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INSERTION PROPERTY BY ESSENTIAL IDEALS

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Abstract. We discuss the condition that if $ab = 0$ for elements a, b in a ring R then $aIb = 0$ for some essential ideal I of R. A ring with such condition is called *IEIP*. We prove that a ring R is IEIP if and only if $D_n(R)$ is IEIP for every $n \geq 2$, where $D_n(R)$ is the ring of n by n upper triangular matrices over R whose diagonals are equal. We construct an IEIP ring that is not Abelian and show that a well-known Abelian ring is not IEIP, noting that rings with the insertion-of-factors-property are Abelian.

1. Introduction

Throughout this note every ring is an associative ring with identity unless otherwise stated. Let R be a ring. We denote the center, the group of all units, and the set of all idempotents of R by $U(R)$, $I(R)$, respectively. We use $N(R)$, $J(R)$, $N_*(R)$, $N^*(R)$ and $W(R)$ to denote the set of all nilpotent elements, Jacobson radical, lower nilradical (i.e., prime radical), upper nilradical (i.e., the sum of all nil ideals) and the Wedderburn radical (i.e., the sum of all nilpotent ideals) of R, respectively. It is well-known that $W(R) \subseteq N_*(R) \subseteq N^*(R) \subseteq N(R)$ and $N^*(R) \subseteq J(R)$. The polynomial ring with an indeterminate x over R is denoted by $R[x]$. $\mathbb{Z}$ ($\mathbb{Z}_n$) denotes the ring of integers (modulo n). Denote the n by n ($n \geq 2$) full (resp., upper triangular) matrix ring over R by $Mat_n(R)$ (resp., $T_n(R)$). Write $D_n(R) = \{(a_{ij}) \in T_n(R) \mid a_{11} = \cdots = a_{nn}\}$ and $N_n(R) = \{(a_{ij}) \in T_n(R) \mid a_{ii} = 0 \text{ for all } i\}$. Use E_{ij} for the matrix with (i, j)-entry 1 and zeros elsewhere. I_n denotes the identity matrix in $Mat_n(R)$.

Due to Bell [2], a ring R (possibly without identity) is called *IFP* if $ab = 0$ for $a, b \in R$ implies $aRb = 0$, i.e., R satisfies the insertion-of-factors-property. It is easily checked that if R is an IFP ring then $W(R) = N_*(R) = N^*(R) = N(R)$. Following the literature, a ring (possibly without identity) is called *reduced* if it contains no nonzero nilpotent elements; and a ring (possibly without identity)

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is called *Abelian* if every idempotent is central. It is easily proved that reduced rings are IFP and IFP rings are Abelian. We will freely use the preceding facts.

2. IEIP rings

In this section we introduce a new condition that is related to the inserting property by essential ideals, and show that the class of rings satisfying this condition is quite large. An ideal I of a ring R is called *essential* in R if $I \cap I' \neq 0$ for each nonzero ideal I' of R. A ring R (possibly without identity) shall be said to satisfy the *insertion property by an essential ideal* (simply, is said to be *IEIP*) provided that if $ab = 0$ for $a, b \in R$ then $aIb = 0$ for some essential ideal I of R. IFP rings are clearly IEIP, but the converse need not hold as the following argument shows.

Lemma 2.1. (1) *Let R be a ring and $n \geq 2$. If J is an ideal of $D_n(R)$ then*

$$J \subseteq \{(a_{ij}) | a_{ij} \in I_{ij}\} \text{ for some ideals } I_{ij} \text{ of } R,$$

where $1 \leq i, j \leq n$ and $i \leq j$.

(2) *Let R be a ring and $n \geq 2$. If J is a nonzero ideal of $D_n(R)$ then J contains IE_{1n} for some nonzero ideal I of R.*

(3) *Let R be a ring and $n \geq 2$. If I is an essential ideal of R then IE_{1n} is an essential ideal of $D_n(R)$.*

(4) *Let R be a ring and $n \geq 2$. If J is an essential ideal of $D_n(R)$ then J contains PE_{1n} for some essential ideal P of R.*

Proof. (1) Let J be a nonzero ideal of $D_n(R)$ and define

$$I_{ij} = \{a_{ij} \mid (a_{ij}) \in J\} \subseteq R$$

for $1 \leq i, j \leq n$ with $i \leq j$. Then since J is an ideal of $D_n(R)$, we see that I_{ij} is also an ideal of R through the computation that $(a_{ij} - b_{ij}) = (a_{ij}) - (b_{ij}) \in J$ and $[rI_n](a_{ij}), (a_{ij})[rI_n] \in J$ for $(a_{ij}), (b_{ij}) \in J$ and $r \in R$. Evidently $J \subseteq \{(a_{ij}) | a_{ij} \in I_{ij}\}$.

