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SIMPLIFIED APPROACH TO VALUATION OF VULNERABLE EXCHANGE OPTION UNDER A REDUCED-FORM MODEL

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Abstract. In this paper, we investigate the valuation of vulnerable exchange option that has credit risk of option issuer. The reduced-form model is used to model credit risk. We assume that credit event is determined by the jump of the counting process with stochastic intensity, which follows the mean reverting process. We propose a simple approach to derive the closed-form pricing formula of vulnerable exchange option under the reduced-form model and provide the pricing formula as the standard normal cumulative function.

1. Introduction

Exchnge option is the most popular exotic option with two underlying assets in the over-the-counter (OTC) market. Since Margrabe [10] first introduce the pricing of exchange option under the Black-Scholes model, there have been various studies on the valuation of exchange option. Geman, Karoui, and Rochet [4] dealt with the pricing formula of exchange option based on the change of numéraire method. Antonelli, Ramponi, and Scarlatti [1] studied the valuation of exchange option under the stochastic volatility model. In addition, Cheang and Chiarella [2] considered the jump-diffusion model to price an exchange option. Kim and Park [7] adopted PDE approach to extend the Margrabe formula with the stochastic volatility model. In this paper, we extend the exchange option pricing model considering credit risk in OTC market.

In general, credit risk is modeled as two types: the reduced-form approach (intensity-based) and the structural approach (firm value-based). Under the reduced-form model, credit event occurs according to a Poisson process with default intensity. That is, the event is triggered by the first jump of the process. On the other hand, the structural model depends on a firm value process of

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option issuer. Under the structural model, credit event occurs if the firm value process stays under a value of the option issuer's liability at maturity. Between the two models, we consider the reduce-form model for modeling credit risk.

Exchange option with credit risk that have been called vulnerable exchange option has been considered under the structural model. Kim and Koo [5] used a partial differential equation (PDE) approach to obtain the pricing formula of vulnerable exchange option and Kim [6] derived the formula of option price based on a probabilistic approach. In this paper, we deal with the valuation of vulnerable exchange option under a reduced-form model. Fard [3] provided the analytic pricing formula of vulnerable european option with a generalized jump model under the reduce-form model. Fard [3] used the method of changing measures via the Esscher transform to obtain the formula. Motivated by the work of [3], we also study the valuation of vulnerable option under the reduced-form model and extend the vulnerable exchange option pricing model. Unlike the approach of [3], we propose a more simple approach without measure change. Finally, we provide the closed-form pricing formula of vulnerable exchange option based on the proposed method and contribute an efficient approach for valuing of vulnerable options under the reduced form model.

The rest of this paper is organized as follows. In section 2, we introduce the model to describe vulnerable exchange option under a reduced-form model. In section 3, we propose our simple approach to drive efficiently the price of vulnerable exchange option and provide the closed-form pricing formula of vulnerable exchange option without any change of measure. Section 4 presents concluding remarks.

2. Model

We assume that the uncertainty in economy is described by a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, Q)$ with a filtration $\{\mathcal{F}_t\}$ satisfying the usual conditions, where Q is a risk-neutral martingale measure. Under the measure Q, the dynamics of two underlying assets are given by

$$\begin{aligned} dS_1(t) &= rS_1(t)dt + \sigma_1 S_1(t)dW_1(t), \\ dS_2(t) &= rS_2(t)dt + \sigma_2 S_2(t)dW_2(t), \end{aligned}$$

where σ_1, σ_2 and r are positive constants and $W_1(t)$ and $W_2(t)$ are the standard Brownian motions in the probability space. To define the default of option issuer under the reduced-form model, we assume that the dynamic of default intensity is given by

$$d\lambda(t) = a(b - \lambda(t))dt + \sigma_3 dW_3(t),$$

where σ_3 is a positive constant and $W_3(t)$ is a standard Brownian motion. Let τ be the default time, then the distribution of τ is defined by $\mathrm{P}(\tau > t) = \mathbb{E}\left[e^{-\int_0^t \lambda(s)ds}\right]$. We also assume that dependent standard Brownian motions

