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GROUP S_3 CORDIAL REMAINDER LABELING FOR PATH AND CYCLE RELATED GRAPHS

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Abstract. Let $G = (V(G), E(G))$ be a graph and let $g : V(G) \to S_3$ be a function. For each edge xy assign the label r where r is the remainder when $o(g(x))$ is divided by $o(g(y))$ or $o(g(y))$ is divided by $o(g(x))$ according as $o(g(x)) \geq o(g(y))$ or $o(g(y)) \geq o(g(x))$. The function g is called a group S_3 cordial remainder labeling of G if $|v_g(i) - v_g(j)| \leq 1$ and $|e_g(1) - e_g(0)| \leq 1$, where $v_g(j)$ denotes the number of vertices labeled with j and $e_g(i)$ denotes the number of edges labeled with i $(i = 0, 1)$. A graph G which admits a group S_3 cordial remainder labeling is called a group S_3 cordial remainder graph. In this paper, we prove that square of the path, duplication of a vertex by a new edge in path and cycle graphs, duplication of an edge by a new vertex in path and cycle graphs and total graph of cycle and path graphs admit a group S_3 cordial remainder labeling.

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1. Introduction

All graphs considered here are finite, simple and undirected. The vertex set and the edge set of a graph G are denoted by $V(G)$ and $E(G)$ so that the order and size of G are $|V(G)|$ and $|E(G)|$ respectively. Terms not defined here are taken from Harary [3]. Graph labeling was first introduced in 1960's. Most of the graph labeling trace their origins in the paper presented by Alex Rosa in 1967 [10]. A labeling of a graph is a map that carries the graph elements to the set of numbers, usually to the set of non-negative or positive integers. If the domain is the set of vertices then the labeling is called vertex labeling. If the domain is the set of edges then the labeling is called edge labeling. If the labels are assigned to both vertices and edges then the labeling is called total labeling. The complete survey of graph labeling is in [2]. Cordial labeling is

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a weaker version of graceful labeling and harmonious labeling introduced by I. Cahit in [1]. Let f be a function from the vertices of G to $\{0,1\}$ and for each edge xy assign the label $|f(x)-f(y)|$. f is called a cordial labeling of G if $|v_f(0)-v_f(1)| \leq 1$ and $|e_f(0)-e_f(1)| \leq 1$. Lourdusamy et al. [5] introduced the concept of group S_3 cordial remainder labeling and they proved that path, cycle, star, bistar, complete bipartite, wheel, fan, comb and crown graphs are group S_3 cordial remainder graphs. In [6, 7, 8], Lourdusamy et al. discussed the behaviour of group S_3 cordial remainder labeling of helm, flower, closed helm, gear, sunflower, triangular snake, quadrilateral snake, lotus inside a circle, double fan, ladder, slanting ladder, triangular ladder, subdivision of star, subdivision of bistar, subdivision of wheel, subdivision of comb, subdivision of crown, subdivision of fan and subdivision of ladder. In [4], Jenifer et al. proved that shadow graph of cycle and path, splitting graph of cycle, armed crown, umbrella graph and dumbbell graph admit a group S_3 cordial remainder labeling. Also they proved that snake related graphs are a group S_3 cordial remainder graphs.

Definition 1.1. Let A be a group. The order of $a \in A$ is the least positive integer n such that $a^n = e$. We denote the order of a by $o(a)$.

Definition 1.2. Consider the symmetric group S_3. Let the elements of S_3 be e, a, b, c, d, f where

$$e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \quad a = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \quad b = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$
$$c = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \quad d = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad f = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

We have $o(e) = 1$, $o(a) = o(b) = o(c) = 2$, $o(d) = o(f) = 3$.

Definition 1.3. Let $G = (V(G), E(G))$ be a graph and let $g : V(G) \to S_3$ be a function. For each edge xy assign the label r where r is the remainder when $o(g(x))$ is divided by $o(g(y))$ or $o(g(y))$ is divided by $o(g(x))$ according as $o(g(x)) \geq o(g(y))$ or $o(g(y)) \geq o(g(x))$. The function g is called a group S_3 cordial remainder labeling of G if $|v_g(i) - v_g(j)| \leq 1$ and $|e_g(1) - e_g(0)| \leq 1$, where $v_g(j)$ denotes the number of vertices labeled with j and $e_g(i)$ denotes the number of edges labeled with i $(i = 0, 1)$. A graph G which admits a group S_3 cordial remainder labeling is called a group S_3 cordial remainder graph.

