THE BERGMAN METRIC AND RELATED BLOCH SPACES ON THE EXPONENTIALLY WEIGHTED BERGMAN SPACE

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Abstract. We estimate the Bergman metric of the exponentially weighted Bergman space and prove many different geometric characterizations for related Bloch spaces. In particular, we prove that the Bergman metric of the exponentially weighted Bergman space is comparable to some Poincaré type metric.

1. Introduction and main results

Let $\mathbb{D}$ denote the unit disc in the complex plane $\mathbb{C}$.

Definition 1 (Class $\mathcal{L}$). Let $\tau \in C^\infty(\mathbb{D})$ be a radial positive function on $\mathbb{D}$. We say that $\tau \in \mathcal{L}$ (see [1] and [6]) if there exist constants $A > 0$ and $B > 0$ such that

(a) $0 < \tau(z) \leq A(1 - |z|^2)$, for $z \in \mathbb{D}$;
(b) $|\tau(z) - \tau(w)| \leq B|z - w|$, for $z, w \in \mathbb{D}$.

The metric ds_τ associated with τ is given by

$$ds_\tau^2 = \frac{1}{\tau(z)^2} dz d\bar{z}.$$

By the condition (a), we know that the metric associated with τ is complete.

Suppose that $\gamma(t)$, $0 \leq t \leq 1$, is a piecewise smooth curve in $\mathbb{D}$. The length of $\gamma(t)$ with respect to the metric is defined by

$$L_\tau(\gamma) = \int_\gamma ds_\tau = \int_0^1 \frac{|\gamma'(t)|}{\tau(\gamma(t))} dt.$$

Then the distance p_τ associated with τ is defined by

$$p_\tau(z, w) = \inf\{L_\tau(\gamma) : \gamma(0) = z, \gamma(1) = w\},$$

where the infimum is taken for all piecewise smooth curves and $z, w \in \mathbb{D}$.

We define

$$Lip(p_\tau) = \{f \in C(\mathbb{D}) : \|f\|_{p_\tau} < +\infty\},$$

where

$$\|f\|_{p_\tau} = \inf\{M : |f(z) - f(w)| \leq M p_\tau(z, w)\}.$$

Let $\mathcal{B}(\tau)$ denote the space of analytic functions f on $\mathbb{D}$ such that

$$\|f\|_{\mathcal{B}(\tau)} = \sup\{\tau(z)|f'(z)| : z \in \mathbb{D}\} < +\infty.$$

For z, w in $\mathbb{D}$, we define the induced distance (see [7] and [8]) for $\mathcal{B}(\tau)$ by

$$d_\tau(z, w) = \sup\{|f(z) - f(w)| : \|f\|_{\mathcal{B}(\tau)} \leq 1\}.$$

We define

$$Lip(d_\tau) = \{f \in C(\mathbb{D}) : \|f\|_{d_\tau} < +\infty\},$$

where

$$\|f\|_{d_\tau} = \inf\{M : |f(z) - f(w)| \leq M d_\tau(z, w)\}.$$

Note that if $\tau(z) = 1 - |z|^2$, then p_τ is the classical Poincaré distance p and $\mathcal{B}(\tau)$ is the Bloch space $\mathcal{B}$. It is well known that the Poincaré distance p is identical to the induced distance d for $\mathcal{B}$ (see [9]).

The following result shows that the functions in $\mathcal{B}(\tau)$ are Lipschitz as mapping from $\mathbb{D}$ with distances p_τ or d_τ to $\mathbb{C}$ with the Euclidean distance.

Theorem 1.1. *Let $\tau \in \mathcal{L}$. Let f be an analytic function on $\mathbb{D}$. Then*

$$\|f\|_{\mathcal{B}(\tau)} = \|f\|_{p_\tau} = \|f\|_{d_\tau}.$$

Given $z \in \mathbb{D}$ and $\rho > 0$, we write

$$D(z, \rho) = \{w \in \mathbb{D} : |w - z| < \rho\}$$

for the Euclidean disc centered at z with radius ρ.