(2) Let J be a nonzero ideal of $D_n(R)$ and take $0 \neq (a_{ij}) \in J$. If $a_{ii} \neq 0$ then $0 \neq a_{ii}E_{1n} = (a_{ij})E_{1n} \in J$ and hence J contains the nonzero ideal

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & Ra_{ii}R \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

of $D_n(R)$. Suppose $a_{ii} = 0$. Then $a_{st} \neq 0$ for some s, t with $1 < s < t < n$. Then $a_{st}E_{1n} = E_{1s}(a_{ij})E_{tn} \in J$ and hence J contains the nonzero ideal

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & Ra_{st}R \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

of $D_n(R)$.

(3) Let I be an essential ideal of R and set $J = IE_{in}$. Let K be a nonzero ideal of $D_n(R)$. Then, by (2), K contains $I'E_{1n}$ for some nonzero ideal I' of R. Since I is essential, $I \cap I' \neq 0$ and hence we have $0 \neq (I \cap I')E_{1n} = IE_{1n} \cap I'E_{1n} \subseteq J \cap K$. Thus J is essential in $D_n(R)$.

(4) Let J be an essential ideal of $D_n(R)$. Then J is nonzero, and so J contains IE_{1n} for some nonzero ideal I of R by (2). Note that if $aE_{1n} \in J$ then $[RaR]E_{1n} \subseteq J$, hence the following argument is possible. Let I_t $(t \in T)$ be a nonzero ideal of R such that I_tE_{1n} is contained in J, and define

$$I_{\max} = \sum_{t \in T} I_t.$$

Then J contains $I_{\max}E_{1n}$. Write $P = I_{\max}$.

Now since J is essential, we see that $J \cap I'E_{1n} \neq 0$ for every nonzero ideal I' of R. This implies

$$0 \neq J \cap I'E_{1n} = PE_{1n} \cap I'E_{1n} = \begin{pmatrix} 0 & 0 & \cdots & 0 & P \cap I' \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

entailing $P \cap I' \neq 0$. Thus P is an essential ideal of R. □

We now obtain the following by help of Lemma 2.1.

Theorem 2.2. *A ring R is IEIP if and only if $D_n(R)$ is IEIP for every $n \geq 2$.*

Proof. Let $n \geq 2$. Let R be an IEIP ring. Suppose that $AB = 0$ for $A = (a_{ij}), B = (b_{ij}) \in D_n(R) \backslash \{0\}$. Then $a_{ii}b_{ii} = 0$.

If $A \in N_n(R)$ or $B \in N_n(R)$, then $A[RE_{1n}]B = 0$. Note that RE_{1n} is an essential ideal of D by Lemma 2.1(3).

Let $A, B \notin N_n(R)$ (i.e., $a_{ii}, b_{ii} \neq 0$). Since R is IEIP and $a_{ii}b_{ii} = 0$, there exists an essential ideal I of R such that $a_{ii}Ib_{ii} = 0$. Set $J = IE_{1n}$. Then J is an essential ideal of $D_n(R)$ by Lemma 2.1(3). Furthermore, we obtain

$$AJB = \begin{pmatrix} 0 & 0 & \cdots & 0 & a_{ii}Ib_{ii} \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} = 0.$$

Therefore $D_n(R)$ is IEIP.

Conversely let $D_n(R)$ be IEIP and suppose that $cd = 0$ for $c, d \in R$. Consider two matrices $C = aI_n$ and $D = bI_n$ in $D_n(R)$. Then $CD = 0$. Since $D_n(R)$ is IEIP, $CJD = 0$ for some essential ideal J of $D_n(R)$. Then J contains PE_{1n} for some essential ideal P of R by Lemma 2.1(3). This yields

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & cPd \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} = C[PE_{1n}]D \subseteq CJD = 0,$$

entailing that $cPd = 0$ for an essential ideal P of R. Thus R is IEIP. □

Let R be an IEIP ring and consider $D_n(R)$ for $n \geq 4$. Then $D_n(R)$ is IEIP by Theorem 2.2, however $D_n(R)$ cannot be IFP by [8, Example 1.3].

Let R be an algebra (possibly without identity) over $\mathbb{Z}$. Following Dorroh [3], the *Dorroh extension* of R by S is the Abelian group $R \times S$ with multiplication given by $(r_1, s_1)(r_2, s_2) = (r_1r_2 + s_1r_2 + s_2r_1, s_1s_2)$, where $r_i \in R$ and $s_i \in S$.

