omega = \fint_\Omega u \, dx = \frac{1}{|\Omega|} \int_\Omega u \, dx.$$

We also denote by $W^{1,q}(\Omega)$ the usual Sobolev space and $W_0^{1,q}(\Omega)$ the completion of $C_0^\infty(\Omega)$ in $W^{1,q}(\Omega)$, where $C_0^\infty(\Omega)$ is the set of all infinitely differentiable functions with compact supports in Ω.

Let $\mathcal{L}$ be a second-order elliptic operator in divergence form

$$\mathcal{L}u = D_\alpha(A^{\alpha\beta} D_\beta u)$$

acting on column vector-valued functions $u = (u^1, \ldots, u^d)^\top$ defined on the domain Ω, where the coefficients $A^{\alpha\beta} = A^{\alpha\beta}(x)$ are $d \times d$ matrix-valued functions on $\mathbb{R}^d$ satisfying the strong ellipticity condition, that is, there is a constant $\lambda \in (0, 1]$ such that for any $x \in \mathbb{R}^d$ and $\xi_\alpha \in \mathbb{R}^d$, $\alpha \in \{1, \ldots, d\}$, we have

$$|A^{\alpha\beta}| \leq \lambda^{-1}, \quad \sum_{\alpha,\beta=1}^{d} A^{\alpha\beta} \xi_\beta \cdot \xi_\alpha \geq \lambda \sum_{\alpha=1}^{d} |\xi_\alpha|^2.$$

The adjoint operator $\mathcal{L}^*$ of $\mathcal{L}$ is defined by

$$\mathcal{L}^* u = D_\alpha((A^{\beta\alpha})^\top D_\beta u),$$

where $(A^{\beta\alpha})^\top$ is the transpose of $A^{\beta\alpha}$.

In the definition below, $G = G(x, y)$ is a $d \times d$ matrix-valued function and $\Pi = \Pi(x, y)$ is a $1 \times d$ vector-valued function in $\Omega \times \Omega$.

Definition 2.1. Let $d \geq 3$ and Ω be a bounded domain in $\mathbb{R}^d$. We say that (G, Π) is a Green function (for the flow velocity) of $\mathcal{L}$ in Ω if it satisfies the following properties.
(a) For any $y \in \Omega$ and $R > 0$,

$$G(\cdot, y) \in W_0^{1,1}(\Omega)^{d\times d} \cap W^{1,2}(\Omega \setminus B_R(y))^{d\times d}$$

and

$$\Pi(\cdot, y) \in \tilde{L}^1(\Omega)^d \cap L^2(\Omega \setminus B_R(y))^d.$$

(b) For any $y \in \Omega$, $(G(\cdot, y), \Pi(\cdot, y))$ satisfies

$$\begin{cases} \mathcal{L}G(\cdot, y) + \nabla \Pi(\cdot, y) = -\delta_y I & \text{in } \Omega, \\ \operatorname{div} G(\cdot, y) = 0 & \text{in } \Omega. \end{cases}$$

(c) If $(u, p) \in W_0^{1,2}(\Omega)^d \times L^2(\Omega)$ is a weak solution of

$$\begin{cases} \mathcal{L}^* u + \nabla p = f & \text{in } \Omega, \\ \operatorname{div} u = g & \text{in } \Omega, \end{cases}$$

where $f \in L^\infty(\Omega)^d$ and $g \in \tilde{L}^\infty(\Omega)$, then for a.e. $y \in \Omega$, we have

$$u(y) = -\int_\Omega G(x, y)^\top f(x)\, dx + \int_\Omega \Pi(x, y)^\top g(x)\, dx. \tag{2.1}$$

Remark 2.2. The property (c) in the definition above together with the solvability result in, for instance, [6, Lemma 3.2] gives the uniqueness of a Green function in the sense that if $(\tilde{G}, \tilde{\Pi})$ is another Green function satisfying the above properties, then for each $\phi \in C_0^\infty(\Omega)^d$ and $\varphi \in C_0^\infty(\Omega)$, we have

$$\int_\Omega \big(G(x, y)^\top - \tilde{G}(x, y)^\top\big)\phi(x)\, dx = \int_\Omega \big(\Pi(x, y)^\top - \tilde{\Pi}(x, y)^\top\big)\varphi(x)\, dx = 0$$

for a.e. $y \in \Omega$.

We make the following assumptions to construct the Green function of $\mathcal{L}$ in Ω.

Assumption 2.3. There exists a constant $K_0 > 0$ such that the following holds. For any $g \in \tilde{L}^2(\Omega)$, there exists $u \in W_0^{1,2}(\Omega)^d$ satisfying

$$\operatorname{div} u = g \text{ in } \Omega, \quad \|Du\|_{L^2(\Omega)} \leq K_0 \|g\|_{L^2(\Omega)}.$$

Remark 2.4. From [1, Theorem 4.1], it follows that Assumption 2.3 holds in a bounded John domain. Moreover, by a scaling argument, we see that if $\Omega = B_R$, then the assumption holds with $q \in (1, \infty)$ in place of 2 and the constant K_0 depending only on d and q.

The following assumption holds, for instance, when the coefficients $A^{\alpha\beta}$ of $\mathcal{L}$ are variably partially BMO; see [2, Theorem 6.2].