For a continuous function f on $\mathbb{D}$, we define a function $\omega_\tau^\rho(f)(z)$ on $\mathbb{D}$ by

$$\omega_\tau^\rho(f)(z) = \sup\{|f(z) - f(w)| : w \in D(z, \rho\tau(z))\}.$$

$\omega_\tau^\rho(f)(z)$ is called the oscillation of f at z in the metric ds_τ. We say that a continuous function f on $\mathbb{D}$ has bounded oscillation in the metric ds_τ if the function $\omega_\tau^\rho(f)(z)$ is bounded in $\mathbb{D}$. We define

$$BO_\tau^\rho = \{f \in C(\mathbb{D}) : \|f\|_{BO_\tau^\rho} < +\infty\},$$

where

$$\|f\|_{BO_\tau^\rho} = \sup\{\omega_\tau^\rho(f)(z) : z \in \mathbb{D}\}.$$

Given a function $f \in L^1(\mathbb{D}, dA)$, we define an averaging function $\hat{f}_\rho(z)$ on $\mathbb{D}$ as follows:

$$\hat{f}_\rho(z) = \frac{1}{|D(z, \rho\tau(z))|} \int_{D(z, \rho\tau(z))} f(w) \, dA(w), \quad z \in \mathbb{D}.$$

We define the mean oscillation of f at z in the metric ds_τ to be

$$MO_\tau^\rho(f)(z) = \left(\frac{1}{|D(z, \rho\tau(z))|} \int_{D(z,\rho\tau(z))} |f(w) - \hat{f}_r(z)|^2 \, dA(w) \right)^{1/2}.$$

We say that a function f has bounded mean oscillation in the metric ds_τ if $MO_\tau^\rho(f)$ is bounded in $\mathbb{D}$. We shall let BMO_τ^ρ denote the space of functions on $\mathbb{D}$ with bounded mean oscillation in the metric ds_τ. Let

$$\|f\|_{BMO_\tau^\rho} = \sup\{MO_\tau^\rho(f)(z) : z \in \mathbb{D}\}.$$

Theorem 1.2. *Let $\tau \in \mathcal{L}$. Let $\rho > 0$ and f be an analytic function on $\mathbb{D}$. Then*

$$\|f\|_{\mathcal{B}(\tau)} \approx \|f\|_{BMO_\tau^\rho} \approx \|f\|_{BO_\tau^\rho}.$$

We note that the above results are well-known for the case $\tau(z) = 1 - |z|^2$ (see [9]).

Definition 2 (Class $\mathcal{L}^*$)**.** Let $\varphi \in C^2(\mathbb{D})$ be a radial function such that $\Delta\varphi(z) \geq C_\varphi > 0$ for some positive constant C_φ depending only on the function φ. We say $\varphi \in \mathcal{L}^*$ (see [1] and [6]) if the function

$$\tau_\varphi(\rho) = (\Delta\varphi(\rho))^{-\frac{1}{2}}, \quad 0 \leq \rho < 1,$$

decreases to 0 as $\rho = |z| \to 1^-$, $\tau_\varphi'(\rho) \to 0$ as $\rho \to 1^-$, and moreover, either there exists a constant $C > 0$ such that $\tau_\varphi(\rho)(1 - \rho)^{-C}$ increases for ρ close to 1 or $\lim_{\rho \to 1^-} \tau_\varphi'(\rho) \log \frac{1}{\tau_\varphi(\rho)} = 0$.

We know that if $\varphi \in \mathcal{L}^*$, then $\tau_\varphi \in \mathcal{L}$. Several examples are given in [6].

The weighted Bergman space A_φ^2 is the space of analytic functions f such that

$$\|f\|_{2,\varphi}^2 = \int_{\mathbb{D}} |f(z)|^2 e^{-2\varphi(z)} \, dA(z) < +\infty.$$

Since the space A_φ^2 is a reproducing kernel Hilbert space, for each $z \in \mathbb{D}$, there are functions $K_z \in A_\varphi^2$ with $f(z) = \langle f, K_z \rangle_\varphi$, where $\langle \cdot, \cdot \rangle_\varphi$ is the usual inner product in L_φ^2. The reproducing kernel

$$K_\varphi(w, z) = \overline{K_z(w)}, \quad z, w \in \mathbb{D},$$

is called the Bergman kernel for A_φ^2.

The Bergman metric on $\mathbb{D}$ associated with φ is given by

$$B_\varphi(z)dzd\bar{z} = \frac{\partial^2}{\partial z \partial \bar{z}} \log K_\varphi(z, z)dzd\bar{z}.$$

The corresponding Bergman distance β_φ is given by

$$\beta_\varphi(z, w) = \inf_\gamma \int_0^1 \sqrt{B_\varphi(\gamma(t))}|\gamma'(t)| \, dt,$$

where γ is a piecewise smooth curve with $\gamma(0) = z$ and $\gamma(1) = w$.