In this paper, we prove that square of the path, duplication of a vertex by a new edge in path and cycle graphs, duplication of an edge by a new vertex in path and cycle graphs and total graph of cycle and path graphs admit a group S_3 cordial remainder labeling.

We use the following definitions in the subsequent sections.

Definition 1.4. For a simple connected graph G the square of graph G is denoted by G^2 and defined as the graph with the same vertex set as of G and two vertices are adjacent in G^2 if they are at a distance 1 or 2 apart in G.

Definition 1.5. [9] Duplication of a vertex u by a new edge $e = vw$ in a graph G produces a new graph $G^{'}$ such that $N(v) \cap N(w) = u$.

Definition 1.6. [9] Duplication of an edge $e = uv$ by a new vertex w in a graph G produces a new graph $G^{'}$ such that $N(w) = \{u, v\}$.

Definition 1.7. [9] The total graph $T(G)$ of a graph G is the graph whose vertex set is $V(G) \bigcup E(G)$ and two vertices are adjacent whenever they are either adjacent or incident in G.

2. Main results

Theorem 2.1. *P_n^2 is a group S_3 cordial remainder graph for every n.*

Proof. Let $v_1, v_2, \cdots, v_n$ be the vertices of the path P_n. Let $E(P_n^2) = \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{v_i v_{i+2} : 1 \leq i \leq n-2\}$. Then P_n^2 is of order n and size $2n-3$. Define $g : V(P_n^2) \to S_3$ as follows:
Case 1. n is odd.

$$g(v_i) = \begin{cases} e & \text{if } i \equiv 1 \ (mod\ 6) \text{ and } 1 \leq i \leq n \\ b & \text{if } i \equiv 2 \ (mod\ 6) \text{ and } 1 \leq i \leq n \\ d & \text{if } i \equiv 3 \ (mod\ 6) \text{ and } 1 \leq i \leq n \\ c & \text{if } i \equiv 4 \ (mod\ 6) \text{ and } 1 \leq i \leq n \\ a & \text{if } i \equiv 5 \ (mod\ 6) \text{ and } 1 \leq i \leq n \\ f & \text{if } i \equiv 0 \ (mod\ 6) \text{ and } 1 \leq i \leq n. \end{cases}$$

It is easy to verify that $|v_g(i) - v_g(j)| \leq 1$ for $i, j \in S_3$ and $|e_g(0) - e_g(1)| \leq 1$. Therefore g is a group S_3 cordial remainder labeling.
Case 2. n is even.

$$g(v_i) = \begin{cases} e & \text{if } i \equiv 1 \ (mod\ 6) \text{ and } 1 \leq i \leq n \\ d & \text{if } i \equiv 2 \ (mod\ 6) \text{ and } 1 \leq i \leq n \\ b & \text{if } i \equiv 3 \ (mod\ 6) \text{ and } 1 \leq i \leq n \\ c & \text{if } i \equiv 4 \ (mod\ 6) \text{ and } 1 \leq i \leq n \\ f & \text{if } i \equiv 5 \ (mod\ 6) \text{ and } 1 \leq i \leq n \\ a & \text{if } i \equiv 0 \ (mod\ 6) \text{ and } 1 \leq i \leq n. \end{cases}$$

It is easy to verify that $|v_g(i) - v_g(j)| \leq 1$ for $i, j \in S_3$ and $|e_g(0) - e_g(1)| \leq 1$. Therefore g is a group S_3 cordial remainder labeling.

Hence, P_n^2 is a group S_3 cordial remainder graph for every n. □

Example 2.2. A group S_3 cordial remainder labeling of P_7^2 is given in FIGURE 1.