Assumption 2.5. There exist constants $R_0 \in (0,1]$ and $A_0 > 0$ such that the following holds. Let $x_0 \in \Omega$ and $0 < R < \min\{R_0, \operatorname{dist}(x_0, \partial\Omega)\}$. If $(u,p) \in W^{1,2}(B_R(x_0))^d \times L^2(B_R(x_0))$ satisfies

$$\begin{cases} \mathcal{L}u + \nabla p = f & \text{in } B_R(x_0), \\ \operatorname{div} u = 0 & \text{in } B_R(x_0), \end{cases}$$

where $f \in L^\infty(B_R(x_0))^d$, then we have $u \in C(\overline{B_{R/2}(x_0)})^d$ (in fact, a version of u belongs to $C(\overline{B_{R/2}(x_0)})^d$) with the estimate

$$\|u\|_{L^\infty(B_{R/2}(x_0))} \le A_0\big(R^{-d/2}\|u\|_{L^2(B_R(x_0))} + R^2\|f\|_{L^\infty(B_R(x_0))}\big).$$

The same statement holds true when $\mathcal{L}$ is replaced by $\mathcal{L}^*$.

In the theorem below and throughout the paper, we denote

$$d_y = \operatorname{dist}(y, \partial\Omega), \quad d_y^* = \min\{R_0, d_y\}.$$

Theorem 2.6. *Let Ω be a bounded domain in $\mathbb{R}^d$, where $d \ge 3$. Then, under Assumptions 2.3 and 2.5, there exist Green functions (G, Π) of $\mathcal{L}$ and (G^*, Π^*) of $\mathcal{L}^*$ such that for any $y \in \Omega$, we have*

$$G(\cdot, y), G^*(\cdot, y) \in C(\Omega \setminus \{y\})^{d\times d},$$

and there exists a measure zero set $N_y \subset \Omega$ such that

$$G(x,y) = G^*(y,x)^\top, \quad G(y,x) = G^*(x,y)^\top \quad \textit{for all } x \in \Omega \setminus N_y. \tag{2.2}$$

Moreover, for any $x, y \in \Omega$ with $0 < |x-y| < d_y^*/2$,

$$|G(x,y)| \le N|x-y|^{2-d}, \tag{2.3}$$

where $N = N(d, \lambda, K_0, A_0)$.

We have the following corollary, the estimates in which can be derived from the proof of Theorem 2.6. Note that the estimates for $\Pi(\cdot, y)$ in (ii) are new even in the case when Ω is a Lipschitz domain; cf. [6, Theorem 2.3].

Corollary 2.7. *Let (G, Π) be the Green function of $\mathcal{L}$ derived from 2.6 under Assumptions 2.3 and 2.5. Then the following hold.*
(i) For any $y \in \Omega$ and $R \in (0, d_y^*)$, we have that

$$\|G(\cdot,y)\|_{L^{2d/(d-2)}(\Omega\setminus\overline{B_R(y)})} + \|DG(\cdot,y)\|_{L^2(\Omega\setminus\overline{B_R(y)})} \le NR^{(2-d)/2}, \tag{2.4}$$

$$\|G(\cdot,y)\|_{L^q(B_R(y))} \le N_q R^{2-d+d/q}, \quad q \in [1, d/(d-2)),$$

$$\|DG(\cdot,y)\|_{L^q(B_R(y))} \le N_q R^{1-d+d/q}, \quad q \in [1, d/(d-1)).$$

Moreover,

$$\big|\{x \in \Omega : |G(x,y)| > t\}\big| \le Nt^{-d/(d-2)}, \quad \forall t > (d_y^*)^{2-d},$$

$$\big|\{x \in \Omega : |D_xG(x,y)| > t\}\big| \le Nt^{-d/(d-1)}, \quad \forall t > (d_y^*)^{1-d}.$$

(ii) For any $y \in \Omega$ *and* $R \in (0, d_y^*)$, *we have that*

$$\|\Pi(\cdot, y)\|_{L^2(\Omega \setminus \overline{B_R(y)})} \le N R^{(2-d)/2}, \tag{2.5}$$

$$\|\Pi(\cdot, y)\|_{L^q(B_R(y))} \le N_q R^{1-d+d/q}, \quad q \in [1, d/(d-1)).$$

Moreover,

$$\big|\{x \in \Omega : |\Pi(x, y)| > t\}\big| \le N t^{-d/(d-1)}, \quad \forall t > (d_y^*)^{1-d}.$$

In the above, the constant N *depends only on* d, λ, K_0, *and* A_0, *and* N_q *depends also on* q.

To obtain the global pointwise bound for $G(x, y)$, we impose the following assumption. By using the $W^{1,q}$-estimates in [8, Theorem 2.4] and a bootstrap argument (see, for instance, the proof of [6, Theorem 2.9]), one can check that the assumption holds when the coefficients $A^{\alpha\beta}$ of $\mathcal{L}$ are variably partially BMO and Ω is a Reifenberg flat domain.