For an analytic function f and $z \in \mathbb{D}$ we define

$$Q_\varphi(f)(z) = \frac{|f'(z)|}{\sqrt{B_\varphi(z)}}.$$

Definition 3. An analytic function f on $\mathbb{D}$ is called a Bloch function associated with φ if

$$\sup\{Q_\varphi(f)(z) : z \in \mathbb{D}\} < +\infty.$$

Theorem 1.3. *Let* $\varphi \in \mathcal{L}^*$. *Then*

$$B_\varphi(z) \approx \frac{1}{\tau_\varphi(z)^2}, \quad z \in \mathbb{D}.$$

Corollary 1.4. *Let* $\varphi \in \mathcal{L}^*$. *There exists a constant* $C > 0$ *such that*

$$C^{-1} p_{\tau_\varphi}(z, w) \leq \beta_\varphi(z, w) \leq C p_{\tau_\varphi}(z, w), \quad z, w \in \mathbb{D}.$$

Corollary 1.5. *Let* $\varphi \in \mathcal{L}^*$. *Let* f *be a Bloch function associated with* φ. *Then*

$$\|f\|_{\mathcal{B}(\tau_\varphi)} \approx \sup\{Q_\varphi(f)(z) : z \in \mathbb{D}\}.$$

Constants. In the rest of the paper we use the same letter C to denote various positive constants which may change at each occurrence. Variables indicating the dependency of constants C will be often specified. We use the notation $X \lesssim Y$ or $Y \gtrsim X$ for nonnegative quantities X and Y to mean $X \leq CY$ for some inessential constant $C > 0$. Similarly, we use the notation $X \approx Y$ if both $X \lesssim Y$ and $Y \lesssim X$ hold.

2. A Hermitian metric

A Hermitian metric ds_τ on $\mathbb{D}$ associated with $\tau \in \mathcal{L}$ is a symmetric tensor represented by

$$ds_\tau^2 = \frac{1}{\tau(z)^2} dz d\bar{z}.$$

Let h be a smooth function on $\mathbb{R}$ such that

$$\frac{1}{\tau(z)} = h(|z|^2).$$

Then we obtain the Gaussian curvature function is given by

$$K_\tau = \frac{4(|z|^2 h'^2 - |z|^2 h h'' - h h')}{h^4}.$$

Setting $t = |z|^2$, we obtain that

$$K_\tau = -\frac{4}{h^2} \left(\frac{t h'(t)}{h} \right)'.$$

Thus $K_\tau \leq 0$ if and only if

$$\frac{th'(t)}{h}$$

is increasing.

Since $g_{11} = g_{22} = h^2(|z|^2)$, and $g_{12} = g_{21} = 0$, we obtain that the Christoffel symbols of our metric are given by:

$$\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^2_{21} = \frac{h_x}{h},$$

$$\Gamma^2_{22} = \Gamma^1_{12} = \Gamma^1_{21} = \frac{h_y}{h},$$

$$\Gamma^2_{11} = -\frac{h_x}{h}, \quad \Gamma^1_{22} = -\frac{h_y}{h}.$$

The parametric equations for the geodesic curve is here:

$$\ddot{x} + 2\frac{xh'}{h}\dot{x}^2 + 4\frac{yh'}{h}\dot{x}\dot{y} - 2\frac{xh'}{h}\dot{y}^2 = 0,$$

$$\ddot{y} - 2\frac{yh'}{h}\dot{x}^2 + 4\frac{xh'}{h}\dot{x}\dot{y} + 2\frac{yh'}{h}\dot{y}^2 = 0.$$

We calculate $p_\tau(0, z)$. Let (ρ, θ) be geodesic polar coordinates about 0. Then

$$ds_\tau^2 = \frac{1}{\tau(\rho)^2}\left\{(d\rho)^2 + \rho^2(d\theta)^2\right\}.$$

Hence

$$g_{11} = \frac{1}{\tau(\rho)^2}, \quad g_{12} = g_{21} = 0, \quad g_{22} = \frac{\rho^2}{\tau(\rho)^2}.$$

This means that it is a Clairaut parametrization and the ρ-parameter curves are geodesics. Thus the segment $\gamma(t) = t|z|$ is the geodesic joining 0 and $|z|$. As the metric is a rotation invariant, it follows that

$$p_\tau(0, z) = p_\tau(0, |z|) = \int_0^{|z|} \frac{1}{\tau(t)}\, dt.$$

Example 1. *We define for $\alpha \geq 1$*

$$\tau_\alpha(z) = (1 - |z|^2)^\alpha, \quad z \in \mathbb{D}.$$

Then $\tau_\alpha \in \mathcal{L}$ and the metric

$$ds_\alpha^2 = \frac{1}{\tau_\alpha(z)^2}\, dz d\bar{z}.$$

is called the weighted Poincaré metric (see [7]). Then

$$h_\alpha(t) = \frac{1}{(1 - t)^\alpha}$$

and the Gaussian curvature function is given by

$$K_\alpha = -4\alpha(1 - |z|^2)^{2(\alpha-1)} < 0.$$

If $\alpha = 1$, it is the Poincaré metric and the curvature is -4.