Proposition 2.3. (1) *Let R be a nil algebra over a field K. If R is IEIP then so is the Dorroh extension of R by K.*

(2) *Let R be an algebra with identity over a commutative domain S. Suppose that every nonzero ideal of the Dorroh extension of R by S contains (c, m) with $c + m \neq 0$. If R is IEIP then so is the Dorroh extension of R by S.*

Proof. (1) Let E be the Dorroh extension of R by K and suppose that $\alpha\beta = 0$ for $0 \neq \alpha, \beta \in D$. Then $\alpha = (a, 0), \beta = (b, 0)$ for some $0 \neq a, b \in R$ because $(r, u) \in U(D)$ when $u \neq 0$. So $ab = 0$. Since R is IEIP, there exists an essential ideal I of R such that $aIb = 0$. Let J be a proper nonzero ideal of E. Then J contains $(c, 0)$ with $0 \neq c \in R$, hence we have $(RcR, 0) = (R, 0)(c, 0)(R, 0) \subseteq J$. Since I is essential in R, we see $RcR \cap I \neq 0$, from which we infer that

$$0 \neq (RcR \cap I, 0) = (RcR, 0) \cap (I, 0) \subseteq J \cap (I, 0).$$

Thus $(I, 0)$ is also an essential ideal of E. Now we have $\alpha(I, 0)\beta = (a, 0)(I, 0)(b, 0) = (aIb, 0) = 0$. Thus E is IEIP.

(2) Let E' be the Dorroh extension of R by S and suppose that $\alpha\beta = 0$ for $0 \neq \alpha = (a, m), \beta = (b, n) \in E'$. Then $m = 0$ or $n = 0$. Let $n = 0$. Then $ab + mb = 0$, hence $(a + m)b = 0$ because S can be considered as a subring of R. Since R is IEIP, there exists an essential ideal I of R such that $(a + m)Ib = 0$. Let J be a proper nonzero ideal of E'. Then, by hypothesis, J contains (c, m) with $c + m \neq 0$. Note $(c, m)(1, 0) = (c + m, 0) \in J$, hence J contains a nonzero ideal $(R(c+m)R, 0)$ of E' through $(R, 0)(c+m, 0)(R, 0) \subseteq J$. Since I is essential in R, we see $R(c + m)R \cap I \neq 0$, from which we infer that

$$0 \neq (R(c + m)R \cap I, 0) = (R(c + m)R, 0) \cap (I, 0) \subseteq J \cap (I, 0).$$

Thus $(I, 0)$ is also an essential ideal of E'. Now we have

$$\alpha(I, 0)\beta = (a, m)(I, 0)(b, 0) = ((a + m)Ib, 0) = 0.$$

Next let $m = 0$. Then $ab + an = 0$, hence $a(b + n) = 0$. The argument is similar to the preceding case, but we write it for completeness. Since R is IEIP, there exists an essential ideal I' of R such that $aI'(b+n) = 0$. By the same manner as above, $(I', 0)$ is also an essential ideal of E'. Now we have

$$\alpha(I', 0)\beta = (a, 0)(I, 0)(b, n) = (aI'(b + n), 0) = 0.$$

Therefore E' is IEIP. □

Consider the Dorroh extensions of $\mathbb{Z}[x]$ and $D_n(\mathbb{Z})$ (for $n \geq 2$) by $\mathbb{Z}$. Then they satisfy the hypothesis of Proposition 2.3 (the proof for $\mathbb{Z}[x]$ is evident, and refer to Lemma 2.1(2) for $D_n(\mathbb{Z})$.) Thus the Dorroh extensions are IEIP by Proposition 2.3 because $\mathbb{Z}[x]$ is a domain (hence IEIP), and $D_n(\mathbb{Z})$ is IEIP by Theorem 2.2.

3. Relationships between IEIP rings and near concepts

In this section we investigate the properties of IEIP rings which are useful in the study of zero divisors. Recall that IFP rings are Abelian. We see that IEIP is another generalization of IFP, by showing that IEIP and Abelian are independent of each other.

Example 3.1. (1) There exists an IEIP ring that is not Abelian. Let $A = \mathbb{Z}_2\langle a, b\rangle$ be the free algebra generated by noncommuting indeterminates a, b over $\mathbb{Z}_2$. Consider the ideal H of A generated by

$$b^2 - b, \ a^2 \ \text{ and } \ bab.$$

Next set $R = A/H$. Identity elements in A with their images in R for simplicity. Then since $a^2 = 0$ and $bab = 0$, we see $(RaR)^3 = 0$. Observing $R/(RaR) \cong \mathbb{Z}_2 + \mathbb{Z}_2 b = \{0, 1, b, 1+b\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, we see $RaR = W(R) = N_*(R) = N^*(R) = N(R)$. Moreover $b^2 = b$ (i.e., $b \in I(R)$) and $ab \neq ba$, so that R is not Abelian. Note that $1 + f \in U(R)$ for $f \in RaR$.

