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ON THE TOUCHARD POLYNOMIALS AND MULTIPLICATIVE PLANE PARTITIONS

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Abstract. For a positive integer n, let $\mu_d(n)$ be the number of multiplicative d-dimensional partitions of $\prod_{i=1}^{n} p_i$, where p_i denotes the ith prime. The number of multiplicative partitions of a square free number with n prime factors is the Bell number $\mu_1(n) = \mathcal{B}_n$. By the definition of the function $\mu_d(n)$, it can be seen that for all positive integers n, $\mu_1(n) = T_n(1) = \mathcal{B}_n$, where $T_n(x)$ is the nth Touchard (or exponential) polynomial. We show that, for a positive n, $\mu_2(n) = 2^n T_n(1/2)$. We also conjecture that for all m, $\mu_3(m) \leq 3^m T_m(1/3)$.

1. Introduction and Notation

A partition of a set S is a collection of disjoint subsets of S whose union is S. For example, one possible partition of $\{1, 2, 3, 4, 5, 6\}$ is $\{\{1, 3\}, \{2\}, \{4, 5, 6\}\}$. The nth Bell number $\mathcal{B}_n$, named after Eric Temple Bell (see [3]), is the number of partitions of a set with n members. Let p_i be the ith prime (i.e., $p_1 = 2$, $p_2 = 3$, etc.). Then, the Bell number $\mathcal{B}_n$ is equal to the number of multiplicative one-dimensional partitions of $\prod_{i=1}^{n} p_i$ (see [5]). The Bell numbers B_k for $k \geq 0$ can be generated by

$$e^{e^x - 1} = \sum_{n=0}^{\infty} \frac{\mathcal{B}_n}{n!} x^n$$

and the first eight Bell numbers are $1, 1, 2, 5, 15, 52, 203, 877$. The Touchard polynomials $T_n(x)$'s (also known as Bell polynomials or exponential polynomials) can be defined by the Stirling transform

$$T_n(x) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^k,$$

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where $\left\{ {n \atop k} \right\}$ denotes the Stirling number of the second kind or can be defined by the exponential generating function

$$e^{x\left(e^t-1\right)} = \sum_{n=0}^{\infty} \frac{T_n(x)}{n!} t^n.$$

Since $T_n(1) = \mathcal{B}_n$, the Touchard polynomials $T_n(x)$ are generalizations of the Bell numbers $\mathcal{B}_n$ for $n \geq 0$. Let $\mu_d(m)$ denote the number of multiplicative d-dimensional partitions of integer $\prod_{i=1}^{m} p_i$. Since $\mu_1(n) = \mathcal{B}_n$, $\mu_d(n)$ are also generalizations of the Bell numbers.

In Section 2, we give some background on Young tableaux to prove that

$$\mu_2(n) = \sum_{k=1}^{n} Y(k) \left\{ {n \atop k} \right\},$$

where $Y(k)$ is the number of standard Young tableaux with k cells.

In Section 3, we study the function $\mu_2(n)$ and we show that, for a positive n, $\mu_2(n) = 2^n T_n(1/2)$. Thus, we extend the result of $\mu_1(n) = T_n(1)$ to the case of

$$\mu_k(n) = k^n T_n(1/k),$$

where $k = 1, 2$.

Finally, in Section 4, we estimate the number of multiplicative solid partitions of $\prod_{i=1}^{m} p_i$. Examining the multiplicative solid partition function for small values of m leads one to conjecture that for all m, $\mu_3(m) \leq 3^m B_m(1/3)$.