Even though we could not solve the geodesic equations by the explicit formula, in (Figure 1) we get the picture of geodesics by using the numerical method.
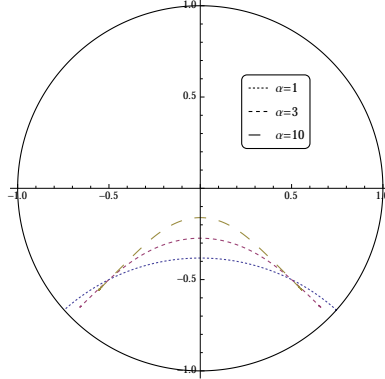

Figure 1. *Geodesics through $(-0.5, -0.5)$ and $(0.5, -0.5)$*
(We use the NDSolve command in the Mathematica program to show the solutions of the geodesic equations for $\alpha = 1, 3$, or 10.)

Let (ρ, θ) be geodesic polar coordinates about 0. Then

$$ds_\alpha^2 = \frac{1}{(1 - \rho^2)^{2\alpha}} \left\{ (d\rho)^2 + \rho^2 (d\theta)^2 \right\}.$$

Hence

$$g_{11} = \frac{1}{(1 - \rho^2)^{2\alpha}}, \quad g_{12} = g_{21} = 0, \quad g_{22} = \frac{\rho^2}{(1 - \rho^2)^{2\alpha}}.$$

Hence it is a Clairaut parametrization and the ρ-parameter curves are geodesics.

Let $p_\alpha(z, w)$ be the weighted Poincaré distance induced by the weighted Poincaré metric. Then

$$p_\alpha(0, z) = \int_0^{|z|} \frac{1}{(1 - t^2)^\alpha} \, dt$$

$$= |z| \, {}_2F_1 \left[\frac{1}{2}, \alpha; \frac{3}{2}; |z|^2 \right].$$

When $\alpha = 2$, then

$$p_2(0, z) = \frac{1}{2} \left[\frac{|z|}{1 - |z|^2} + \tanh^{-1}(|z|) \right].$$

Here $F(a, b; c|z)$ is the hypergeometric function given by

$$F(a, b; c|z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_0^1 t^{b-1} (1 - t)^{c-b-1} (1 - zt)^{-a} \, dt, \quad |z| < 1,$$

where Γ is the Gamma function and $a \in \mathbb{R}$, $c > b > 0$.

3. Preliminaries

In this section we let $\tau \in \mathcal{L}$. The following notations will be frequently used:

$$m_\tau = \frac{1}{4} \min\left\{1, \frac{1}{A}, \frac{1}{B}\right\},$$

where A and B are the constants in the conditions (a) and (b) in Definition 1.

Lemma 3.1. *Let* $0 < \rho \leq m_\tau$ *and* $w \in \mathbb{D}$. *Then,*

$$\frac{3}{4}\tau(w) \leq \tau(z) \leq \frac{5}{4}\tau(w), \quad z \in D(w, \rho\tau(w)).$$

Proof. Let $z \in D(w, \rho\tau(w))$. By the conditon (b) in Definition 1, we have

$$\tau(w) \leq \tau(z) + B|z - w| \leq \tau(z) + B\rho\tau(w) \leq \tau(z) + \frac{1}{4}\tau(w).$$

Therefore

$$\frac{3}{4}\tau(w) \leq \tau(z), \quad z \in D(w, \rho\tau(w)).$$

Similarly, it can be proved that

$$\tau(z) \leq \frac{5}{4}\tau(w), \quad z \in D(w, \rho\tau(w)).$$

$\square$

As a consequence of Lemma 3.1 we have

Corollary 3.2. *Let* $0 < \rho_1, \rho_2 \leq m_\tau$ *and* $z, w \in \mathbb{D}$. *If* $D(z, \rho_1\tau(z)) \cap D(w, \rho_2\tau(w)) \neq \emptyset$. *then* $\tau(z) \sim \tau(w)$.

Corollary 3.3. *Let* $0 < \frac{8}{3}\rho \leq m_\tau$ *and* $z \in D(z_0, \rho\tau(z_0))$. *Then* $D(z_0, \rho\tau(z_0)) \subset D(z, \frac{8}{3}\rho\tau(z))$.