$(W_1(t), W_2(t), W_3(t))$ under the measure Q have the covariance matrix

$$\begin{pmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{pmatrix} t.$$

From these assumptions, vulnerable exchange option price C at time 0 with maturity T under the reduced-form model is defined as

$$C = e^{-rT}\mathbb{E}^Q\left[w(S_1(T) - S_2(T))^+\mathbf{1}_{\{\tau\le T\}} + (S_1(T) - S_2(T))^+\mathbf{1}_{\{\tau>T\}}|\mathcal{F}_0\right], \quad (1)$$

where w is a recovery rate. As in Lando [9], Eq. (1) can be decomposed into the following

$$\begin{aligned} C &= we^{-rT}\mathbb{E}^Q\left[(S_1(T) - S_2(T))^+|\mathcal{F}_0\right] \\ &\quad +(1-w)\mathbf{1}_{\{\tau>0\}}e^{-rT}\mathbb{E}^Q\left[e^{-\int_0^t \lambda(s)ds}(S_1(T) - S_2(T))^+|\mathcal{F}_0\right]. \quad (2) \end{aligned}$$

3. Valuation of vulnerable exchange option

In this section, we propose a simplified approach for valuation of vulnerable exchange option. Firstly, for the simplification of the price C, we rewrite Eq. (2) as

$$C = we^{-rT}I_1 + (1-w)e^{-rT}I_2. \quad (3)$$

In Eq. (3), we can obtain easily the formula for I_1 because I_1 is Margrabe's formula for exchange option in [10]. Therefore, the calculation of I_2 completes the valuation of vulnerable exchange option under the reduced-form model. We now introduce a key lemma to calculate I_2.

Lemma 3.1. *Suppose that X and Y are normal variables with parameters $(0, \sigma_X^2)$,$(0, \sigma_Y^2)$ and correlation ρ. Then, for some constant C,*

$$\mathbb{E}[e^X 1_{\{Y\ge C\}}] = e^{\frac{\sigma_X^2}{2}}\Phi\left(\frac{Cov(X,Y) - C}{\sigma_Y}\right),$$

where Φ is the cumulative standard normal distribution function and $Cov(X,Y)$ is the covariance between X and Y.

Proof. If we consider the conditional distribution of X given Y, then the random variable X is normally distributed with mean $\rho\frac{\sigma_X}{\sigma_Y}Y$ and variance $\sigma_X^2(1-\rho^2)$. By the moment generating function of normal variable, we have

$$\begin{aligned} \mathbb{E}[e^X 1_{\{Y\ge C\}}] &= \mathbb{E}[\mathbb{E}[e^X|Y]1_{\{Y\ge C\}}] \\ &= \mathbb{E}[e^{\rho\frac{\sigma_X}{\sigma_Y}Y+\frac{1}{2}\sigma_X^2(1-\rho^2)}1_{\{Y\ge C\}}] \\ &= e^{\frac{1}{2}\sigma_X^2(1-\rho^2)}\mathbb{E}[e^{\rho\frac{\sigma_X}{\sigma_Y}Y}1_{\{Y\ge C\}}], \end{aligned}$$

where C is some constant. Let $Y = \sigma_Y Z$, then

$$\begin{aligned}\mathbb{E}[e^{\rho\frac{\sigma_X}{\sigma_Y}Y}1_{\{Y\geq C\}}] &= \mathbb{E}[e^{\rho\sigma_X Z}1_{\{Z\geq\frac{C}{\sigma_Y}\}}] \\ &= \int_{\frac{C}{\sigma_Y}}^{\infty}\frac{e^{\rho\sigma_X Z}}{\sqrt{2\pi}}e^{-\frac{1}{2}Z^2}dZ \\ &= e^{\frac{1}{2}\rho^2\sigma_X^2}\int_{\frac{C}{\sigma_Y}}^{\infty}\frac{e^{-\frac{1}{2}(Z-\rho\sigma_X)^2}}{\sqrt{2\pi}}dZ.\end{aligned}$$