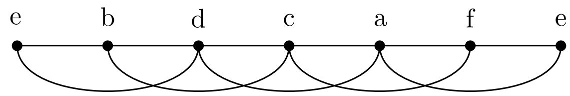

Figure 1

Theorem 2.3. *The graph obtained by duplication of each vertex by an edge in path P_n is a group S_3 cordial remainder graph.*

Proof. Let $V(G) = \{u_i, v_i, w_i : 1 \leq i \leq n\}$ and $E(G) = \{u_iv_i, u_iw_i, v_iw_i : 1 \leq i \leq n\} \bigcup \{u_iu_{i+1} : 1 \leq i \leq n-1\}$. Therefore G is of order $3n$ and size $4n - 1$. Define $g : V(G) \to S_3$ as follows:

$$g(u_i) = \begin{cases} a & \text{if } i \equiv 1 \ (mod\ 4) \text{ and } 1 \leq i \leq n \\ b & \text{if } i \equiv 2 \ (mod\ 4) \text{ and } 1 \leq i \leq n \\ d & \text{if } i \equiv 3 \ (mod\ 4) \text{ and } 1 \leq i \leq n \\ f & \text{if } i \equiv 0 \ (mod\ 4) \text{ and } 1 \leq i \leq n \ ; \end{cases}$$

$$g(v_i) = \begin{cases} e & \text{if } i \equiv 1 \ (mod\ 4) \text{ and } 1 \leq i \leq n \\ c & \text{if } i \equiv 2 \ (mod\ 4) \text{ and } 1 \leq i \leq n \\ a & \text{if } i \equiv 3 \ (mod\ 4) \text{ and } 1 \leq i \leq n \\ b & \text{if } i \equiv 0 \ (mod\ 4) \text{ and } 1 \leq i \leq n \ ; \end{cases}$$

$$g(w_i) = \begin{cases} d & \text{if } i \equiv 1 \ (mod\ 4) \text{ and } 1 \leq i \leq n \\ f & \text{if } i \equiv 2 \ (mod\ 4) \text{ and } 1 \leq i \leq n \\ e & \text{if } i \equiv 3 \ (mod\ 4) \text{ and } 1 \leq i \leq n \\ c & \text{if } i \equiv 0 \ (mod\ 4) \text{ and } 1 \leq i \leq n \ . \end{cases}$$

Nature of n	$v_g(a)$	$v_g(b)$	$v_g(c)$	$v_g(d)$	$v_g(e)$	$v_g(f)$	$e_g(0)$	$e_g(1)$
$2k-1$ $(k \geq 1)$	k	$k-1$	$k-1$	k	k	$k-1$	$4k-2$	$4k-3$
$2k$ $(k \geq 1)$	k	k	k	k	k	k	$4k$	$4k-1$

Table 1

From TABLE 1, it is easy to verify that $|v_g(i) - v_g(j)| \leq 1$ for $i, j \in S_3$ and $|e_g(0) - e_g(1)| \leq 1$. Therefore g is a group S_3 cordial remainder labeling.

□

Theorem 2.4. *The graph obtained by duplication of each vertex by an edge in cycle C_n is a group S_3 cordial remainder graph.*

Proof. Let $V(G) = \{u_i, v_i, w_i : 1 \leq i \leq n\}$ and $E(G) = E(C_n) \bigcup \{u_iv_i, u_iw_i, v_iw_i : 1 \leq i \leq n\}$. Therefore, G is of order $3n$ and size $4n$. Define $g : V(G) \to S_3$ as follows:

Case 1. $n = 3$.

$$g(u_i)=\begin{cases} a & \text{if } i=1\\ b & \text{if } i=2\\ d & \text{if } i=3; \end{cases} \qquad g(v_i)=\begin{cases} e & \text{if } i=1\\ c & \text{if } i=2\\ a & \text{if } i=3; \end{cases}$$

$$g(w_i)=\begin{cases} d & \text{if } i=1\\ f & \text{if } i=2\\ e & \text{if } i=3. \end{cases}$$

Here we have $v_g(b)=v_g(c)=v_g(f)=1, v_g(a)=v_g(d)=v_g(e)=2$ and $e_g(0)=e_g(1)=6$. Therefore $|v_g(i)-v_g(j)|\leq 1$ for $i,j\in S_3$ and $|e_g(0)-e_g(1)|\leq 1$.

Case 2. $n\geq 4$.

Subcase 2.1. $n\equiv 0\ (mod\ 4)$.

