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STRUCTURE JACOBI OPERATORS AND REAL HYPERSURFACES OF TYPE(A) IN COMPLEX SPACE FORMS

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Abstract. Let M be a real hypersurface with almost contact metric structure (ϕ, ξ, η, g) in a nonflat complex space form $M_n(c)$. We denote S and R_ξ by the Ricci tensor of M and by the structure Jacobi operator with respect to the vector field ξ respectively. In this paper, we prove that M is a Hopf hypersurface of type (A) in $M_n(c)$ if it satisfies $R_\xi\phi = \phi R_\xi$ and at the same time satisfies $(\nabla_{\phi\nabla_\xi\xi}R_\xi)\xi = 0$ or $R_\xi\phi S = S\phi R_\xi$.

1. Introduction

A complex n-dimensional Kähler manifold with Kähler structure J of constant holomorphic sectional curvature $4c$ is called a complex space form and denoted by $M_n(c)$. As is well known, a connected complete and simply connected complex space form is complex analytically isometric to a complex projective space $P_n\mathbb{C}$, a complex Euclidean space $\mathbb{C}$ or a complex hyperbolic space $H_n\mathbb{C}$ if $c > 0$, $c = 0$ or $c < 0$, respectively.

The study of real hypersurfaces in complex projective space $P_n\mathbb{C}$ was initiated by Takagi [17], who proved that all homogeneous real hypersurfaces in $P_n\mathbb{C}$ could be divided into six types which are said to be of type A_1, A_2, B, C, D and E. He showed also in [16] and [17] that if a real hypersurface M in $P_n\mathbb{C}$ has two or three distinct constant principal curvatures, then M is locally congruent to one of the homogeneous ones of type A_1, A_2 or B. In particular, real hypersurfaces of type A_1, A_2 and B in $P_n\mathbb{C}$ have been studied by several authors (see, Cecil and Ryan [3], [4] and Okumura [17]).

In the case of complex hyperbolic space $H_n\mathbb{C}$, Montiel and Romero started the study of real hypersurfaces in [14] and constructed some homogeneous real hypersurfaces in $H_n\mathbb{C}$ which are said to be of type A_0, A_1 and A_2. Those hypersurfaces have a lot of nice geometric properties (see Berndt [1] and Montiel

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and Romero [15]). In 2007 Berndt and Tamaru [2] classified all homogeneous real hypersurfaces in $H_n\mathbb{C}$.

Let M be a real hypersurface of type A_1 or type A_2 in a complex projective space $P_n\mathbb{C}$ or that of type A_0, A_1 or A_2 in a complex hyperbolic space $H_n\mathbb{C}$. Then M is said to be of *type* (A) for simplicity. By a theorem due to Okumura [15] and to Montiel and Romero [14] we have

Theorem O-MR ([14], [15]). *If the shape operator A and the structure tensor ϕ commute to each other, then a real hypersurface of a complex space form $M_n(c)$, $c \neq 0$ is locally congruent to be of type (A).*

Characterization problems for a real hypersurface of type (A) in a complex space form were studied by many authors (cf. [5] $\sim$ [11], [13] etc.).

We denote by S and R_ξ be the Ricci tensor and the structure Jacobi operator with respect to the vector field ξ of M respectively.

To investigate of real hypersurfaces with respect to the structure Jacobi operator it is a very important problem to study real hypersurfaces satisfying $R_\xi\phi = \phi R_\xi$ in $M_n(c)$.

Under the condition $R_\xi A = AR_\xi$ we know that the following theorem ([5]):

Theorem CK ([5]). *Let M be a real hypersurface of $M_n(c)$, $c \neq 0$. If M satisfies $R_\xi\phi = \phi R_\xi$ and at the same time satisties $R_\xi A = AR_\xi$, then M is a Hopf hypersurface. Further M is of type (A) or a Hopf hypersurface with $g(A\xi, \xi) = 0$.*

In this paper we discuss real hypersurfaces satistying $R_\xi\phi = \phi R_\xi$ and at the same time $\nabla_{\phi\nabla_\xi\xi} R_\xi = 0$ in a nonflat complex space form $M_n(c)$. From the different point of view of Theorem CK, we give also another characterizations of real hypersurfaces of type (A) in $M_n(c)$ by using the Ricci tensor and the structure Jacobi operator.