Every element of R is expressed by

$$h_0 + h_1 b + h_2 a + h_3 ba + h_4 ab + h_5 aba,$$

where $h_i \in \mathbb{Z}_2$. Consider the ideal $RabaR$ and any nonzero ideal J of R. Take $0 \neq f = h_0 + h_1 b + h_2 a + h_3 ba + h_4 ab + h_5 aba \in J$. If $h_0 = 1$ then J contains $faba = aba$. If $h_0 = 0$ and $h_1 = 1$ then J contains $afa = aba$.

Next let $h_0 = h_1 = 0$, i.e., $f = h_2 a + h_3 ba + h_4 ab + h_5 aba$. If $h_3 = 1$ then J contains $af = aba$. If $h_4 = 1$ then J contains $fa = aba$. If $h_3 = h_4 = 0$, i.e., $f = h_2 a + h_5 aba$, then J contains $fba = aba$ when $h_2 = 1$, and J contains $f = aba$ because f is nonzero when $h_2 = 0$.

Thus $RabaR$ is an essential ideal of R.

Suppose $fg = 0$ for $f = h_0 + h_1 b + h_2 a + h_3 ba + h_4 ab + h_5 aba, g = k_0 + k_1 b + k_2 a + k_3 ba + k_4 ab + k_5 aba \in R\backslash\{0\}$. Then $h_0 = 0$ or $k_0 = 0$ clearly. Write $f_1 = h_2 a + h_3 ba + h_4 ab + h_5 aba$ and $g_1 = k_2 a + k_3 ba + k_4 ab + k_5 aba$.

`Case 1.` Let $h_0 = 1$ and $k_0 = 0$. Then

$$f = 1 + b + f_1 \text{ and } g = k_1 b + g_1$$

because $f \in U(R)$ when $h_1 = 0$. Since $(RaR)^3 = 0$, we see $f_1(RabaR) = 0$ and $(RabaR)g_1 = 0$, from which we obtain that

$$\begin{aligned} f(RabaR)g &= (1 + b + f_1)(RabaR)(k_1 b + g_1) \\ &= (1 + b)(RabaR)k_1 b + (1 + b)(RabaR)g_1 + f_1(RabaR)k_2 b + f_1(RabaR)g_1 \\ &= (1 + b)(RabaR)k_1 b. \end{aligned}$$

If $k_1 = 1$ then $f(RabaR)g = (1 + b)(RabaR)b = 0$ because $bab = 0$. Therefore $f(RabaR)g = 0$ when $h_0 = 1$ and $k_0 = 0$.

`Case 2.` Let $h_0 = 0$ and $k_0 = 1$. Then

$$f = h_1 b + f_1 \text{ and } g = 1 + b + g_1$$

because $g \in U(R)$ when $k_1 = 0$. By the same procedure as in Case 1, we obtain that

$$\begin{aligned} f(RabaR)g &= (h_1 b + f_1)(RabaR)(1 + b + g_1) \\ &= h_1 b(RabaR)(1 + b) + h_1 b(RabaR)g_1 + f_1(RabaR)(1 + b) + f_1(RabaR)g_1 \\ &= h_1 b(RabaR)(1 + b). \end{aligned}$$

If $h_1 = 1$ then $f(RabaR)g = b(RabaR)(1 + b) = 0$ because $bab = 0$. Therefore $f(RabaR)g = 0$ when $h_0 = 0$ and $k_0 = 1$.

`Case 3.` Let $h_0 = 0$ and $k_0 = 0$. Then

$$f = h_1 b + f_1 \text{ and } g = k_1 b + g_1.$$

By the same procedure as above, we obtain that

$$\begin{aligned} f(RabaR)g &= (h_1 b + f_1)(RabaR)(k_1 b + g_1) \\ &= h_1 b(RabaR)k_1 b + h_1 b(RabaR)g_1 + f_1(RabaR)k_1 b + f_1(RabaR)g_1 \\ &= h_1 b(RabaR)k_1 b. \end{aligned}$$

If $h_1 = 1$ or $k_1 = 1$ then $f(RabaR)g = h_1 b(RabaR)k_1 b = 0$ because $bab = 0$. Therefore $f(RabaR)g = 0$ when $h_0 = 0$ and $k_0 = 0$.