Let $n=4k$ and $k\geq 1$.

$$g(u_i)=\begin{cases} a & \text{if } i\equiv 1\ (mod\ 4) \text{ and } 1\leq i\leq 4k\\ b & \text{if } i\equiv 2\ (mod\ 4) \text{ and } 1\leq i\leq 4k\\ d & \text{if } i\equiv 3\ (mod\ 4) \text{ and } 1\leq i\leq 4k\\ f & \text{if } i\equiv 0\ (mod\ 4) \text{ and } 1\leq i\leq 4k\ ; \end{cases}$$

$$g(v_i)=\begin{cases} e & \text{if } i\equiv 1\ (mod\ 4) \text{ and } 1\leq i\leq 4k\\ c & \text{if } i\equiv 2\ (mod\ 4) \text{ and } 1\leq i\leq 4k\\ a & \text{if } i\equiv 3\ (mod\ 4) \text{ and } 1\leq i\leq 4k\\ b & \text{if } i\equiv 0\ (mod\ 4) \text{ and } 1\leq i\leq 4k\ ; \end{cases}$$

$$g(w_i)=\begin{cases} d & \text{if } i\equiv 1\ (mod\ 4) \text{ and } 1\leq i\leq 4k\\ f & \text{if } i\equiv 2\ (mod\ 4) \text{ and } 1\leq i\leq 4k\\ e & \text{if } i\equiv 3\ (mod\ 4) \text{ and } 1\leq i\leq 4k\\ c & \text{if } i\equiv 0\ (mod\ 4) \text{ and } 1\leq i\leq 4k\ . \end{cases}$$

Here we have $v_g(a)=v_g(b)=v_g(c)=v_g(d)=v_g(e)=v_g(f)=2k$ and $e_g(0)=e_g(1)=8k$. Therefore $|v_g(i)-v_g(j)|\leq 1$ for $i,j\in S_3$ and $|e_g(0)-e_g(1)|\leq 1$.

Subcase 2.2. $n\equiv 3\ (mod\ 4)$.

Let $n=4k+3$ and $k\geq 1$. We assign the labels to the vertices u_i, v_i and w_i for $1\leq i\leq 4k$ as in Subcase (2.1) and for the remaining vertices assign the following labels:

$$g(u_i)=\begin{cases} a & \text{if } i=4k+1\\ b & \text{if } i=4k+2\\ d & \text{if } i=4k+3; \end{cases} \qquad g(v_i)=\begin{cases} e & \text{if } i=4k+1\\ c & \text{if } i=4k+2\\ a & \text{if } i=4k+3; \end{cases}$$

$$g(w_i)=\begin{cases} d & \text{if } i=4k+1\\ f & \text{if } i=4k+2\\ e & \text{if } i=4k+3. \end{cases}$$

Here we have $v_g(b)=v_g(c)=v_g(f)=2k+1, v_g(a)=v_g(d)=v_g(e)=2k+2$ and $e_g(0)=e_g(1)=8k+6$. Therefore $|v_g(i)-v_g(j)|\leq 1$ for $i,j\in S_3$ and $|e_g(0)-e_g(1)|\leq 1$.

Subcase 2.3. $n\equiv 2\ (mod\ 4)$.

Let $n = 4k + 2$ and $k \geq 1$. We assign the labels to the vertices u_i, v_i and w_i for $1 \leq i \leq 4k$ as in Subcase (2.1) and for the remaining vertices assign the following labels:

$$g(u_i) = \begin{cases} b & \text{if } i = 4k + 1 \\ d & \text{if } i = 4k + 2; \end{cases} \qquad g(v_i) = \begin{cases} a & \text{if } i = 4k + 1 \\ f & \text{if } i = 4k + 2; \end{cases}$$

$$g(w_i) = \begin{cases} e & \text{if } i = 4k + 1 \\ c & \text{if } i = 4k + 2. \end{cases}$$

Here we have $v_g(a) = v_g(b) = v_g(c) = v_g(d) = v_g(e) = v_g(f) = 2k + 1$ and $e_g(0) = e_g(1) = 8k + 4$. Therefore $|v_g(i) - v_g(j)| \leq 1$ for $i, j \in S_3$ and $|e_g(0) - e_g(1)| \leq 1$.