Assumption 2.8. There exist constants $R_0 \in (0, 1]$ and $A_1 > 0$ such that the following holds. Let $x_0 \in \partial\Omega$ and $R \in (0, R_0)$. If $(u, p) \in W^{1,2}(\Omega_R(x_0))^d \times L^2(\Omega_R(x_0))$ satisfies

$$\begin{cases} \mathcal{L}u + \nabla p = f & \text{in } \Omega_R(x_0), \\ \operatorname{div} u = 0 & \text{in } \Omega_R(x_0), \\ u = 0 & \text{on } \partial\Omega \cap B_R(x_0), \end{cases}$$

where $f \in L^\infty(\Omega_R(x_0))^d$, then we have

$$\|u\|_{L^\infty(\Omega_{R/2}(x_0))} \le A_1 \big(R^{-d/2} \|u\|_{L^2(\Omega_R(x_0))} + R^2 \|f\|_{L^\infty(\Omega_R(x_0))}\big).$$

The same statement holds true when $\mathcal{L}$ is replaced by $\mathcal{L}^*$.

Theorem 2.9. *Let* Ω *be a bounded domain in* $\mathbb{R}^d$, *where* $d \ge 3$. *Let* (G, Π) *be the Green function constructed in Theorem 2.6 under Assumptions 2.3 and 2.5. If we assume Assumption 2.8 (in addition to Assumptions 2.3 and 2.5), then for any* $x, y \in \Omega$ *with* $0 < |x - y| < R_0$, *we have*

$$|G(x, y)| \le N |x - y|^{2-d},$$

where $N = N(d, \lambda, K_0, A_0, A_1)$.

We end this section with the following remark.

Remark 2.10. It is not clear us if the estimates in Corollary 2.7 still hold for $R \in (0, R_0)$ under Assumption 2.8 in addition to Assumptions 2.3 and 2.5. In fact, the proofs of Lemmas 3.2–3.5 in Section 3 does not work for $y \in \Omega$ and $d_y < R < R_0$, because the divergence equation is not necessarily solvable in $\Omega_R(x)$.

3. Approximated Green functions

Hereafter in the paper, we use the following notation.

Notation 3.1. For nonnegative (variable) quantities A and B, we denote $A \lesssim B$ if there exists a generic positive constant N such that $A \le NB$. We add subscript letters like $A \lesssim_{a,b} B$ to indicate the dependence of the implicit constant N on the parameters a and b.

In this section, we assume that the hypotheses of Theorem 2.6 hold. Under the hypotheses, we shall construct approximated Green functions for the flow velocity of the Stokes system. We mainly follow the arguments in [12] (see also [2, 6]).

Let $y \in \Omega$, $\varepsilon \in (0,1]$, $k \in \{1,\dots,d\}$, and

$$\Phi_{\varepsilon,y}(x) = -\frac{1}{|\Omega_\varepsilon(y)|} I_{\Omega_\varepsilon(y)}(x). \tag{3.1}$$

By [6, Lemma 3.2], there exists a unique $(v,\pi) = (v_{\varepsilon,y,k}, \pi_{\varepsilon,y,k}) \in W_0^{1,2}(\Omega)^d \times \tilde{L}^2(\Omega)$ satisfying

$$\begin{cases} \mathcal{L}v + \nabla\pi = \Phi_{\varepsilon,y} e_k & \text{in } \Omega, \\ \operatorname{div} v = 0 & \text{in } \Omega, \end{cases} \tag{3.2}$$

where e_k is the kth unit vector in $\mathbb{R}^d$. Moreover, we have

$$\|Dv\|_{L^2(\Omega)} + \|\pi\|_{L^2(\Omega)} \lesssim_{d,\lambda,K_0} |\Omega_\varepsilon(y)|^{(2-d)/(2d)}. \tag{3.3}$$

Due to Assumption 2.5, there is a version $\tilde{v}$ of v such that $\tilde{v} = v$ a.e. in Ω and $\tilde{v}$ is continuous in Ω. We define the approximated Green function $(G_\varepsilon(\cdot,y), \Pi_\varepsilon(\cdot,y))$ (for the flow velocity) of $\mathcal{L}$ by

$$G_\varepsilon^{jk}(\cdot,y) = \tilde{v}^j = \tilde{v}^j_{\varepsilon,y,k}, \quad \Pi_\varepsilon^k(\cdot,y) = \pi = \pi_{\varepsilon,y,k}.$$

In the lemma below, we obtain a pointwise bound for $G_\varepsilon(\cdot,y)$.

Lemma 3.1. *Let $\varepsilon \in (0,1]$ and $x, y \in \Omega$ with*

$$0 < 2\varepsilon < \frac{|x-y|}{2} < \frac{d_y^*}{3}.$$

Then we have

$$|G_\varepsilon(x,y)| \lesssim_{d,\lambda,K_0,A_0} |x-y|^{2-d}.$$

Proof. Denote

$$(v,\pi) = (G_\varepsilon^{\cdot k}(\cdot,y), \Pi_\varepsilon^k(\cdot,y)), \tag{3.4}$$

where $G_\varepsilon^{\cdot k}(\cdot,y)$ is the kth column of $G_\varepsilon(\cdot,y)$. Let $x \in \Omega$ and $0 < 2\varepsilon < R < d_y^*$. Find $(u,p) \in W_0^{1,2}(\Omega)^d \times \tilde{L}^2(\Omega)$ satisfying