All manifolds in the present paper are assume to be connected and of class C^∞ and the real hypersurfaces supposed to be orientable.

2. Basic properties of real hypersurfaces

Let M be a real hyperusurface immersed in a complex space form $M_n(c)$, $c \neq 0$ with almost complex structure J, and N be a unit normal vector field on M. The Riemannian connection $\tilde{\nabla}$ in $M_n(c)$ and ∇ in M are related by the following formulas for any vector fields X and Y on M :

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \tilde{\nabla}_X N = -AX,$$

where g denotes the Riemannian metric tensor of M induced from that of $M_n(c)$, and A denotes the shape operator of M in the direction N.

For any vector field X tangent to M, we put

$$JX = \phi X + \eta(X)N, \quad JN = -\xi.$$

We call ξ the structure vector field (or the Reeb vector field) and its flow also denoted by the same latter ξ. The Reeb vector field ξ is said to be *principal* if $A\xi = \alpha\xi$, where $\alpha = \eta(A\xi)$.

A real hypersurface M is said to be a *Hopf hypersurface* if the Reeb vector field ξ is principal. It is known that the aggregate (ϕ, ξ, η, g) is an almost contact metric structure on M, that is, we have

$$\begin{aligned} &\phi^2 X = -X + \eta(X)\xi, \ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \\ &\eta(\xi) = 1, \ \phi\xi = 0, \ \eta(X) = g(X, \xi) \end{aligned}$$

for any vector fields X and Y on M. From Kähler condition $\tilde{\nabla} J = 0$, and taking account of above equations, we see that

$$\nabla_X \xi = \phi AX, \tag{2.1}$$

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi \tag{2.2}$$

for any vector fields X and Y tangent to M.

Since we consider that the ambient space is of constant holomorphic sectional curvature $4c$, equations of the Gauss and Codazzi are respectively given by

$$\begin{aligned} R(X,Y)Z = \ & c\{g(Y,Z)X - g(X,Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ & -2g(\phi X, Y)\phi Z\} + g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{2.3}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = c\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\} \tag{2.4}$$

for any vector fields X, Y and Z on M, where R denotes the Riemannian curvature tensor of M.

In what follows, to write our formulas in convention forms, we denote by $\alpha = \eta(A\xi)$, $\beta = \eta(A^2\xi)$, $\gamma = \eta(A^3\xi)$ and $h = \mathrm{Tr}A$, and for a function f we denote by ∇f the gradient vector field of f.

From the Gauss equation (2.3), the Ricci tensor S of M is given by

$$SX = c\{(2n+1)X - 3\eta(X)\xi\} + hAX - A^2X \tag{2.5}$$

for any vector field X on M, which implies

$$S\xi = 2c(n-1)\xi + hA\xi - A^2\xi. \tag{2.6}$$

Now, we put

$$A\xi = \alpha\xi + \mu W, \tag{2.7}$$

where W is a unit vector field orthogonal to ξ. In the sequel, we put $U = \nabla_\xi \xi$, then by (2.1) we see that $U = \mu\phi W$ and hence U is orthogonal to W. So we have $g(U, U) = \mu^2$. Using (2.7), it is clear that

$$\phi U = -A\xi + \alpha\xi, \tag{2.8}$$

which shows that $g(U, U) = \beta - \alpha^2$. Thus it is seen that

$$\mu^2 = \beta - \alpha^2. \tag{2.9}$$

Making use of (2.1), (2.7) and (2.8), it is verified that

$$\mu g(\nabla_X W, \xi) = g(AU, X), \tag{2.10}$$

$$g(\nabla_X \xi, U) = \mu g(AW, X) \tag{2.11}$$

because W is orthogonal to ξ.