2. Standard Young Tableaux

A multiplicative plane (or two-dimensional) partition is a decomposition of a set into a product of relatively small positive integers, which are arranged on a plane. The ordering property generalizes to the products being non-increasing along both rows and columns. For example,the multiplicative plane partition $77_{(0,0)} \cdot 65_{(1,0)} \cdot 23_{(2,0)} \cdot 19_{(0,1)} \cdot 17_{(1,1)} \cdot 6_{(0,2)}$ is represented as

$$\begin{array}{lll} 6 & & \\ 19 & 17 & \\ 77 & 65 & 23\ . \end{array}$$

Generalization to a d-dimensional partition is straightforward. The multiplicative d-dimensional partition of a positive integer n is an array whose multiplication is

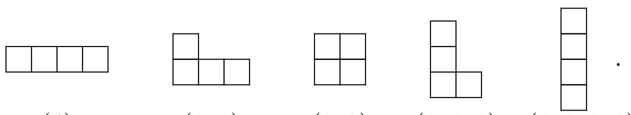
$$n = \prod_{i_1,.,i_d \geqslant 0} n_{i_1 i_2 \cdots i_d},$$

where $n_{i_1 i_2 \cdots i_d}$ are positive integers satisfying $n_{i_1 i_2 \cdots i_d} \geqslant n_{j_1 j_2 \cdots j_d}$ if $i_1 \leq j_1, i_2 \leq j_2, ..., i_d \leq j_d$ (see [1, p.179]).

Young tableaux are ubiquitous combinatorial objects that have made important and inspiring appearances in representation theory, geometry and algebra (see, for example [4], [12], and [13]). First we need to define some notations and conventions regarding partitions and Young diagrams. A partition of a positive integer n is a sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ satisfying $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k > 0$ and $n = \lambda_1 + \lambda_2 + ... + \lambda_k$. Similarly, a multiplicative partition of an integer n that is greater than 1 is a sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ satisfying $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k > 1$ and $n = \lambda_1 \cdot \lambda_2 \cdot ... \cdot \lambda_k$. We write $\lambda \vdash n$ to denote that λ is a partition of n and $\lambda' \vdash^* n$ to denote that λ' is a multiplicative partition of n. By convention, we let $P(n)$ denote the number of partitions of n. For example, it is easy to see that $P(4) = 5$ since the partitions of 4 are

$$(4), (3, 1), (2, 2), (2, 1, 1), (1, 1, 1, 1).$$

Partitions can be graphically visualized with Young diagrams. A Young diagram is a finite collection of cells arranged in left-justified rows, with the row lengths weakly decreasing. Listing the number of boxes in each row gives a partition λ of a non-negative integer n, the total number of boxes in the diagram. In this study, Young diagrams will be drawn using the French notation with the longest row on the bottom and will be identified with the partition itself by considering a partition as a collection of cells. For example, the Young diagrams corresponding to the partitions of 4 are



(4) (3, 1) (2, 2) (2, 1, 1) (1, 1, 1, 1)

Since there is a clear one-to-one correspondence between partitions and Young diagrams, we use the two terms interchangeably, and we will use the Greek letter λ to denote them. A Young filling of λ assigns a positive integer to each box of λ, e.g.,

$$\begin{array}{lllll} 3 & 4 & & & \\ 2 & 2 & 3 & 6 & \\ 1 & 3 & 3 & 4 & 8 \end{array} .$$

A set of Young tableaux has n distinct entries, arbitrarily assigned to cells in the Young filling. A tableau is considered standard if the entries in each row and each column show an increasing trend and if it is a bijective assignment of $\{1, 2, ..., n\}$. A standard tableau filling of shape λ is a labeling of the cells of the standard Young diagram of λ with the numbers 1 to n. We denote the number of standard tableaux of shape λ by f^λ. For example, $f^{(3,2)} = 5$ since there are 5 standard tableaux of shape $(3, 2)$:

$$\begin{array}{lll} 4 & 5 & \\ 1 & 2 & 3 \end{array} \quad \begin{array}{lll} 3 & 5 & \\ 1 & 2 & 4 \end{array} \quad \begin{array}{lll} 3 & 4 & \\ 1 & 2 & 5 \end{array} \quad \begin{array}{lll} 2 & 4 & \\ 1 & 3 & 5 \end{array} \quad \begin{array}{lll} 2 & 5 & \\ 1 & 3 & 4 \end{array} .$$