Lemma 3.4. *Let* f *be an analytic function in* $\mathbb{D}$. *Then*

$$|f'(z)| \leq \frac{1}{2\pi} \frac{3}{(\rho\tau(z))^3} \int_{D(z, \rho\tau(z))} |f(w)| \, dA(w).$$

Proof. By the Cauchy integral formula, it follows that

$$f'(z) = \frac{1}{2\pi i} \int_{|z-\zeta|=r} \frac{f(\zeta)}{(\zeta - z)^2} \, d\zeta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{f(z + re^{i\theta})}{re^{i\theta}} \, d\theta.$$

Hence

$$|f'(z)| \int_0^{\rho\tau(z)} r^2 \, dr \leq \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\rho\tau(z)} |f(z + re^{i\theta})| r \, dr d\theta$$

or

$$|f'(z)| \leq \frac{1}{2\pi} \frac{3}{(\rho\tau(z))^3} \int_{D(z,\rho\tau(z))} |f(w)| \, dA(w).$$

$\square$

4. Geometric characterizations of the τ-Bloch space

We recall that the τ-Bloch space $\mathcal{B}(\tau)$ is defined by

$$\mathcal{B}(\tau) = \{f \in H(\mathbb{D}) : \|f\|_{\mathcal{B}(\tau)} < +\infty\},$$

where

$$\|f\|_{\mathcal{B}(\tau)} = \sup\{\tau(z)|f'(z)| : z \in \mathbb{D}\}$$

and the induced distance d_τ (see [7] and [8]) is defined by

$$d_\tau(z,w) = \sup\{|f(z) - f(w)| : \|f\|_{\mathcal{B}(\tau)} \leq 1\}.$$

Lemma 4.1. *Let $\tau \in \mathcal{L}$. Then*

$$\limsup_{w \to z} \frac{d_\tau(z,w)}{|z-w|} \leq \frac{1}{\tau(z)}.$$

Proof. For $w \in \mathbb{D}$ with $z \neq w$ we have

$$|f(z) - f(w)| \leq |z - w| \int_0^1 |f'(tz + (1-t)w)| dt$$

$$\leq |z - w| \int_0^1 \frac{1}{\tau(tz + (1-t)w)} dt$$

for all $f \in \mathcal{B}(\tau)$ with $\|f\|_{\mathcal{B}(\tau)} \leq 1$. Taking the supremum over all such f, we get

$$\frac{d_\tau(z,w)}{|z-w|} \leq \int_0^1 \frac{1}{\tau(tz + (1-t)w)} dt.$$

Since $\frac{1}{\tau}$ is uniformly continuous on a neighborhood of z, we have

$$\limsup_{w \to z} \frac{d_\tau(z,w)}{|z-w|} \leq \lim_{w \to z} \int_0^1 \frac{1}{\tau(tz + (1-t)w)} dt = \frac{1}{\tau(z)}.$$

$\square$

We recall that BO_τ^ρ denotes the space of continuous functions on $\mathbb{D}$ with bounded oscillation with the semi-norm

$$\|f\|_{BO_\tau^\rho} = \sup\{\omega_\tau^\rho(f)(z) : z \in \mathbb{D}\}.$$

Example 2. *Let $z = |z|e^{i\theta}$ and $w \in D(z, \rho\tau(z))$. Then $te^{i\theta} \in D(z, \rho\tau(z))$ for any t between $|z|$ and $|w|$. Since $\tau(t) = \tau(te^{i\theta}) \approx \tau(z)$, it follows that*

$$|p_\tau(0,z) - p_\tau(0,w)| = \left| \int_{|z|}^{|w|} \frac{1}{\tau(t)} dt \right| \lesssim \frac{|w-z|}{\tau(z)} \leq \rho.$$

Hence $p_\tau(0, \cdot) \in BO_\tau^\rho$.

Theorem 4.2. *Let $\tau \in \mathcal{L}$. Let f be an analytic function on $\mathbb{D}$. Then*

$$\|f\|_{\mathcal{B}(\tau)} = \|f\|_{p_\tau} = \|f\|_{d_\tau}$$

and

$$\|f\|_{\mathcal{B}(\tau)} \approx \|f\|_{BMO_\tau^\rho} \approx \|f\|_{BO_\tau^\rho}.$$

Proof. Suppose that there exists a constant $C > 0$ such that

$$|f(z) - f(w)| \leq C p_\tau(z, w), \quad z, w \in \mathbb{D}.$$

Fix z in $\mathbb{D}$ and let $\gamma(s)$ be the geodesic parametrized by arc-length such that $\gamma(0) = z$. Since $p_\tau(\gamma(0), \gamma(s)) = s$, we have

$$|f(\gamma(0)) - f(\gamma(s))| \leq C s.$$

This means that $|f'(z)||\gamma'(0)| \leq C$. Now

$$1 = \frac{1}{s} p_\tau(\gamma(0), \gamma(s)) = \frac{1}{s} \int_0^s \frac{|\gamma'(t)|}{\tau(\gamma(t))}\, dt.$$

Thus we have $\tau(z) = |\gamma'(0)|$. Hence $\tau(z)|f'(z)| \leq C$. This means that $\|f\|_{\mathcal{B}(\tau)} \leq \|f\|_{p_\tau}$.