Now, differentiating (2.8) covariantly and taking account of (2.1) and (2.2), we find

$$(\nabla_X A)\xi = -\phi\nabla_X U + g(AU + \nabla\alpha, X)\xi - A\phi AX + \alpha\phi AX, \tag{2.12}$$

which together with (2.4) implies that

$$(\nabla_\xi A)\xi = 2AU + \nabla\alpha. \tag{2.13}$$

Applying (2.12) by ϕ and making use of (2.11), we obtain

$$\phi(\nabla_X A)\xi = \nabla_X U + \mu g(AW, X)\xi - \phi A\phi AX - \alpha AX + \alpha g(A\xi, X)\xi, \tag{2.14}$$

which connected to (2.1) and (2.13) gives

$$\nabla_\xi U = 3\phi AU + \alpha A\xi - \beta\xi + \phi\nabla\alpha. \tag{2.15}$$

Using (2.3), the structure Jacobi operator R_ξ is given by

$$R_\xi(X) = R(X, \xi)\xi = c\{X - \eta(X)\xi\} + \alpha AX - \eta(AX)A\xi \tag{2.16}$$

for any vector field X on M, which implies that

$$R_\xi\xi = 0, \tag{2.17}$$

$$R_\xi U = cU + \alpha AU, \quad R_\xi AU = cAU + \alpha A^2U. \tag{2.18}$$

Differentiating (2.16) covariantly along M and using (2.1), we find

$$\begin{aligned} g((\nabla_X R_\xi)Y, Z) &= g(\nabla_X(R_\xi Y) - R_\xi(\nabla_X Y), Z) \\ &= -c\{\eta(Z)g(\phi AX, Y) + \eta(Y)g(\phi AX, Z)\} + (X\alpha)g(AY, Z) \\ &+ \alpha g((\nabla_X A)Y, Z) - \eta(AZ)\{g((\nabla_X A)\xi, Y) + g(A\phi AX, Y)\} \\ &- \eta(AY)\{g((\nabla_X A)\xi, Z) + g(A\phi AX, Z)\}. \end{aligned} \tag{2.19}$$

From (2.5) we obtain

$$SU = c(2n+1)U + hAU - A^2U, \tag{2.20}$$

$$SA\xi = c\{(2n+1)A\xi - 3\alpha\xi\} + hA^2\xi - A^3\xi. \tag{2.21}$$

Because of (2.5) and (2.7), we also have

$$\mu SW = hA^2\xi - A^3\xi - \alpha(hA\xi - A^2\xi) + c(2n+1)(A\xi - \alpha\xi). \tag{2.22}$$

3. Structure Jacobi operator of a real hypersurface

Let M be a real hypersurface in complex space form $M_n(c)$, $c \neq 0$ satisfying $R_\xi \phi = \phi R_\xi$, which means that the eigenspace of R_ξ is invariant by the structure operator ϕ. Then by (2.16) we have

$$\alpha(\phi AX - A\phi X) = g(A\xi, X)U + g(U, X)A\xi. \tag{3.1}$$

We set $\Omega = \{p \in M : \mu(p) \neq 0\}$, and suppose that Ω is nonvoid, that is, ξ is not principal curvature vector on M. In the sequel, we discuss our arguments on the open subset Ω of M unless otherwise stated. Then, it is, using (3.1), clear that $\alpha \neq 0$ on Ω. So a function λ given by $\beta = \alpha\lambda$ is defined. Thus, replacing X by U in (3.1) and using (2.8), we find

$$\alpha(\phi AU - A^2\xi + \alpha A\xi) = \mu^2 A\xi,$$

which connected to (2.9) yields

$$\phi AU = \lambda A\xi - A^2\xi \tag{3.2}$$

because $\alpha \neq 0$ on Ω.