Subcase 2.4. $n \equiv 1 \ (mod\ 4)$.

Let $n = 4k+1$ and $k \geq 1$. We assign the labels to the vertices u_i, v_i and w_i for $1 \leq i \leq 4k$ as in Subcase (2.1), except for the vertices $u_{4k+1}, v_{4k+1}, w_{4k+1}$ are labeled by f, b, c respectively. Here we have $v_g(b) = v_g(c) = v_g(f) = 2k+1, v_g(a) = v_g(d) = v_g(e) = 2k$ and $e_g(0) = e_g(1) = 8k + 2$. Therefore $|v_g(i) - v_g(j)| \leq 1$ for $i, j \in S_3$ and $|e_g(0) - e_g(1)| \leq 1$.

It is easy to verify that $|v_g(i) - v_g(j)| \leq 1$ for $i, j \in S_3$ and $|e_g(0) - e_g(1)| \leq 1$. Therefore g is a group S_3 cordial remainder labeling. □

Example 2.5. A group S_3 cordial remainder labeling of the graph obtained by duplication of each vertex by an edge in cycle C_7 is given in FIGURE 2.

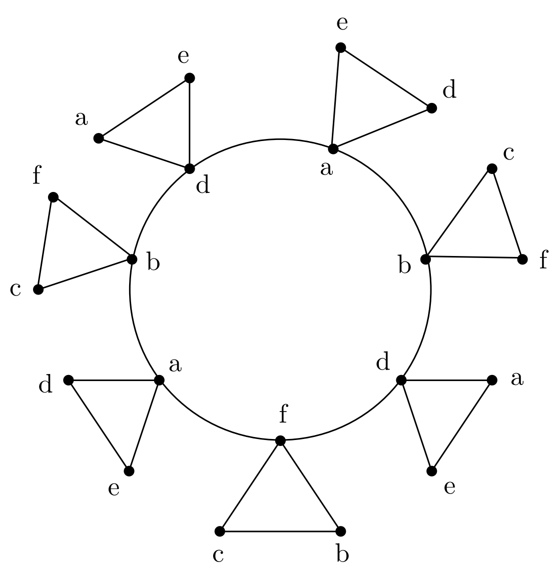

FIGURE 2

Theorem 2.6. *The graph obtained by duplication of each edge of cycle C_n by a vertex is a group S_3 cordial remainder graph.*

Proof. Let $u_1, u_2, \cdots, u_n$ be the vertices of cycle C_n and G be the graph obtained by duplication of each edge u_iu_{i+1} and u_nu_1 of cycle C_n by vertex $v_i (1 \leq i \leq n)$. Then $V(G) = \{u_i, v_i : 1, 2, \cdots, n\}$ and $E(G) = \{u_iu_{i+1}, u_{i+1}v_i : 1 \leq i \leq n - 1\} \bigcup \{u_iv_i : 1 \leq i \leq n\} \bigcup \{u_nu_1, v_nu_1\}$. Clearly $|V(G)| = 2n$ and $|E(G)| = 3n$. Define $g : V(G) \to S_3$ as follows:

Case 1. $n = 3$.

$$g(u_i) = \begin{cases} a & \text{if } i = 1 \\ d & \text{if } i = 2 \\ e & \text{if } i = 3 \ ; \end{cases} \qquad g(v_i) = \begin{cases} c & \text{if } i = 1 \\ d & \text{if } i = 2 \\ f & \text{if } i = 3 \ . \end{cases}$$

Here we have $v_g(a) = v_g(b) = v_g(c) = v_g(d) = v_g(e) = v_g(f) = 1$ and $e_g(0) = 5, e_g(1) = 4$. Therefore $|v_g(i) - v_g(j)| \leq 1$ for $i, j \in S_3$ and $|e_g(0) - e_g(1)| \leq 1$.

Case 2. $n = 4$.

$$g(u_i) = \begin{cases} a & \text{if } i = 1 \\ b & \text{if } i = 2 \\ d & \text{if } i = 3 \\ f & \text{if } i = 4 \ ; \end{cases} \qquad g(v_i) = \begin{cases} e & \text{if } i = 1 \\ b & \text{if } i = 2 \\ c & \text{if } i = 3 \\ a & \text{if } i = 4 \ . \end{cases}$$

Here we have $v_g(a) = v_g(b) = 2, v_g(c) = v_g(d) = v_g(e) = v_g(f) = 1$ and $e_g(0) = 6, e_g(1) = 6$. Therefore $|v_g(i) - v_g(j)| \leq 1$ for $i, j \in S_3$ and $|e_g(0) - e_g(1)| \leq 1$.