$$\begin{cases} \mathcal{L}u + \nabla p = -f & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \end{cases}$$

where $f = I_{\Omega_R(x)}(\operatorname{sgn} v^1, \dots, \operatorname{sgn} v^d)^\top$. Then by Assumption 2.5, the Sobolev inequality, and the $W^{1,2}$-estimate, we have

$$\begin{aligned} \|u\|_{L^\infty(B_{R/2}(y))} &\lesssim_{A_0} R^{-d/2}\|u\|_{L^2(B_R(y))} + R^2 \\ &\lesssim_{d,A_0} R^{1-d/2}\|Du\|_{L^2(\Omega)} + R^2 \\ &\lesssim_{d,\lambda,K_0,A_0} R^2. \end{aligned} \tag{3.5}$$

Observe that

$$\int_{\Omega_R(x)} f \cdot v \, dz = \int_\Omega A^{\alpha\beta} D_\beta v \cdot D_\alpha u \, dz = \fint_{B_\varepsilon(y)} u^k \, dz. \tag{3.6}$$

Since $\varepsilon < R/2$, combining (3.5) and (3.6), we get

$$\|v\|_{L^1(\Omega_R(x))} \lesssim_{d,\lambda,K_0,A_0} R^2 \tag{3.7}$$

for any $x \in \Omega$ and $0 < 2\varepsilon < R < d_y^*$.

We now ready to prove the lemma. Let

$$0 < 2\varepsilon < R := \frac{|x-y|}{2} < \frac{d_y^*}{3}.$$

Since $B_R(x) \subset \Omega$ and $B_R(x) \cap B_\varepsilon(y) = \emptyset$, (3.2) yields that

$$\begin{cases} \mathcal{L}v + \nabla\pi = 0 & \text{in } B_R(x), \\ \operatorname{div} v = 0 & \text{in } B_R(x). \end{cases} \tag{3.8}$$

For any $z \in B_R(x)$ and $r \in (0, R]$ with $B_r(z) \subset B_R(x)$, by Assumption 2.5 applied to (3.8) in $B_r(z)$, we have

$$\|v\|_{L^\infty(B_{r/2}(z))} \lesssim r^{-d/2}\|v\|_{L^2(B_r(z))},$$

and thus, by a standard iteration argument (see, for instance, [11, pp. 80–82]), we obtain

$$\|v\|_{L^\infty(B_{R/2}(x))} \lesssim R^{-d}\|v\|_{L^1(B_R(x))}. \tag{3.9}$$

From this together with (3.7) and continuity of v, we get the desired estimate. The lemma is proved. $\square$

From Lemma 3.1, by following the steps in the proof of [6, Lemma 4.3], we obtain the following uniform estimates for $G_\varepsilon(\cdot, y)$ and $DG_\varepsilon(\cdot, y)$. The corresponding estimates for $\Pi_\varepsilon(\cdot, y)$ can be found in Lemma 3.5 below.

Lemma 3.2. *Let $\varepsilon \in (0, 1]$, $y \in \Omega$, and $R \in (0, d_y^*)$. Then we have*

$$\|G_\varepsilon(\cdot, y)\|_{L^{2d/(d-2)}(\Omega\setminus\overline{B_R(y)})} + \|DG_\varepsilon(\cdot, y)\|_{L^2(\Omega\setminus\overline{B_R(y)})} \le NR^{(2-d)/2}, \tag{3.10}$$

$$\|G_\varepsilon(\cdot, y)\|_{L^{q_1}(B_R(y))} \le N_1 R^{2-d+d/q_1}, \quad q_1 \in [1, d/(d-2)),$$

$$\|DG_\varepsilon(\cdot, y)\|_{L^{q_2}(B_R(y))} \le N_2 R^{1-d+d/q_2}, \quad q_2 \in [1, d/(d-1)). \tag{3.11}$$

Moreover, we have

$$\left|\{x \in \Omega : |G_\varepsilon(x, y)| > t\}\right| \le N t^{-d/(d-2)}, \quad \forall t > (d_y^*)^{2-d},$$

$$\big|\{x \in \Omega : |D_x G_\varepsilon(x,y)| > t\}\big| \le N t^{-d/(d-1)}, \quad \forall t > (d_y^*)^{1-d}.$$

In the above, the constant N depends only on d, λ, K_0, and A_0, and N_i depend also on q_i.

Lemma 3.3. *Let $\varepsilon \in (0,1]$, $y \in \Omega$, and $R \in (0, d_y^*)$. Then we have*

$$\big\|\Pi_\varepsilon(\cdot,y) - (\Pi_\varepsilon(\cdot,y))_{B_R(y)}\big\|_{L^1(B_R(y))} \lesssim_{d,\lambda,K_0,A_0} R.$$