Applying by ϕ, we have

$$\phi A^2\xi = AU + \lambda U, \tag{3.3}$$

which together with (2.7) yields

$$\mu\phi AW = AU + (\lambda - \alpha)U. \tag{3.4}$$

Since W is orthogonal to U, we see from the last equation

$$g(AW, U) = 0. \tag{3.5}$$

If we replace X by AU in (3.1) and take account of (3.2), then we find

$$\alpha\phi A^2U - \alpha(\lambda A^2\xi - A^3\xi) = g(AU, U)A\xi, \tag{3.6}$$

which enables us to obtain

$$g(AU, U) = \gamma - \alpha\lambda^2. \tag{3.7}$$

Theorem 3.1. *Let M be a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$ such that $R_\xi\phi = \phi R_\xi$ holds on M. If it satisfies $R_\xi U = 0$, then M is a Hopf hypersurface. Furthermore, M is locally congruent to one of the following real hypersurface of type (A) or a Hopf hypersurface with $\eta(A\xi) = 0$.*

(I) *In case that $P_n\mathbb{C}$*

- (A_1) *a tube of radius r over a hyperplane $P_{n-1}\mathbb{C}$, $0 < r < \pi/2$,*
- (A_2) *a tube of radius r over a totally geodesic $P_k\mathbb{C}$ $(1 \leq k \leq n-2)$, where $0 < r < \pi/2$,*
- (T) *a tube of radius $\pi/4$ over a certain complex submanifold in $P_n\mathbb{C}$,*

(II) *In case $H_n\mathbb{C}$*

- (A_0) *a horosphere in $H_n\mathbb{C}$, i.e., a Montiel tube,*
- (A_1) *a geodesic hypersphere or a tube over a hyperplane $H_{n-1}\mathbb{C}$,*

(A_2) *a tube over a totally geodesic* $H_k\mathbb{C}$ $(1 \leq k \leq n-2)$.

Proof. Since $\alpha \neq 0$ on Ω, the first equation of (2.18) implies that $AU = -\frac{c}{\alpha}U$. Thus (3.2) reformed as $A^2\xi = \sigma A\xi + c\xi$ because of (2.8), where we have put $\sigma = \lambda - \frac{c}{\alpha}$.

By the way, from (2.16) we have

$$g(R_\xi Y, AX) - g(R_\xi X, AY)$$
$$= g(A^2\xi, Y)g(A\xi, X) - g(A^2\xi, X)g(A\xi, Y) + c\{g(A\xi, Y)\eta(X) - g(A\xi, X)\eta(Y)\}$$

for any vector fields X and Y, which together with the last equation gives $R_\xi A = AR_\xi$. According to Theorem CK, we conclude that our assertion. This completes the proof. □

Further, we now assume that

$$R_\xi S\xi = 0 \tag{3.8}$$

on M. Because of (3.1), we then have $R_\xi \phi S\xi = 0$, which together with (2.6) gives $R_\xi\phi(hA\xi - A^2\xi) = 0$. Thus, it follows that

$$R_\xi AU = (h - \lambda)R_\xi U \tag{3.9}$$

by virtue of (3.3). Because of (2.18) we can write (3.9) as

$$hAU - A^2U = (\lambda + \frac{c}{\alpha})AU - \frac{c}{\alpha}(h - \lambda)U \tag{3.10}$$

since $\alpha \neq 0$ on Ω. Applying this by ϕ and using (2.8) and (3.2), we find

$$\alpha\phi A^2U = \{\alpha(h - \lambda) - c\}(\lambda A\xi - A^2\xi) - c(h - \lambda)(A\xi - \alpha\xi).$$