Let $f \in \mathcal{B}(\tau)$. For $z, w \in \mathbb{D}$ let γ be the geodesic in $\mathbb{D}$ such that $\gamma(0) = z$ and $\gamma(1) = w$. Then

$$|f(z) - f(w)| \leq \int_0^1 |f'(\gamma(t))||\gamma'(t)|\, dt$$

$$\leq \|f\|_{\mathcal{B}(\tau)} \int_0^1 \frac{|\gamma'(t)|}{\tau(\gamma(t))}\, dt = p_\tau(z, w)\|f\|_{\mathcal{B}(\tau)}.$$

Hence we have $\|f\|_{p_\tau} \leq \|f\|_{\mathcal{B}(\tau)}$.

Let $f \in \mathcal{B}(\tau)$. Take $F = f/\|f\|_{\mathcal{B}(\tau)}$. Then $\|F\|_{\mathcal{B}(\tau)} = 1$. By the definition of d_τ we have

$$|F(z) - F(w)| \leq d_\tau(z, w)$$

or

$$|f(z) - f(w)| \leq \|f\|_{\mathcal{B}(\tau)} d_\tau(z, w).$$

Hence it follows that $\|f\|_{d_\tau} \leq \|f\|_{\mathcal{B}(\tau)}$.

Suppose that there exists a constant $C > 0$ such that

$$|f(z) - f(w)| \leq C d_\tau(z, w), \quad z, w \in \mathbb{D}.$$

By Lemma 4.1, we have

$$|f'(z)| = \limsup_{w \to z} \frac{|f(z) - f(w)|}{|z - w|} = \limsup_{w \to z} \frac{|f(z) - f(w)|}{d_\tau(z, w)} \frac{d_\tau(z, w)}{|z - w|} \leq C \frac{1}{\tau(z)}.$$

This means that $\|f\|_{\mathcal{B}(\tau)} \leq \|f\|_{d_\tau}$. Consequently, we have $\|f\|_{\mathcal{B}(\tau)} = \|f\|_{p_\tau} = \|f\|_{d_\tau}$.

Note that

$$\widehat{|f|^2_\rho}(z) - |\hat{f}_\rho(z)|^2$$
$$= \frac{1}{2|D(z,\rho\tau(z))|^2} \int_{D(z,\rho\tau(z))} \int_{D(z,\rho\tau(z))} |f(w) - f(\zeta)|^2 \, dA(w) dA(\zeta).$$

Hence we have

$$MO^\rho_\tau(f)(z) \leq \left(\frac{1}{|D(z,\rho\tau(z))|} \int_{D(z,\rho\tau(z))} |f(w) - f(z)|^2 \, dA(w) \right)^{1/2}$$
$$\leq \omega^\rho_\tau(f)(z).$$

By the sub-mean value property and Hölder's inequality, we have

$$|f(w)| \lesssim \left(\frac{1}{|D(w,\rho\tau(w))|} \int_{D(w,\rho\tau(w))} |f(\zeta)|^2 \, dA(\zeta) \right)^{1/2}.$$

Replace f by $f - f(z)$, then

$$|f(z) - f(w)| \lesssim \left(\frac{1}{D(w,\rho\tau(w))} \int_{D(w,\rho\tau(w))} |f(z) - f(\zeta)|^2 \, dA(\zeta) \right)^{1/2}$$

for analytic function f and $z, w \in \mathbb{D}$. If $w \in D(z,\rho\tau(z))$, then $D(w,\rho\tau(w)) \subset D(z,3\rho\tau(z))$. Thus

$$|f(z) - f(w)| \lesssim \left(\frac{1}{|D(w,\rho\tau(w))|} \int_{D(w,\rho\tau(w))} |f(z) - f(\zeta)|^2 \, dA(\zeta) \right)^{1/2}$$
$$\lesssim \left(\frac{1}{\tau(z)^2} \int_{D(z,3\rho\tau(z))} |f(z) - f(\zeta)|^2 \, dA(\zeta) \right)^{1/2}.$$

Now for $\zeta \in D(z, 3\rho\tau(z))$ we have

$$|f(z) - f(\zeta)| \leq |\zeta - z| \int_0^1 |f'(z + t(\zeta - z))| \, dt$$
$$\leq |\zeta - z| \|f\|_{\mathcal{B}(\tau)} \int_0^1 \frac{1}{\tau(z + t(\zeta - z))} \, dt$$
$$\approx \frac{|\zeta - z|}{\tau(z)} \|f\|_{\mathcal{B}(\tau)}.$$