Proof. Recall the notation (3.4). Let us fix $q \in (1, d/(d-1))$ and $q' \in (d,\infty)$ with $\frac{1}{q} + \frac{1}{q'} = 1$. By the existence of solutions to the divergence equation in a ball (see, for instance, [1]), there exists $\phi \in W_0^{1,q'}(B_R(y))^d$ such that

$$\operatorname{div}\phi = \operatorname{sgn}(\pi - (\pi)_{B_R(y)}) - \big(\operatorname{sgn}(\pi - (\pi)_{B_R(y)})\big)_{B_R(y)} \quad \text{in } B_R(y)$$

and

$$\|D\phi\|_{L^{q'}(B_R(y))} \le N R^{d/q'}.$$

Here, by a scaling argument, one can check that the constant N in the above inequality depends only on d and q'. We extend ϕ by zero in $\Omega \setminus B_R(y)$ and apply ϕ as a test function to (3.2) to get

$$\int_\Omega \pi \operatorname{div}\phi\, dx = -\int_\Omega A^{\alpha\beta} D_\beta v \cdot D_\alpha \phi\, dx + \fint_{B_\varepsilon(y)} \phi\, dx.$$

Observe that

$$\int_\Omega \pi \operatorname{div}\phi\, dx = \int_\Omega (\pi - (\pi)_{B_R(y)}) \operatorname{div}\phi\, dx = \int_{B_R(y)} |\pi - (\pi)_{B_R(y)}|\, dx$$

and

$$\|\phi\|_{L^\infty(\Omega)} = \|\phi\|_{L^\infty(B_R(y))} \lesssim_d R^{1-d/q'} \|Du\|_{L^{q'}(B_R(y))} \lesssim R.$$

Combining these together and using Hölder's inequality and (3.11), we have

$$\begin{aligned}\|\pi - (\pi)_{B_R(y)}\|_{L^1(B_R(y))} &\lesssim \|Dv\|_{L^q(B_R(y))} \|D\phi\|_{L^{q'}(B_R(y))} + R \\ &\lesssim_{d,\Lambda,K_0,A_0} R.\end{aligned}$$

The lemma is proved. □

Lemma 3.4. *Let $\varepsilon \in (0, R/16]$, $y \in \Omega$, and $R \in (0, d_y^*/2)$. For $k \in \{1,\ldots,d\}$, we set*

$$\tilde\Pi_\varepsilon^k(\cdot,y) = \Pi_\varepsilon^k(\cdot,y) - (\Pi_\varepsilon^k(\cdot,y))_{B_R(y)\setminus\overline{B_{R/2}(y)}}.$$

Then we have

$$\|\tilde\Pi_\varepsilon^k(\cdot,y)\|_{L^2(B_R(y)\setminus\overline{B_{R/2}(y)})} \lesssim_{d,\lambda,K_0,A_0} R^{(2-d)/2},$$

where $G_\varepsilon^{\cdot k}(\cdot,y)$ is the kth column of $G_\varepsilon(\cdot,y)$.

Proof. Recall the notation 3.4, and set

$$\tilde{\pi} = \pi - (\pi)_{B_R(y)\setminus\overline{B_{R/2}(y)}}.$$

It suffices to show that

$$\|\tilde{\pi}\|_{L^2(B_R(y)\setminus\overline{B_{R/2}(y)})} \lesssim R^{-1}\|v\|_{L^2(B_{5R/4}(y)\setminus\overline{B_{R/4}(y)})} \tag{3.12}$$

because by Lemma 3.1, we have

$$\|v\|_{L^2(B_{5R/4}(y)\setminus\overline{B_{R/4}(y)})} \lesssim_{d,\lambda,K_0,A_0} R^{2-d/2}.$$

From the existence of solutions to the divergence equation, there exists a function $\phi \in W_0^{1,2}(B_R(y)\setminus\overline{B_R(y)})^d$ such that

$$\operatorname{div}\phi = \tilde{\pi} \quad \text{in } \; B_R(y)\setminus\overline{B_{R/2}(y)}$$

and

$$\|D\phi\|_{L^2(B_R(y)\setminus B_{R/2}(y))} \lesssim_d \|\tilde{\pi}\|_{L^2(B_R(y)\setminus B_{R/2}(y))}. \tag{3.13}$$

We extend ϕ by zero in $\Omega\setminus(B_R(y)\setminus\overline{B_{R/2}(y)})$ and apply ϕ as a test function to (3.2) to get

$$\int_\Omega \pi \operatorname{div}\phi\, dx = -\int_\Omega A^{\alpha\beta}D_\beta v\cdot D_\alpha\phi\, dx.$$

Thus by using Hölder's inequality, (3.13), and the fact that

$$\int_\Omega \pi \operatorname{div}\phi\, dx = \int_\Omega \tilde{\pi}\operatorname{div}\phi\, dx = \int_{B_R(y)\setminus\overline{B_{R/2}(y)}} |\tilde{\pi}|^2\, dx,$$

we have

$$\int_{B_R(y)\setminus\overline{B_{R/2}(y)}} |\tilde{\pi}|^2\, dx \lesssim_{d,\lambda} \int_{B_R(y)\setminus\overline{B_{R/2}(y)}} |Dv|^2\, dx. \tag{3.14}$$

Now we let $z\in B_R(y)\setminus\overline{B_{R/2}(y)}$. Since $B_{R/4}(z)\cap B_\varepsilon(y)=\emptyset$, it holds that

$$\begin{cases} \mathcal{L}v + \nabla\pi = 0 & \text{in } \; B_{R/4}(z),\\ \operatorname{div} v = 0 & \text{in } \; B_{R/4}(z). \end{cases}$$