If we combine this to (3.6), then we get

$$\alpha A^3\xi = (\alpha h - c)A^2\xi + (\gamma - \alpha h\lambda + ch)A\xi + c\alpha(\lambda - h)\xi. \tag{3.11}$$

where we have used (3.7), which tells us that

$$\alpha(hA^2\xi - A^3\xi) = cA^2\xi + (\alpha\lambda h - \gamma - ch)A\xi + c\alpha(h - \lambda)\xi. \tag{3.12}$$

Tranforming this by A and making use of (3.11), we have

$$\begin{aligned}\alpha(hA^3\xi - A^4\xi) = \{\lambda\alpha h - \gamma - \frac{c^2}{\alpha}\}A^2\xi \\ + c\{\frac{\gamma}{\alpha} - \lambda h + \frac{ch}{\alpha} + \alpha(h - \lambda)\}A\xi + c^2(\lambda - h)\xi.\end{aligned} \tag{3.13}$$

From (2.21) and (3.12) we get

$$\alpha SA\xi = cA^2\xi + \{c(2n+1)\alpha - \gamma + \alpha\lambda h - ch\}A\xi + c\alpha(h - \lambda - 3\alpha)\xi. \tag{3.14}$$

Combining (2.20) to (3.10), we find

$$SU = (\lambda + \frac{c}{\alpha})AU + \{c(2n+1) + \frac{c}{\alpha}(\lambda - h)\}U. \tag{3.15}$$

Now, we see from (2.22) and (3.12) that

$$\mu SW = (\alpha + \frac{c}{\alpha})A^2\xi + \{c(2n+1) + h(\lambda - \alpha) + \frac{1}{\alpha}(\gamma + ch)\}A\xi$$
$$+ \{c(h - \lambda) + c(2n+1)\alpha\}\xi,$$

which connected to (3.3) implies that

$$\mu\phi SW = (\alpha + \frac{c}{\alpha})AU + \{\alpha\lambda + \frac{c\lambda}{\alpha} + c(2n+1) + h(\lambda - \alpha) - \frac{1}{\alpha}(\gamma + ch)\}U. \tag{3.16}$$

Replacing X by W in (2.19), we find

$$\begin{aligned} g((\nabla_W R_\xi)Y, Z) = (W\alpha)g(AY, Z) - c\{\eta(Z)g(\phi AW, Y) + \eta(Y)g(\phi AW, Z)\} \\ + ag((\nabla_W A)Y, Z) - \eta(AZ)\{g((\nabla_W A)\xi, Y) + g(A\phi AW, Y)\} \\ - \eta(AY)\{g((\nabla_W A)\xi, Z) + g(A\phi AW, Z)\}. \end{aligned}$$

Now, suppose that $(\nabla_{\phi\nabla_\xi\xi})R_\xi = 0$. Then we have $(\nabla_W R_\xi)\xi = 0$. Putting $Y = \xi$ in the last relationship, and using (2.13), we then have

$$\alpha A\phi AW + c\phi AW = 0 \tag{3.17}$$

because of U and W are mutually orthogonal. Because of (3.4) we can write (3.17) as

$$\alpha A^2U + \{\alpha(\lambda - \alpha) + c\}AU + c(\lambda - \alpha)U = 0. \tag{3.18}$$

which together with (2.16) implies that $R_\xi AU + (\lambda - \alpha)R_\xi U = 0$. Combining this to (3.9), it follows that

$$(h - \alpha)R_\xi U = 0. \tag{3.19}$$

Using above discussions we can prove the following :

Theorem 3.2. *Let M be a real hypersurface of a complex space form $M_n(c)$, $c \neq 0$ which satisfies $R_\xi\phi = \phi R_\xi$ and at the same time $R_\xi S\xi = 0$. If $(\nabla_{\phi\nabla_\xi\xi}R_\xi)\xi = 0$, then M is the same type as those stated in Theorem 2.1 provided that the scalar curvature $\bar{r}$ of M is satisfied $\bar{r} - 4c(n^2 - 1) \geq 0$, where S denotes the Ricci tensor of M.*

Proof. $R_\xi U \neq 0$ on Ω, then we have $h - \alpha = 0$ on this open subset by virtue of (3.19). So we have

$$T_r({}^tAA) - h^2 = \|A - h\eta \otimes \xi\|^2$$

on the subset.