The hook length of a cell $c = (i, j) \in \lambda$ is the number of cells weakly above and strictly to the right of c, i.e., the number of boxes directly to the right or above

c (including c itself). We denote this by $h_\lambda(c)$. The Frame-Robinson-Thrall hook-length formula states that if $\lambda \vdash n$, then the number of standard Young tableaux of shape λ is

$$f^\lambda = \frac{n!}{\prod_{c\in\lambda} h_\lambda(c)}.$$

If $\lambda = (3,2) \vdash 5$ then the hook lengths of λ are given in the diagram

$$\lambda = (3,2): \qquad \begin{array}{ccc} 2 & 1 & \\ 4 & 3 & 1 \end{array}.$$

Thus, for example,

$$f^{(3,2)} = \frac{5!}{2\cdot 1\cdot 4\cdot 3\cdot 1} = 5,$$

which agrees with our previous computation.

For a positive integer n, let $Y(n)$ be the number of standard Young tableaux with n cells. This say that $Y(n) = \sum_{\lambda\vdash n} f^\lambda$. For example, $Y(3) = f^{(3)} + f^{(2,1)} + f^{(1,1,1)} = 4$ since there are four standard Young tableaux:

$$\begin{array}{|c|} \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array} \qquad \begin{array}{|c|c} \hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array} \qquad \begin{array}{|c|c} \hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \qquad \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline \end{array} \quad .$$

The number of standard Young tableaux of size $1, 2, 3, ...$ are

$$1, 2, 4, 10, 26, 76, 232, 764, 2620, 9496, ...$$

The Robinson-Schensted-Knuth correspondence proves the fact that the number of standard Young tableaux of $\{1, 2, ..., n\}$ is equal to the number of involutions of order n (see [6] and [7]).

Lemma 2.1 (see [9, p. 324])**.** *$Y(n)$ is equal to the number of involutions. These numbers can be generated by the recurrence relation*

$$Y(n) = Y(n-1) + (n-1)Y(n-2)$$

with $Y(1) = 1$ and $Y(2) = 2$.

By convention, let $\mu_2(0) = Y(0) = 1$ and $Y(k) = 0$ for $k < 0$.

3. Touchard Polynomials

Let $\tau = (\tau_{i_1}, \ldots, \tau_{i_k})$ be a multiplicative plane partition of $\prod_{i=1}^{n} p_i$. Since any two factors of $\prod_{i=1}^{n} p_i$ are different, the number of the standard Young tableaux of set $\{\tau_{i_1}, \ldots, \tau_{i_k}\}$ is equal to $Y(k)$. There are five multiplicative partitions with product 30:

$$(30) \quad (15,2) \quad (10,3) \quad (6,5) \quad (5,3,2),$$

and there are eleven multiplicative plane partitions with product 30:

$$Y(1) = 1: \quad \begin{array}{|c|}\hline 30 \\ \hline\end{array};$$

$$Y(2) = 6: \quad \begin{array}{|c|c|}\hline 15 & 2 \\ \hline\end{array} \quad \begin{array}{|c|}\hline 2 \\ \hline 15 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 10 & 3 \\ \hline\end{array} \quad \begin{array}{|c|}\hline 3 \\ \hline 10 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 6 & 5 \\ \hline\end{array} \quad \begin{array}{|c|}\hline 5 \\ \hline 6 \\ \hline\end{array};$$

$$Y(3) = 4: \quad \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline 5 \\ \hline\end{array} \quad \begin{array}{|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 5 & 3 \\ \hline\end{array} \quad \begin{array}{|c|c|}\cline{1-1} 3 & \multicolumn{1}{c}{} \\ \hline 5 & 2 \\ \hline\end{array} \quad \begin{array}{|c|c|c|}\hline 5 & 3 & 2 \\ \hline\end{array}.$$