By the Caccioppoli inequality (see, for instance, [6, Lemma 3.3]) applied to the above system and the fact that $B_{R/4}(z)\subset\big(B_{5R/4}(y)\setminus\overline{B_{R/4}(y)}\big)$, we have

$$\int_{B_{R/8}(z)} |Dv|^2\, dx \lesssim \frac{1}{R^2}\int_{B_{5R/4}(y)\setminus\overline{B_{R/4}(y)}} |v|^2\, dx$$

Since the above inequality holds for all $z\in B_R(y)\setminus\overline{B_{R/2}(y)}$, by a covering argument, we obtain that

$$\int_{B_R(y)\setminus\overline{B_{R/2}(y)}} |Dv|^2\, dx \lesssim \frac{1}{R^2}\int_{B_{5R/4}(y)\setminus\overline{B_{R/4}(y)}} |v|^2\, dx,$$

which together with (3.14) gives (3.12). The lemma is proved. □

From the above two lemmas, we get the following uniform estimates for $\Pi_\varepsilon(\cdot,y)$.

Lemma 3.5. *Let $\varepsilon \in (0,1]$, $y \in \Omega$, and $R \in (0, d_y^*)$. Then we have*

$$\|\Pi_\varepsilon(\cdot, y)\|_{L^2(\Omega\setminus\overline{B_R(y)})} \le NR^{(2-d)/2}, \tag{3.15}$$

$$\|\Pi_\varepsilon(\cdot, y)\|_{L^q(B_R(y))} \le N_q R^{1-d+d/q_2}, \quad q \in [1, d/(d-1)).$$

Moreover, we have

$$\left|\{x \in \Omega : |\Pi_\varepsilon(x,y)| > t\}\right| \le Nt^{-d/(d-1)}, \quad \forall t > (d_y^*)^{1-d}.$$

In the above, the constant N depends only on d, λ, K_0, and A_0, and N_q depends also on q.

Proof. We only prove (3.15) because the others are its easy consequences; see, for instance, [2, Lemmas 4.4 and 4.5]. Recall the notation (3.4). We may assume that $0 < R < d_y^*/2$. If $R/16 \le \varepsilon \le 1$, then by (3.3) we have

$$\|\pi\|_{L^2(\Omega)} \lesssim |B_{R/2}(y)|^{(2-d)/(2d)} \lesssim R^{(2-d)/2}.$$

Assume $0 < \varepsilon \le R/16$, and let η be an infinitely differentiable function in $\mathbb{R}^d$ satisfying

$$0 \le \eta \le 1, \quad \eta \equiv 1 \text{ in } B_{R/2}(y), \quad \operatorname{supp} \eta \subset B_R(y), \quad |\nabla \eta| \lesssim_d R^{-1}.$$

By Assumption 2.3, there exists $\phi \in W_0^{1,2}(\Omega)^d$ such that

$$\operatorname{div} \phi = \pi I_{\Omega\setminus\overline{B_R(y)}} - \left(\pi I_{\Omega\setminus\overline{B_R(y)}}\right)_\Omega \quad \text{in } \Omega \tag{3.16}$$

and

$$\|\phi\|_{L^{2d/(d-2)}(\Omega)} + \|D\phi\|_{L^2(\Omega)} \lesssim_{d,K_0} \|\pi\|_{L^2(\Omega\setminus\overline{B_R(y)})}. \tag{3.17}$$

We apply $(1-\eta)\phi$ as a test function to (3.2) to obtain

$$\int_\Omega \pi \operatorname{div}((1-\eta)\phi)\, dx = -\int_\Omega A^{\alpha\beta} D_\beta v \cdot D_\alpha((1-\eta)\phi)\, dx.$$

Set $\tilde{\pi} = \pi - (\pi)_{B_{R(y)}\setminus\overline{B_{R/2}(y)}}$, and note that by (3.16) and $\int_\Omega \pi\, dx = 0$, we have

$$\begin{aligned}
&\int_\Omega \pi \operatorname{div}((1-\eta)\phi)\, dx \\
&= \int_\Omega \pi \operatorname{div} \phi\, dx - \int_\Omega \tilde{\pi} \operatorname{div}(\eta\phi)\, dx \\
&= \int_{\Omega\setminus\overline{B_R(y)}} |\pi|^2\, dx - \int_{B_R(y)\setminus\overline{B_{R/2}(y)}} \tilde{\pi} \nabla\eta \cdot \phi\, dx - \left(\pi I_{\Omega\setminus\overline{B_R(y)}}\right)_\Omega \int_{B_R(y)} \tilde{\pi}\eta\, dx.
\end{aligned}$$

We also note that

$$\left|\left(\pi I_{\Omega\setminus\overline{B_R(y)}}\right)_\Omega \int_{B_R(y)} \tilde{\pi}\eta\, dx\right| \lesssim R^{-d/2} \|\pi\|_{L^2(\Omega\setminus\overline{B_R(y)})} \left\|\pi - (\pi)_{B_R(y)}\right\|_{L^1(B_R(y))}.$$