Hence

$$|f(z) - f(w)| \lesssim \left(\frac{1}{\tau(z)^2} \int_{D(z,3\rho\tau(z))} |f(z) - f(\zeta)|^2 \, dA(\zeta) \right)^{1/2}$$

$$\lesssim \left(\frac{\|f\|_{\mathcal{B}(\tau)}^2}{\tau(z)^4} \int_{D(z,3\rho\tau(z))} |\zeta - z|^2 \, dA(\zeta) \right)^{1/2}$$

$$\lesssim \|f\|_{\mathcal{B}(\tau)}$$

or

$$\omega_\tau^\rho(f)(z) \lesssim \|f\|_{\mathcal{B}(\tau)}.$$

Thus

$$MO_\tau^\rho(f)(z) \le \omega_\tau^\rho(f)(z) \lesssim \|f\|_{\mathcal{B}(\tau)}.$$

By Lemma 3.4 and Hölder's inequality, we have

$$|f'(z)| \lesssim \frac{1}{\tau(z)^2} \left(\int_{D(z,\rho\tau(z))} |f(w)|^2 \, dA(w) \right)^{1/2}$$

for an analytic function in $\mathbb{D}$. Replace f by $f - \hat{f}_\rho(z)$. Then

$$|f'(z)| \lesssim \frac{1}{\tau(z)^2} \left(\int_{D(z,\rho\tau(z))} |f(w) - \hat{f}_\rho(z)|^2 \, dA(w) \right)^{1/2}.$$

Thus

$$\tau(z)|f'(z)| \lesssim \left(\frac{1}{|D(z,\rho\tau(z))|} \int_{D(z,\rho\tau(z))} |f(w) - \hat{f}_\rho(z)|^2 \, dA(w) \right)^{1/2}$$

$$= MO_\tau^\rho(f)(z).$$

Hence we have $\|f\|_{\mathcal{B}(\tau)} \approx \|f\|_{BMO_\tau^\rho} \approx \|f\|_{BO_\tau^\rho}.$ □

5. Estimates for the Bergman metric

We have the following family of analytic peak functions with precise growth conditions. The result was proved in ([2], [6]).

Lemma 5.1. *Let $\varphi \in \mathcal{L}^*$ and $\tau_\varphi = (\Delta\varphi)^{-\frac{1}{2}}$. Let $\rho \in (0, m_{\tau_\varphi}]$ and $n \in \mathbb{N} \setminus \{0\}$. Let $\delta > 0$ be sufficiently close to 1. Given $z \in \mathbb{D}$ with $\delta \le |z| < 1$, there exists a function $F_z \in H(\mathbb{D})$ such that*

(a) $|F_z(w)| \approx e^{\varphi(w)}, \quad w \in D(z, \rho\tau_\varphi);$

(b) $|F_z(w)| \lesssim e^{\varphi(w)} \min \left[1, \frac{\min\{\tau_\varphi(z), \tau_\varphi(w)\}}{|z-w|} \right]^{3n}, \quad w \in \mathbb{D}.$

For $\varphi \in \mathcal{L}^*$ and under the additional condition: there are constants $\rho > 0$ and $0 < t < 1$ such that

$$\tau_\varphi(w) \leq \tau_\varphi(z) + t|z - w| \quad \text{for} \quad w \notin D(z, \rho\tau(z)),$$

Lin and Rochberg [4] constructed a family of analytic peak functions with the growth condition (a) in Lemma 5.1.

The weighted Bergman space A_φ^2 is the space of functions $f \in H(\mathbb{D})$ such that

$$\|f\|_{2,\varphi}^2 = \int_{\mathbb{D}} |f(z)|^2 e^{-2\varphi(z)} \, dA(z) < +\infty.$$

Corollary 5.2 ([2], [6]). *Let $\delta > 0$ be sufficiently close to 1.*
  (a) $\|F_z\|_{2,\varphi}^2 \approx \tau_\varphi(z)^2, \quad |z| \geq \delta.$
  (b) $K(z,z) \approx \frac{e^{2\varphi(z)}}{\tau_\varphi(z)^2}, \quad |z| \geq \delta.$

Lemma 5.3. [5] *Let $f \in H(\mathbb{D})$ with $f(z) = 0$. Then*

$$|f'(z)|^2 e^{-2\varphi(z)} \lesssim \frac{1}{\tau_\varphi(z)^4} \int_{D(z,\rho\tau_\varphi(z))} |f(w)|^2 e^{-2\varphi(w)} \, dA(w).$$