Lemma 3.1. *For a non-negative integer n, we have*

$$\mu_2(n) = \sum_{k=0}^{n} Y(k) \left\{ {n \atop k} \right\}.$$

Proof. For a multiplicative partition λ of n with exactly k parts, let $||\lambda|| = k$. If $n = 0$, then $\mu_2(0) = Y(0)S(0,0) = 1 \cdot 1 = 1$. Thus, we may assume $n \geq 1$. Because $\left\{ {n \atop k} \right\}$ is the number of ways of partitioning a set of n elements into k non-empty sets, we get

$$\begin{aligned}
\mu_2(n) &= \sum_{\lambda \vdash^* m} Y(||\lambda||) \\
&= \sum_{k=1}^{n} \sum_{\substack{\lambda \vdash^* m \\ ||\lambda|| = k}} Y(k) \\
&= \sum_{k=1}^{n} Y(k) \cdot \Big(\sum_{\substack{\lambda \vdash^* m \\ ||\lambda|| = k}} 1 \Big) \\
&= \sum_{k=1}^{n} Y(k) \left\{ {n \atop k} \right\} \\
&= \sum_{k=0}^{n} Y(k) \left\{ {n \atop k} \right\}.
\end{aligned}$$

□

The following lemma deals with the exponential generating function of a sequence $\left\{ {n \atop k} \right\}$.

Lemma 3.2 (see [10, Theorem 13.6])**.** *Let k be a non-negative integer. The exponential generating function of a sequence $\left\{ {n \atop k} \right\}$ is*

$$\sum_{n=k}^{\infty} \left\{ {n \atop k} \right\} \frac{x^n}{n!} = \frac{1}{k!} (e^x - 1)^k.$$

Exponential generating functions are particularly useful when building objects from an underlying set (the label set).

The following two lemmas show that the sequences $\mu_2(n)$ and $2^n B_n(1/2)$ have the same exponential generating function.

Lemma 3.3. *The exponential generating function of a sequence $\mu_2(n)$ is*

$$\Phi(x) = \sum_{m=0}^{\infty} \mu_2(m)\frac{x^m}{m!} = e^{(e^{2x}-1)/2}.$$

Proof. By Lemmas 3.1 and 3.2,

$$\begin{aligned}
\Phi(x) &= \sum_{m=0}^{\infty} \mu_2(m)\frac{x^m}{m!} \\
&= \sum_{m=0}^{\infty} \frac{1}{m!}\sum_{k=0}^{m} S(m,k)Y(k)x^m \\
&= \sum_{k=0}^{\infty} Y(k)\sum_{m=k}^{\infty} S(m,k)\frac{x^m}{m!} \\
&= \sum_{k=0}^{\infty} Y(k)\frac{(e^x-1)^k}{k!}\ .
\end{aligned}$$

If we differentiate both sides with respect to x, then from Lemma 2.1, we get

$$\begin{aligned}
\frac{d}{dx}\Phi(x) &= \sum_{k=1}^{\infty} Y(k)\frac{e^x(e^x-1)^{k-1}}{(k-1)!} \\
&= \sum_{k=1}^{\infty}(Y(k-1)+(k-1)Y(k-2))\frac{e^x(e^x-1)^{k-1}}{(k-1)!} \\
&= \left(\sum_{\ell=1}^{\infty} Y(\ell-1)\frac{e^x(e^x-1)^{\ell-1}}{(\ell-1)!}\right) + \left(\sum_{k=2}^{\infty}(k-1)Y(k-2)\frac{e^x(e^x-1)^{k-1}}{(k-1)!}\right) \\
&= e^x\Phi(x) + \sum_{k=2}^{\infty} Y(k-2)\frac{e^x(e^x-1)^{k-1}}{(k-2)!} \\
&= e^x\Phi(x) + e^x(e^x-1)\Phi(x) \\
&= e^{2x}\Phi(x).
\end{aligned}$$