Let $Z - \rho\sigma_1 = t$, then

$$\mathbb{E}[e^X 1_{\{Y\geq C\}}] = e^{\frac{\sigma_X^2}{2}}\int_{\frac{C}{\sigma_Y}-\rho\sigma_X}^{\infty}\frac{e^{-\frac{1}{2}t^2}}{\sqrt{2\pi}}dt.$$

Since $Cov(X, Y) = \rho\sigma_X\sigma_Y$, the proof is completed. $\square$

From Lemma 3.1, we can obtain I_2 in Eq. (3).

Lemma 3.2. *Under the measure Q, I_2 is given by*

$$\begin{aligned}I_2 &= \mathbb{E}^Q\left[e^{-\int_0^t\lambda(s)ds}(S_1(T)-S_2(T))^+|\mathcal{F}_0\right] \\ &= S_1(0)e^{(r-b)T-\frac{\lambda(0)-b}{a}(1-e^{-aT})-\frac{\sigma_1\sigma_3\rho_{13}}{a}M_1+\frac{\sigma_3^2}{2a^2}M_2}\Phi\left(\frac{\Lambda_1-\frac{1}{2}(\sigma_1^2-\sigma_2^2)T}{\sigma\sqrt{T}}\right) \\ &\quad -S_2(0)e^{(r-b)T-\frac{\lambda(0)-b}{a}(1-e^{-aT})-\frac{\sigma_2\sigma_3\rho_{23}}{a}M_1+\frac{\sigma_3^2}{2a^2}M_2}\Phi\left(\frac{\Lambda_2-\frac{1}{2}(\sigma_1^2-\sigma_2^2)T}{\sigma\sqrt{T}}\right)\end{aligned}$$

where

$$\begin{aligned}&M_1 = \int_0^T(1-e^{-a(T-s)})ds, \\ &M_2 = \int_0^T(1-e^{-a(T-s)})^2ds, \\ &\Lambda_1 = (\sigma_1^2-\sigma_1\sigma_2\rho_{12})T+\left(\frac{\sigma_2\sigma_3\rho_{23}}{a}-\frac{\sigma_1\sigma_3\rho_{13}}{a}\right)M_1, \\ &\Lambda_2 = (\sigma_1\sigma_2\rho_{12}-\sigma_2^2)T+\left(\frac{\sigma_2\sigma_3\rho_{23}}{a}-\frac{\sigma_1\sigma_3\rho_{13}}{a}\right)M_1, \\ &\sigma^2 = \sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho_{12}.\end{aligned}$$

Proof. Firstly, we divide I_2 into two terms as follows.

$$\begin{aligned}I_2 &= \mathbb{E}^Q\left[e^{-\int_0^t\lambda(s)ds}S_1(T)\mathbf{1}_{\{S_1(T)>S_2(T)\}}|\mathcal{F}_0\right] \\ &\quad -\mathbb{E}^Q\left[e^{-\int_0^t\lambda(s)ds}S_2(T)\mathbf{1}_{\{S_1(T)>S_2(T)\}}|\mathcal{F}_0\right] \\ &:= J_1 - J_2 \qquad (4)\end{aligned}$$

Let us consider J_1. Since

$$\int_0^T \lambda(s)ds = bT + \frac{\lambda(0)-b}{a}(1-e^{-aT}) + \int_0^T \frac{\sigma_3}{a}(1-e^{-a(T-s)})dW_3(s)$$

and $S_1(T) = S_1(0)e^{(r-\frac{1}{2}\sigma_1^2)T+\sigma_1 W_1(T)}$, J_1 can be written as

$$\begin{aligned} J_2 &= S_1(0)e^{(r-b-\frac{1}{2}\sigma_1^2)T-\frac{\lambda(0)-b}{a}(1-e^{-aT})} \\ &\quad \times \mathbb{E}\left[e^{\sigma_1 W_1(T)-\frac{\sigma_3}{a}\int_0^T(1-e^{-a(T-s)})dW_3(s)}\mathbf{1}_{\{S_1(T)>S_2(T)\}}|\mathcal{F}_0\right]. \end{aligned} \tag{5}$$