Case 3. $n = 5$.

$$g(u_i) = \begin{cases} a & \text{if } i = 1 \\ d & \text{if } i = 2 \\ b & \text{if } i = 3 \\ c & \text{if } i = 4 \\ e & \text{if } i = 5 \ ; \end{cases} \qquad g(v_i) = \begin{cases} a & \text{if } i = 1 \\ c & \text{if } i = 2 \\ d & \text{if } i = 3 \\ b & \text{if } i = 4 \\ f & \text{if } i = 5 \ . \end{cases}$$

Here we have $v_g(a) = v_g(b) = v_g(c) = v_g(d) = 2, v_g(e) = v_g(f) = 1$ and $e_g(0) = 8, e_g(1) = 7$. Therefore $|v_g(i) - v_g(j)| \leq 1$ for $i, j \in S_3$ and $|e_g(0) - e_g(1)| \leq 1$.

Case 4. $n \geq 6$.

Subcase 4.1. $n \equiv 0 \ (mod\ 6)$.

Let $n = 6k$ and $k \geq 1$.

$$g(u_i) = \begin{cases} a & \text{if } i \equiv 1 \ (mod\ 6) \text{ and } 1 \leq i \leq 6k \\ d & \text{if } i \equiv 2 \ (mod\ 6) \text{ and } 1 \leq i \leq 6k \\ b & \text{if } i \equiv 3 \ (mod\ 6) \text{ and } 1 \leq i \leq 6k \\ c & \text{if } i \equiv 4 \ (mod\ 6) \text{ and } 1 \leq i \leq 6k \\ e & \text{if } i \equiv 5 \ (mod\ 6) \text{ and } 1 \leq i \leq 6k \\ f & \text{if } i \equiv 0 \ (mod\ 6) \text{ and } 1 \leq i \leq 6k \ ; \end{cases}$$

$$g(v_i) = \begin{cases} a & \text{if } i \equiv 1 \ (mod\ 6) \text{ and } 1 \le i \le 6k \\ c & \text{if } i \equiv 2 \ (mod\ 6) \text{ and } 1 \le i \le 6k \\ d & \text{if } i \equiv 3 \ (mod\ 6) \text{ and } 1 \le i \le 6k \\ f & \text{if } i \equiv 4 \ (mod\ 6) \text{ and } 1 \le i \le 6k \\ e & \text{if } i \equiv 5 \ (mod\ 6) \text{ and } 1 \le i \le 6k \\ b & \text{if } i \equiv 0 \ (mod\ 6) \text{ and } 1 \le i \le 6k\ . \end{cases}$$

Here we have $v_g(a) = v_g(b) = v_g(c) = v_g(d) = v_g(e) = v_g(f) = 2k$ and $e_g(0) = e_g(1) = 9k$. Therefore $|v_g(i) - v_g(j)| \le 1$ for $i, j \in S_3$ and $|e_g(0) - e_g(1)| \le 1$.

Subcase 4.2. $n \equiv 5\ (mod\ 6)$.

Let $n = 6k + 5$ and $k \ge 1$. We assign the labels to the vertices u_i, v_i for $1 \le i \le 6k$ as in Subcase (4.1) and for the remaining vertices assign the following labels:

$$g(u_i) = \begin{cases} a & \text{if } i = 6k+1 \\ d & \text{if } i = 6k+2 \\ b & \text{if } i = 6k+3 \\ c & \text{if } i = 6k+4 \\ e & \text{if } i = 6k+5\ ; \end{cases} \qquad g(v_i) = \begin{cases} a & \text{if } i = 6k+1 \\ c & \text{if } i = 6k+2 \\ d & \text{if } i = 6k+3 \\ b & \text{if } i = 6k+4 \\ f & \text{if } i = 6k+5\ . \end{cases}$$

Here we have $v_g(a) = v_g(b) = v_g(c) = v_g(d) = 2k + 2, v_g(e) = v_g(f) = 2k + 1$ and $e_g(0) = 9k + 8, e_g(1) = 9k + 7$. Therefore $|v_g(i) - v_g(j)| \le 1$ for $i, j \in S_3$ and $|e_g(0) - e_g(1)| \le 1$.