Therefore, from Hölder's inequality, (3.17), and the fact that

$$\int_{B_R(y)} |\tilde{\pi}|\, dx \lesssim \int_{B_R(y)} \left|\pi - (\pi)_{B_R(y)}\right| dx,$$

we get

$$\|\pi\|_{L^2(\Omega\setminus\overline{B_R(y)})} \lesssim \|Dv\|_{L^2(\Omega\setminus\overline{B_R(y)})} + \|\tilde{\pi}\|_{L^2(B_R(y)\setminus\overline{B_{R/2}(y)})} + R^{-d/2}\big\|\pi - (\pi)_{B_R(y)}\big\|_{L^1(B_R(y))}.$$

Finally, by using (3.10) and Lemmas 3.3 and 3.4, we conclude (3.15). $\square$

4. Proofs of main theorems

Throughout this section, we denote by $(G_\varepsilon(\cdot,y),\Pi_\varepsilon(\cdot,y))$ the approximated Green function constructed in Section 3.

Proof of Theorem 2.6. Let $\varepsilon\in(0,1]$ and $y\in\Omega$. By Lemmas 3.2 and 3.5, the weak compactness theorem, and a diagonalization process, we see that there exist a sequence $\{\varepsilon_\rho\}_{\rho=1}^\infty$ tending to zero and a pair $(G(\cdot,y),\Pi(\cdot,y))$ such that for any $R\in(0,d_y^*)$,

$$\begin{aligned}(1-\eta_R)G_{\varepsilon_\rho}(\cdot,y) \rightharpoonup (1-\eta_R)G(\cdot,y) \quad \text{weakly in } W_0^{1,2}(\Omega)^{d\times d},\\ \Pi_{\varepsilon_\rho}(\cdot,y) \rightharpoonup \Pi(\cdot,y) \quad \text{weakly in } L^2(\Omega\setminus\overline{B_R(y)})^d,\end{aligned} \tag{4.1}$$

where η_R is any smooth function in $\mathbb{R}^d$ satisfying $\eta_R=1$ in $B_R(y)$, and that for fixed $q\in(1,d/(d-1))$,

$$\begin{aligned}G_{\varepsilon_\rho}(\cdot,y) \rightharpoonup G(\cdot,y) \quad \text{weakly in } W^{1,q}(B_{d_y^*}(y))^{d\times d},\\ \Pi_{\varepsilon_\rho}(\cdot,y) \rightharpoonup \Pi(\cdot,y) \quad \text{weakly in } L^q(B_{d_y^*}(y))^d.\end{aligned} \tag{4.2}$$

Then one can check that $(G(\cdot,y),\Pi(\cdot,y))$ satisfies the properties (a)–(c) in Definition 2.1, which means that (G,Π) is the Green function of $\mathcal{L}$ in Ω. Due to the property (b) and Assumption 2.5, the function $G(\cdot,y)$ is continuous in $\Omega\setminus\{y\}$. More precisely, we choose a version of G which is continuous in $\Omega\setminus\{y\}$ and denote it again by G.

Note that the identity (2.1) holds for *all* $y\in\Omega$ if $g\equiv 0$. Indeed, if $(u,p)\in W_0^{1,2}(\Omega)^d\times L^2(\Omega)$ is a weak solution of (2.1) with $g\equiv 0$, then by Assumption 2.5 there is a continuous version of u in Ω, denoted again by u, satisfying

$$\fint_{\Omega_{\varepsilon_\rho}(y)} u\,dx = -\int_\Omega G_{\varepsilon_\rho}(x,y)^\top f(x)\,dx \quad \text{for all } y\in\Omega.$$

By taking $\rho\to\infty$ and using the continuity of u, we have

$$u(y) = -\int_\Omega G(x,y)^\top f(x)\,dx \quad \text{for all } y\in\Omega. \tag{4.3}$$

Now we shall prove the estimates in Corollary 2.7 and the pointwise bound (2.3). The estimates (2.4) and (2.5) are simple consequences of (3.10), (3.15), and the weak semi-continuity. Then by following the same steps used in, for instance, [2, Lemmas 4.4 and 4.5], one can prove that the others in the corollary also hold. To show (2.3), let $x, y \in \Omega$ with $0 < |x-y| < d_y^*/2$, and denote $r = |x-y|/2$. Since $(G(\cdot,y), \Pi(\cdot,y))$ satisfies

$$\begin{cases} \mathcal{L}G(\cdot,y) + \nabla \Pi(\cdot,y) = 0 & \text{in } B_r(x), \\ \operatorname{div} G(\cdot,y) = 0 & \text{in } B_r(x), \end{cases}$$

by Assumption 2.5, Hölder's inequality, and the estimates in Corollary 2.7, we have

$$\begin{aligned} |G(x,y)| &\lesssim r^{(2-d)/2} \|G(\cdot,y)\|_{L^{2d/(d-2)}(B_r(x))} \\ &\lesssim r^{(2-d)/2} \|G(\cdot,y)\|_{L^{2d/(d-2)}(\Omega \setminus \overline{B_r(y)})} \\ &\lesssim r^{2-d}, \end{aligned}$$

which gives (2.3).