By Ito isometry and the properties of Brownian motion, we can find that $\sigma_1 W_1(T)$ and $\frac{\sigma_3}{a}\int_0^T(1-e^{-a(T-s)})dW_3(s)$ are normal variables and satisfy the following

$$\begin{aligned} Z_1 &:= \sigma_1 W_1(T) \sim \mathrm{N}\left(0, \sigma_1^2 T\right), \\ Z_2 &:= \frac{\sigma_3}{a}\int_0^T (1-e^{-a(T-s)})dW_3(s) \sim \mathrm{N}\left(0, \frac{\sigma_3^2}{a^2}\int_0^T(1-e^{-a(T-s)})^2 ds\right). \end{aligned}$$

Moreover, let $Z_3 := \sigma_2 W_2(T)$, $\mathbb{E}[\mathbf{1}_{\{S_1(T)>S_2(T)\}}|\mathcal{F}_0] = \mathrm{P}(S_1(T) > S_2(T))$ is represented by

$$\mathrm{P}(S_1(T) > S_2(T)) = \mathrm{P}\left(Z_1 - Z_3 > \frac{1}{2}(\sigma_1^2 - \sigma_2^2)T\right). \tag{6}$$

The variables $Z_1 - Z_2$ and $Z_1 - Z_3$ are obviously normal variables and satisfy the following

$$\begin{aligned} X_1 &:= Z_1 - Z_2 \\ &\sim \mathrm{N}\left(0, \sigma_1^2 T + \frac{\sigma_3^2}{a^2}\int_0^T (1-e^{-a(T-s)})^2 ds - 2\frac{\sigma_1\sigma_3\rho_{13}}{a}\int_0^T(1-e^{-a(T-s)})ds\right), \\ X_2 &:= Z_1 - Z_3 \sim \mathrm{N}\left(0, (\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho_{12})T\right). \end{aligned}$$

Then, J_2 is simplified as

$$J_2 = S_1(0)e^{(r-b-\frac{1}{2}\sigma_1^2)T-\frac{\lambda(0)-b}{a}(1-e^{-aT})}\mathbb{E}\left[e^{X_1}\mathbf{1}_{\{X_2>K\}}|\mathcal{F}_0\right], \tag{7}$$

where $K = \frac{1}{2}(\sigma_1^2 - \sigma_2^2)T$. Since the covariance between X_1 and X_2 is

$$\begin{aligned} Cov(X_1, X_2) &= \sigma_1^2 T - \sigma_1\sigma_2\rho_{12}T + \frac{\sigma_2\sigma_3\rho_{23}}{a}\int_0^T(1-e^{-a(T-s)})ds \\ &\quad - \frac{\sigma_1\sigma_3\rho_{13}}{a}\int_0^T(1-e^{-a(T-s)})ds. \end{aligned} \tag{8}$$

Now, we can apply Lemma 3.1 directly. Then, we have

$$\begin{aligned} J_2 &= S_1(0)e^{(r-b)T-\frac{\lambda(0)-b}{a}(1-e^{-aT})+\frac{\sigma_3^2}{2a^2}\int_0^T(1-e^{-a(T-s)})^2ds-\frac{\sigma_1\sigma_3\rho_{13}}{a}\int_0^T(1-e^{-a(T-s)})ds} \\ &\quad \times\Phi\left(\frac{Cov(X_1,X_2)-\frac{1}{2}(\sigma_1^2-\sigma_2^2)T}{\sqrt{(\sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho_{12})T}}\right), \end{aligned} \tag{9}$$

where $Cov(X_1, X_2)$ is defined in Eq. (8). In a similar way, we can derive J_2 in Eq. (4). □

Combining the result of Lemma 3.2 and Margrabe's formula, we can obtain the price of vulnerable exchange option under the reduced-form model. The result is presented in the following theorem.