Subcase 4.3. $n \equiv 4\ (mod\ 6)$.

Let $n = 6k + 4$ and $k \ge 1$. We assign the labels to the vertices u_i, v_i for $1 \le i \le 6k$ as in Subcase (4.1) and for the remaining vertices assign the following labels:

$$g(u_i) = \begin{cases} a & \text{if } i = 6k+1 \\ b & \text{if } i = 6k+2 \\ d & \text{if } i = 6k+3 \\ f & \text{if } i = 6k+4\ ; \end{cases} \qquad g(v_i) = \begin{cases} e & \text{if } i = 6k+1 \\ b & \text{if } i = 6k+2 \\ c & \text{if } i = 6k+3 \\ a & \text{if } i = 6k+4\ . \end{cases}$$

Here we have $v_g(a) = v_g(b) = 2k + 2, v_g(c) = v_g(d) = v_g(e) = v_g(f) = 2k + 1$ and $e_g(0) = 9k + 6, e_g(1) = 9k + 6$. Therefore $|v_g(i) - v_g(j)| \le 1$ for $i, j \in S_3$ and $|e_g(0) - e_g(1)| \le 1$.

Subcase 4.4. $n \equiv 3\ (mod\ 6)$.

Let $n = 6k + 3$ and $k \ge 1$. We assign the labels to the vertices u_i, v_i for $1 \le i \le 6k$ as in Subcase (4.1) and for the remaining vertices assign the following labels:

$$g(u_i) = \begin{cases} a & \text{if } i = 6k+1 \\ d & \text{if } i = 6k+2 \\ e & \text{if } i = 6k+3\ ; \end{cases} \qquad g(v_i) = \begin{cases} c & \text{if } i = 6k+1 \\ d & \text{if } i = 6k+2 \\ f & \text{if } i = 6k+3\ . \end{cases}$$

Here we have $v_g(a) = v_g(b) = v_g(c) = v_g(d) = v_g(e) = v_g(f) = 2k + 1$ and $e_g(0) = 9k + 5, e_g(1) = 9k + 4$. Therefore $|v_g(i) - v_g(j)| \le 1$ for $i, j \in S_3$ and

$|e_g(0) - e_g(1)| \leq 1$.

Subcase 4.5. $n \equiv 2\ (mod\ 6)$.

Let $n = 6k + 2$ and $k \geq 1$.

$$g(u_i) = \begin{cases} a & \text{if } i \equiv 1\ (mod\ 6) \text{ and } 1 \leq i \leq 6k \\ d & \text{if } i \equiv 2\ (mod\ 6) \text{ and } 1 \leq i \leq 6k \\ b & \text{if } i \equiv 3\ (mod\ 6) \text{ and } 1 \leq i \leq 6k \\ c & \text{if } i \equiv 4\ (mod\ 6) \text{ and } 1 \leq i \leq 6k \\ f & \text{if } i \equiv 5\ (mod\ 6) \text{ and } 1 \leq i \leq 6k \\ e & \text{if } i \equiv 0\ (mod\ 6) \text{ and } 1 \leq i \leq 6k; \end{cases}$$

$$g(v_i) = \begin{cases} a & \text{if } i \equiv 1\ (mod\ 6) \text{ and } 1 \leq i \leq 6k \\ c & \text{if } i \equiv 2\ (mod\ 6) \text{ and } 1 \leq i \leq 6k \\ d & \text{if } i \equiv 3\ (mod\ 6) \text{ and } 1 \leq i \leq 6k \\ f & \text{if } i \equiv 4\ (mod\ 6) \text{ and } 1 \leq i \leq 6k \\ b & \text{if } i \equiv 5\ (mod\ 6) \text{ and } 1 \leq i \leq 6k \\ e & \text{if } i \equiv 0\ (mod\ 6) \text{ and } 1 \leq i \leq 6k; \end{cases}$$

for the remaining vertices assign the following labels:

$$g(u_i) = \begin{cases} f & \text{if } i = 6k+1 \\ a & \text{if } i = 6k+2; \end{cases} \qquad g(v_i) = \begin{cases} e & \text{if } i = 6k+1 \\ d & \text{if } i = 6k+2. \end{cases}$$