Summarizing, we now have $f(RabaR)g = 0$ when $fg = 0$. Since $RabaR$ is essential in R, we see that R is an IEIP ring.

(2) There exists an Abelian ring that is not IEIP. We follow the construction in [1, Example 4.8] for our purpose. Let K be a field and $A = K\langle a, b\rangle$ be the free algebra generated by noncommuting indeterminates a, b over K. Let I be the ideal of A generated by b^2 and set $R = A/I$. Every element of A is identified with its image in R for simplicity. Then R is Abelian by [1, Theorem 4.7] and [6, Corollary 8].

Suppose that $fg = 0$ for $0 \neq f, g \in R$. Then $f = f_1 b$ and $g = bg_1$ for some $f_1, g_1 \in R$ by [7, Remark after Theorem 1.2]. Let I be any nonzero ideal of R

and take $0 \neq h \in I$. By the argument in the proof of [7, Theorem 1.2], we can write $h = kb + h_0$ with $k \in K$ and $h_0 \in R$ such that a occurs in every nonzero term of h_0. Note

$$aha = a(kb + h_0)a = kaba + ah_0a \neq 0 \ \text{ and } \ aha \in I.$$

Thus we have

$$f(aha)g = f(kaba + ah_0a)g = kfabag + fah_0ag \neq 0, \text{ so that } fIg \neq 0,$$

by applying the method in the proof of [7, Theorem 1.2]. This concludes that R is not IEIP.

We will consider the structure of IEIP rings under some conditions which play roles in ring theory.

Proposition 3.2. (1) *Let R be a prime ring. Then the following conditions are equivalent:*
(i) *R is IEIP;*
(ii) *R is IFP;*
(iii) *R is a domain.*
(2) *Let R be a semiprime IEIP ring and suppose that $ab = 0$ for $a, b \in R$. Then $IbIa = 0$ and $bIaI = 0$ for some essential ideal I of R.*
(3) *Let R be a semiprime IEIP ring and suppose that $a^2 = 0$ for $a \in R$. Then $aI = 0$ and $Ia = 0$ for some essential ideal I of R.*

Proof. (1) It suffices to prove (i) $\Rightarrow$ (iii). Let R be IEIP and suppose $ab = 0$ for $a, b \in R$. Then $aIb = 0$ for some essential ideal I of R, whence $aR(Ib) = aIb = 0$. Assume $a \neq 0$. Then since R is prime, we obtain $Ib = 0$ and $IRb = 0$ follows. Since R is prime and $I \neq 0$, we get $b = 0$. Thus R is a domain.

(2) Since R is IEIP and $ab = 0$, $aIb = 0$ for some essential ideal I of R. This yields

$$(IbIaR)(IbIaR) = IbI(aRIb)IaR = IbI(aIb)IaR = 0$$

and

$$(RbIaI)(RbIaI) = RbI(aIRb)IaI = RbI(aIb)IaI = 0.$$

Since R is semiprime, we see that $IbIa = 0$ and $bIaI = 0$.

(3) By (2), we get $(Ia)^2 = 0$ and $(aI)^2 = 0$. Since R is semiprime, we obtain $Ia = 0$ and $aI = 0$. □

From Proposition 3.2(1), it is immediately clear that $Mat_n(R)$ (for $n \geq 2$) cannot be IEIP over any prime ring R. One may ask whether factor rings of IEIP rings are IEIP. But the answer is negative by the following.

Example 3.3. There exists a domain which has a non-IEIP factor ring. We apply the ring in [5, Example 3]. Let S be the localization of $\mathbb{Z}$ at the prime ideal $p\mathbb{Z}$, where p is an odd prime.

Next set R be the quaternions over S. Then R is clearly a domain (hence IEIP) and $J(R) = pR$. But $R/J(R)$ is isomorphic to $Mat_2(\mathbb{Z}_p)$ by the argument in [4, Exercise 2A]. But $Mat_2(\mathbb{Z}_p)$ is not IEIP by the argument above, concluding that $R/J(R)$ is not IEIP.

From Proposition 3.2(1), we can obtain the following useful result.

Corollary 3.4. (1) *Let R be an IEIP ring such that R/P is IEIP for every minimal prime ideal P of R. Then $R/N_*(R)$ is a reduced ring.*

(2) *Let R be a semiprime ring such that R/P is IEIP for every minimal prime ideal P of R. Then if R is IEIP then R is reduced.*

Proof. (1) R/P is a prime IEIP ring by hypothesis, whence R/P is a domain by Proposition 3.2(1). Then $R/N_*(R)$ is a subdirect product of domains, and so $R/N_*(R)$ is reduced. (2) is an immediate consequence of (1). □

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