For $\sigma \in (0,1]$ and $x \in \Omega$, let $(G^*_\sigma(\cdot,x), \Pi^*_\sigma(\cdot,x))$ be the approximated Green function for $\mathcal{L}^*$, i.e., if we set $w = w_{\sigma,x,\ell}$ as the ℓ-th column of $G^*_\sigma(\cdot,x)$ and $\kappa = \kappa_{\sigma,x,\ell}$ as the ℓ-th component of $\Pi^*_\sigma(\cdot,x)$, then $(w,\kappa) \in W^{1,2}_0(\Omega)^d \times \tilde{L}^2(\Omega)$ satisfies

$$\begin{cases} \mathcal{L}^* w + \nabla \kappa = \Phi_{\sigma,x} e_\ell & \text{in } \Omega, \\ \operatorname{div} w = 0 & \text{in } \Omega, \end{cases}$$

where $\Phi_{\sigma,x}$ is given in (3.1). By proceeding similarly as above, we can find a sequence $\{\sigma_\tau\}_{\tau=1}^\infty$ tending to zero and a unique Green function $(G^*(\cdot,x), \Pi^*(\cdot,x))$ of $\mathcal{L}^*$ in Ω such that $\big(G^*_{\sigma_\tau}(\cdot,x), \Pi^*_{\sigma_\tau}(\cdot,x)\big)$ and $(G^*(\cdot,x), \Pi^*(\cdot,x))$ satisfy the natural counterparts of (4.1), (4.2), and the properties of the Green function of $\mathcal{L}$. Notice from (4.3) that

$$G^*_\sigma(y,x) = \fint_{\Omega_\sigma(x)} G(z,y)^\top \, dz \quad \text{for all } x, y \in \Omega.$$

Then by the continuity of $G(\cdot,y)$ on $\Omega \setminus \{y\}$, we have

$$\lim_{\sigma \to 0} G^*_\sigma(y,x) = G(x,y)^\top \quad \text{for all } x, y \in \Omega \text{ with } x \neq y. \tag{4.4}$$

Now we prove (2.2). Let $y \in \Omega$ be given. Then there exists a measure zero set $N_y \subset \Omega$ containing y such that, by passing to a subsequence,

$$\lim_{\rho \to \infty} G_{\varepsilon_\rho}(x,y) = G(x,y) \quad \text{for all } x \in \Omega \setminus N_y. \tag{4.5}$$

Indeed, since it holds that

$$\|G_{\varepsilon_\rho}(\cdot,y)\|_{W^{1,1}(\Omega)} \le_{d,\lambda,K_0,A_0,R_0,\operatorname{dist}(y,\partial\Omega)} 1,$$

by the Rellich-Kondrachov compactness theorem, for a sufficiently small $\delta > 0$, there exists a subsequence of $\{G_{\varepsilon_\rho}(\cdot,y)\}$ which converges a.e. to $G(\cdot,y)$ on Ω_δ, where Ω_δ is a smooth subdomain satisfying

$$\{x \in \Omega : B_\delta(x) \subset \Omega\} \subset \Omega_\delta \subset \Omega.$$

Thus by a diagonalization process, one can easily see that (4.5) holds. Combining (4.5) and the counterpart of (4.4), we have

$$G(x,y) = G^*(y,x)^\top$$

for all $x \in \Omega \setminus N_y$. Similarly, we see that the above identity holds for all $y \in \Omega \setminus N_x$. This gives (2.2). The theorem is proved. $\square$

We now prove Theorem 2.9.

Proof of Theorem 2.9. The proof is similar to that of Lemma 3.1. Note that by utilizing Assumptions 2.5 and 2.8, and following the steps used in deriving (3.7), we have

$$\|G_\varepsilon(\cdot,y)\|_{L^1(\Omega_R(x))} \lesssim_{d,\lambda,K_0,A_0,A_1} R^2 \tag{4.6}$$

for all $x,y \in \Omega$ and $0 < 2\varepsilon < R < R_0$.

Let $x,y \in \Omega$ with $0 < |x-y| < R_0$, and set $R = |x-y|/2$. Then for $\sigma \in (0,R)$, since $(G^*_\sigma(\cdot,x), \Pi^*_\sigma(\cdot,x))$ satisfies

$$\begin{cases} \mathcal{L}^* G^*_\sigma(\cdot,x) + \nabla \Pi^*_\sigma(\cdot,x) = 0 & \text{in } \Omega_R(y), \\ \operatorname{div} G^*_\sigma(\cdot,x) = 0 & \text{in } \Omega_R(y), \end{cases}$$

by Assumptions 2.5 and 2.8, we have

$$\|G^*_\sigma(\cdot,x)\|_{L^\infty(\Omega_{r/2}(z))} \lesssim_{d,A_0,A_1} r^{-d/2} \|G^*_\sigma(\cdot,x)\|_{L^2(\Omega_r(z))}$$

for any $z \in \Omega_R(y)$ and $0 < r < \operatorname{dist}(z, \partial B_R(y))$. From this together with a standard iteration argument, we get

$$\|G^*_\sigma(\cdot,x)\|_{L^\infty(\Omega_{R/2}(y))} \lesssim R^{-d} \|G^*_\sigma(\cdot,x)\|_{L^1(\Omega_R(y))} \lesssim R^{2-d}$$

where we used the counterpart of (4.6) in the second inequality. Finally, by using (4.4) we obtain the desired global bound for $G(x,y)$. The theorem is proved. $\square$

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