On the other hand, the scalar curvature $\bar{r}$ of M is, using (2.5), given by $\bar{r} = 4c(n^2-1) + h^2 - T_r({}^tAA)$. Thus, it follows that

$$\bar{r} - 4c(n^2-1) + \|A - h\eta \otimes \xi\|^2 = 0.$$

Hence, it follows that $AX = \alpha\eta(X)\xi$ for any vector field X because we assumed $\bar{r} - 4c(n^2-1) \geq 0$, which implies $AU = 0$ on the set. Thus, we have $R_\xi U = 0$ on M because of (3.18). Therefore we arrive at the conclusion by virtue of Theorem 2.1. This completes the proof. □

4. Real hypersurfaces satisfying $R_\xi\phi S = S\phi R_\xi$

Theorem 4.1. *Let M be a real hypersurface in a complex space form $M_n(c)$, $(c \neq 0, n \geq 2)$. If it satisfies $R_\xi\phi = \phi R_\xi$ and at the same time $R_\xi\phi S = S\phi R_\xi$, then M is the same type as that stated in Theorem 2.1, where S denotes the Ricci tensor of M.*

Proof. From the assumption

$$R_\xi\phi S = S\phi R_\xi, \tag{4.1}$$

we have $R_\xi\phi S\xi = 0$, which together with (2.6) gives $R_\xi\phi(hA\xi - A^2\xi) = 0$. Thus, we have (3.9) because of (3.3). Consequently (3.10) $\sim$ (3.16) are accomplished on Ω.

Now, from (3.10) we have

$$hA^2U - A^3U = (\lambda + \frac{c}{\alpha})A^2U - \frac{c}{\alpha}(h-\lambda)AU,$$

which together with (2.5) and (2.20) yields

$$SAU = ASU. \tag{4.2}$$

If we take account of (2.20) and (3.9), then we obtain

$$R_\xi SU = SR_\xi U, \tag{4.3}$$

where we have used (4.2), which together with (3.15) gives

$$R_\xi SU = \{(2n+1)c + \lambda(h-\lambda)\}R_\xi U. \tag{4.4}$$

On the other hand, putting $X = \mu W$ in $R_\xi\phi SX = S\phi R_\xi X$ and using (2.8), we have

$$\mu R_\xi\phi SW = SR_\xi U,$$

or, using (4.3) and (4.4),

$$\mu R_\xi\phi SW = \{c(2n+1) + \lambda(h-\lambda)\}R_\xi U. \tag{4.5}$$

If we use (3.9) and (3.16), then the left hand side of (4.5) is given by

$$\mu R_\xi \phi SW = \{c(2n+1) + h\lambda - \frac{\gamma}{\alpha}\} R_\xi U.$$

Combining the last two relationships, it follows that $(\gamma - \alpha\lambda^2)R_\xi U = 0$ and hence $g(AU, U)R_\xi U = 0$ by virtue of (3.7). According to Theorem 2.1, it follows that

$$g(AU, U) = \gamma - \alpha\lambda^2 = 0. \tag{4.6}$$

In the next place, from our assumption we have

$$R_\xi \phi S A^2\xi = S\phi R_\xi A^2\xi, \tag{4.7}$$

which together with the fact hat $R_\xi\phi = \phi R_\xi$ and (3.3) gives

$$R_\xi \phi S A^2\xi = SR_\xi(AU + \lambda U),$$

or using (3.9), (4.3) and (4.4)