Theorem 3.3. *The price of vulnerable exchange option at time 0 under the reduced-form model is given by*

$$\begin{aligned} C &= wS_1(0)\Phi\left(\frac{\ln\left(\frac{S_1(0)}{S_2(0)}\right)+\frac{1}{2}\sigma^2}{\sigma\sqrt{T}}\right) - wS_2(0)\Phi\left(\frac{\ln\left(\frac{S_1(0)}{S_2(0)}\right)-\frac{1}{2}\sigma^2}{\sigma\sqrt{T}}\right) \\ &\quad +(1-w)S_1(0)e^{-bT-\frac{\lambda(0)-b}{a}(1-e^{-aT})-\frac{\sigma_1\sigma_3\rho_{13}}{a}M_1+\frac{\sigma_3^2}{2a^2}M_2}\Phi\left(\frac{\Lambda_1-\frac{1}{2}(\sigma_1^2-\sigma_2^2)T}{\sigma\sqrt{T}}\right) \\ &\quad -(1-w)S_2(0)e^{-bT-\frac{\lambda(0)-b}{a}(1-e^{-aT})-\frac{\sigma_2\sigma_3\rho_{23}}{a}M_1+\frac{\sigma_3^2}{2a^2}M_2}\Phi\left(\frac{\Lambda_2-\frac{1}{2}(\sigma_1^2-\sigma_2^2)T}{\sigma\sqrt{T}}\right), \end{aligned}$$

where all parameters and notations are defined in Lemmas 3.2.

Remark 1. The proposed approach applies easily to vulnerable european option pricing under the reduced-form model which has the default intensity process with mean-reverting.

4. Concluding remarks

Since exchange option is one of the popular exotic options in the OTC market, many researchers have studied on pricing for exchange option. In this paper, we study the valuation of exchange option with credit risk. Specifically, we derive closed-form pricing formula for vulnerable exchange option, which has credit risk of option issuer. Credit risk is modeled as the reduced-form model of Fard [3]. Finally, we propose a simple approach based on the variable properties to obtain the formula and present the pricing formula with the standard normal cumulative distribution function. Our approach will surely be efficient not only for the pricing of vulnerable exchange option but also for the pricing of vulnerable european option.

References

[1] F. Antonelli , A. Ramponi, S. Scarlatti, *Exchange option pricing under stochastic volatility: a correlation expansion*, Rev. Deriv. Res., 13(1) (2010), 45-73.

[2] G.H.L. Cheang, C. Chiarella, *Exchange options under jump-Diffusion dynamics*, Appl. Math. Financ., 18(3) (2011), 245-76.

[3] F.A. Fard, *Analytical pricing of vulnerable options under a generalized jumpdiffusion model*, Insur. Math. Econ., 60 (2015), 19-28.

[4] H. Geman, N. Karoui, J.C. Rochet, *Changes of Numéraire, Changes of Probability Measure and Option Pricing*, J. Appl. Probab., 32(2) (1995), 443-58.

[5] G. Kim, E. Koo, *Closed-form pricing formula for exchange option with credit risk*, Chaos Soliton Fract., 91 (2016), 221-227.

[6] G. Kim *A Probabilistic Approach for Valuing Exchange Option with Default Risk*, East Asian Math. J., 36(1) (2020), 55-60.
[7] J.-H. Kim, C.R. Park, *A multiscale extension of the Margrabe formula under stochastic volatility*, Chaos Soliton Fract., 97 (2017), 59-65.
[8] P. Klein, *Pricing Black-Scholes options with correlated credit risk*, J. Bank. Financ., 50 (1996), 1211-1229.
[9] D.Lando, *On Cox processes and credit risky securities*, Rev. Deriv. Res., 2 (1998), 99-120.
[10] W. Margrabe, *The value of an option to exchange one asset for another*, J. Financ., 33(1) (1978), 177-86.

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