We recall that

$$B_\varphi(z) = \frac{\partial^2}{\partial z \partial \bar{z}} \log K_\varphi(z,z),$$

where $K_\varphi(w,z)$ is the Bergman kernel for A_φ^2. It is well-known that (see [3])

$$B_\varphi(z) = \frac{1}{K_\varphi(z,z)} \sup\{|f'(z)|^2 : f \in A_\varphi^2, \|f\|_{2,\varphi} = 1, f(z) = 0\}.$$

Theorem 5.4. *Let $\varphi \in \mathcal{L}^*$. Then*

$$B_\varphi(z) \approx \frac{1}{\tau_\varphi(z)^2}, \quad z \in \mathbb{D}.$$

Proof. By Lemma 5.3 and Corollary 5.2 (b), we have

$$B_\varphi(z) \lesssim \frac{1}{\tau_\varphi(z)^2}.$$

For the converse inequality, we take

$$f_z(w) = \frac{w - z}{\tau_\varphi(z)^2} F_z(w),$$

where F_z is the peak function constructed in Lemma 5.1. Then

$$\|f_z\|_{2,\varphi}^2 = \frac{1}{\tau_\varphi(z)^4} \int_{\mathbb{D}} |w - z|^2 |F_z(w)|^2 e^{-2\varphi(w)} \, dA(w).$$

Let $\rho \in (0, m_{\tau_\varphi}]$ and $\delta \leq |z| < 1$. Let

$$I(z) = \int_{D(z,\rho\tau_\varphi(z))} |w - z|^2 |F_z(w)|^2 e^{-2\varphi(w)} \, dA(w)$$

and

$$J(z) = \int_{\mathbb{D} \setminus D(z, \rho\tau_\varphi(z))} |w - z|^2 |F_z(w)|^2 e^{-2\varphi(w)} \, dA(w).$$

By Corollary 5.2, we have

$$I(z) \lesssim \tau_\varphi(z)^2 \|F_z\|_{2,\varphi}^2 \lesssim \tau_\varphi(z)^4.$$

On the other hand, by Lemma 5.1 with $n = 2$, we have

$$J(z) \lesssim \tau_\varphi(z)^6 \int_{\mathbb{D} \setminus D(z, \rho\tau_\varphi(z))} \frac{dA(w)}{|w - z|^4}$$

$$\lesssim \tau_\varphi(z)^6 \int_{\rho\tau_\varphi(z)}^1 \frac{dr}{r^3} \lesssim \tau_\varphi(z)^4.$$

Thus $f_z \in A_\varphi^2$ and $\|f_z\|_{2,\varphi}^2 \lesssim 1$. We take

$$g_z(\zeta) = \frac{f_z(\zeta)}{\|f_z\|_{2,\varphi}}.$$

Then $g_z \in A_\varphi^2$, $\|g\|_{2,\varphi} = 1$, and

$$|g_z'(z)|^2 \approx |f_z'(z)|^2 \approx \frac{|F_z(z)|^2}{\tau_\varphi(z)^4} \approx \frac{e^{2\varphi(z)}}{\tau_\varphi(z)^4} \approx \frac{K_\varphi(z, z)}{\tau_\varphi(z)^2}.$$

Hence it follows that

$$B_\varphi(z) \geq \frac{|g_z'(z)|^2}{K_\varphi(z, z)} \gtrsim \frac{1}{\tau_\varphi(z)^2} \quad \text{for} \quad |z| \geq \delta.$$

Now if we choose

$$g_z(w) = \frac{w - z}{\|w - z\|_{2,\varphi}}.$$

Then $g_z \in A_\varphi^2$, $\|g_z\|_{2,\varphi} = 1$ and $g_z(z) = 0$. Now we have

$$|g_z'(z)|^2 = \frac{1}{\|w - z\|_{2,\varphi}^2} \geq \frac{1}{(1 + \delta)^2 \|1\|_{2,\varphi}^2}, \quad |z| \leq \delta$$

and

$$K_\varphi(z, z) \leq C, \quad |z| \leq \delta,$$

for some constant $C > 0$. Hence

$$B_\varphi(z) \geq \frac{1}{C(1 + \delta)^2 \|1\|_{2,\varphi}^2}, \quad |z| \leq \delta.$$

Since $\tau_\varphi \in C(\mathbb{D})$ with $\tau_\varphi(z) \neq 0, z \in \mathbb{D}$, it follows that $\frac{1}{\tau_\varphi} \in C(\mathbb{D})$. Thus

$$\frac{1}{\tau_\varphi(z)^2} \leq C', \quad |z| \leq \delta,$$

for some constant $C' > 0$. Thus

$$B_\varphi(z) \gtrsim \frac{1}{\tau_\varphi(z)^2} \quad \text{for} \quad |z| \le \delta, .$$

Thus we get the result.   □

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Competing Interests
The authors declare that there is no conflict of interests regarding the publication of this paper.

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