Therefore, $\Phi(x) = e^{(e^{2x})/2}\dfrac{\mu_2(0)}{e^{1/2}} = e^{(e^{2x}-1)/2}$. □

The generating function of the Bell numbers (see [11]) is

$$\sum_{n=0}^{\infty} \frac{B_n(1)}{n!}t^n = e^{\left(e^t-1\right)}.$$

Similarly, we have

Lemma 3.4. *Let c be a positive real number. The exponential generating function of a sequence $c^nB_n(1/c)$ is*

$$\sum_{m=0}^{\infty} c^m\frac{B_m(1/c)}{m!}x^m = e^{(e^{cx}-1)/c}.$$

Proof. By Lemma 3.2,

$$\begin{aligned}
\sum_{m=0}^{\infty} c^m B_n(1/c)\frac{x^m}{m!} &= \sum_{m=0}^{\infty} \frac{(cx)^m}{m!} \sum_{k=0}^{m} S(m,k)\left(\frac{1}{c}\right)^k \\
&= \sum_{k=0}^{\infty} \frac{1}{c^k} \sum_{m=k}^{\infty} S(m,k)\frac{(cx)^m}{m!} \\
&= \sum_{k=0}^{\infty} \frac{1}{c^x}(e^{cx}-1)^k \frac{1}{k!} \\
&= \sum_{k=0}^{\infty} \left(\frac{e^{cx}-1}{c}\right)^k \frac{1}{k!} \\
&= e^{(e^{cx}-1)/c}.
\end{aligned}$$

□

From Lemmas 3.3 and 3.4, we derive the following theorem.

Theorem 3.5. *If m is a non-negative integer, then*

$$\mu_2(m) = 2^m B_m\left(\frac{1}{2}\right).$$

4. Concluding Remarks

In this section, we first describe how the values displayed in Table 1 are determined. A solid standard Young tableaux (SSYT) with n cells is a way of arranging the integers 1 through n as a three-dimensional Young diagram of a plane partition, with the entries increasing from left to right, back to front, and bottom to top. Recently, Ekhad and Zeilberger (see [8]), who first introduced SSYTs with n cells, investigated the number of SSYTs with n cells. They obtained the first thirty terms of the SSYTs with n cells. The first five terms of SSYT were found to be 1, 3, 9, 33, 135. Similar to Lemma 3.1 one can obtain $\mu_3(n) = \sum_{k=0}^{n} \hat{Y}(n)\left\{ {n \atop k} \right\}$, where $\hat{Y}(n)$ is the number of SSYTs with n cells. Accordingly, the values in Table 1 are thereby determined.

From Theorem 3.5, we see that for all positive integers n,

$$\begin{aligned}
\mu_1(n) &= \mathcal{B}_n = 1^n T_n(1/1), \\
\mu_2(n) &= 2^n T_n(1/2).
\end{aligned}$$

This would initially lead us to believe that comparably interesting discoveries are awaited for high-dimensional multiplicative partitions. We guess $\mu_k(n) = k^n B_n(1/k)$ for all positive k. However, Table 1 demonstrates that such hopes are in vain. Nevertheless, we anticipate that it can still help us deduce the correct leading asymptotic behavior.

We now state an additional problem.

We see from the table that $\mu_3(n) \le 3^n B_n(1/3)$ for $n \le 13$. Does this condition hold for all positive $n > 13$?

n	$\mu_2(n)$	$\mu_3(n)$	$3^n B_n(1/3)$
1	1	1	1
2	3	4	4
3	11	19	19
4	49	109	109
5	257	736	742
6	1539	5707	5815
7	10299	49849	51193
8	75905	482722	498118
9	609441	5115169	5296321
10	5284451	58704493	60987817
11	49134923	723833920	754940848
12	487026929	9528008713	9983845261
13	5120905441	133208376217	140329768789

TABLE 1. $\mu_2(n)$, $\mu_3(n)$, and $3^n B_n(1/3)$ for $1 \le n \le 13$

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