$$R_\xi \phi S A^2\xi = \{(2n+1)ch + h\lambda(h-\lambda)\}R_\xi U. \tag{4.8}$$

By the way, using (4.6) we can write (3.14) as

$$\alpha(hA^3\xi - A^4\xi) = (\lambda\alpha h - \alpha\lambda^2 - \frac{c^2}{\alpha})A^2\xi + c(\lambda^2 - \lambda h + \frac{ch}{\alpha} + \alpha(h-\lambda))A\xi + c^2(\lambda - h)\xi,$$

which together with (2.5) yields

$$\phi SA^2\xi = c(2n+1)\phi A^2\xi + (\lambda h - \lambda^2 - (\frac{c}{\alpha})^2)\phi A^2\xi + \{\frac{c}{\alpha}(\lambda^2 - \lambda h + \frac{c}{\alpha}h) + c(h-\lambda)\}U.$$

If we use (3.3) and (3.9) to this, then we obtain

$$R_\xi \phi S A^2\xi = \{c(2n+1)h + h^2\lambda - h\lambda^2 + \frac{c}{\alpha}(\lambda^2 - \lambda h) + c(h-\lambda)\}R_\xi U.$$

Comparing this with (4.8), we obtain $(h-\lambda)(\alpha-\lambda) = 0$, where we used Theorem 2.1, which enables us to obtain $h - \lambda = 0$ because $\alpha - \lambda \neq 0$ on Ω. According to (3.9), we obtain $R_\xi AU = 0$, that is, $\alpha A^2U + cAU = 0$ because of the second equation of (2.18). Hence we have $g(A^2U, U) = 0$ by virtue of (4.6) and thus $AU = 0$. So (3.2) becomes $A^2\xi = \lambda A\xi$. Differentiating this covariantly along Ω and taking account of (2.1), we find

$$\begin{aligned} &g((\nabla_X A)A\xi, Y) + g(A(\nabla_X A)\xi, Y) + g(A^2\phi AX, Y) \\ &\quad = (X\lambda)g(A\xi, Y) + \lambda g((\nabla_X A)\xi, Y) + \lambda g(A\phi AX, Y), \end{aligned} \tag{4.9}$$

which together with (2.13) and the fact that $AU = 0$ implies that

$$2g((\nabla_X A)\xi, A\xi) = \lambda(X\alpha) + \alpha(X\lambda),$$

or, using (2.4),

$$(\nabla_\xi A)A\xi = \frac{1}{2}\nabla\beta - cU.$$

Replacing X by ξ in (4.9) and making use of (2.13) and the fact that $AU = 0$, we find

$$\frac{1}{2}\nabla\beta = -A\nabla\alpha + \lambda\nabla\alpha + (\xi\lambda)A\xi + cU,$$

where we have used the last relationship. If we take the inner product with U to this, then we obtain

$$\frac{1}{2}U\beta = \lambda(U\alpha) + c\mu^2, \tag{4.10}$$

which shows that

$$\alpha(U\lambda) - \lambda(U\alpha) = 2c\mu^2 \tag{4.11}$$

by virtue of $\beta = \alpha\lambda$.

On the other hand, if we put $X = A\xi$ in (4.9) and make use of (2.4), (2.7) and (2.13), then we get

$$\frac{1}{2}(A\nabla\beta - \lambda\nabla\beta) + (\alpha^2 + \mu^2)\nabla\lambda = g(A\xi, \nabla\lambda)A\xi + c(3\alpha - 2\lambda)U.$$

If we take the inner product with U to this and make use of (4.10) and the fact that $AU = 0$, then $\lambda\{\alpha(U\lambda) - \lambda(U\alpha)\} = c(3\alpha - \lambda)\mu^2$, which together with (4.11) gives $c(\lambda - \alpha)\mu^2 = 0$, a contradiction. Thus, Ω is empty set. that is, M is a Hopf hypersurface. So α is constant (see, [12]). From (3.1) we have $\alpha(A\phi - \phi A) = 0$ and hence $A\xi = 0$ or $A\phi = \phi A$. Owing to Theorem O-MR, we arrive at the conclusion. This completes the proof.

□

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