Here we have $v_g(b) = v_g(c) = 2k, v_g(a) = v_g(d) = v_g(e) = v_g(f) = 2k + 1$ and $e_g(0) = 9k + 3, e_g(1) = 9k + 3$. Therefore $|v_g(i) - v_g(j)| \leq 1$ for $i, j \in S_3$ and $|e_g(0) - e_g(1)| \leq 1$.

Subcase 4.6. $n \equiv 1\ (mod\ 6)$.

Let $n = 6k + 1$ and $k \geq 1$. We assign the labels to the vertices u_i, v_i for $1 \leq i \leq 6k$ as in Subcase (4.5), except for the two vertices u_{6k+1}, v_{6k+1} are labeled by f, a respectively. Here we have $v_g(b) = v_g(c) = v_g(d) = v_g(e) = 2k, v_g(a) = v_g(f) = 2k + 1$ and $e_g(0) = 9k + 1, e_g(1) = 9k + 2$. Therefore $|v_g(i) - v_g(j)| \leq 1$ for $i, j \in S_3$ and $|e_g(0) - e_g(1)| \leq 1$.

Hence g is a group S_3 cordial remainder labeling. □

Example 2.7. A group S_3 cordial remainder labeling of the graph obtained by duplication of each edge of cycle C_8 by a vertex is given in FIGURE 3.

Corollary 2.8. *The graph obtained by duplication of each edge of path P_n by a vertex is a group S_3 cordial remainder graph.*

Theorem 2.9. *The total graph of path $T(P_n)$ is a group S_3 cordial remainder graph for every n.*

Proof. Let $V(T(P_n)) = \{v_i : 1 \leq i \leq n\} \bigcup \{u_i : 1 \leq i \leq n-1\}$ and $E(T(P_n)) = \{u_iu_{i+1} : 1 \leq i \leq n-2\} \bigcup \{v_iu_{i-1} : 2 \leq i \leq n\} \bigcup \{v_iv_{i+1}, v_iu_i : 1 \leq i \leq n-1\}$. Then $T(P_n)$ is of order $2n - 1$ and size $4n - 5$. Define $g : V(T(P_n)) \to S_3$ is as follows:

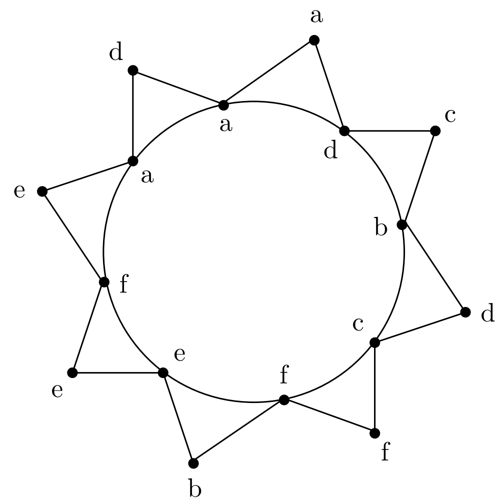

Figure 3

$$g(v_i) = \begin{cases} a & \text{if } i \equiv 1 \ (mod\ 6) \text{ and } 1 \leq i \leq n \\ b & \text{if } i \equiv 2 \ (mod\ 6) \text{ and } 1 \leq i \leq n \\ f & \text{if } i \equiv 3 \ (mod\ 6) \text{ and } 1 \leq i \leq n \\ c & \text{if } i \equiv 4 \ (mod\ 6) \text{ and } 1 \leq i \leq n \\ e & \text{if } i \equiv 5 \ (mod\ 6) \text{ and } 1 \leq i \leq n \\ d & \text{if } i \equiv 0 \ (mod\ 6) \text{ and } 1 \leq i \leq n \ ; \end{cases}$$

$$g(u_i) = \begin{cases} d & \text{if } i \equiv 1 \ (mod\ 6) \text{ and } 1 \leq i \leq n-1 \\ e & \text{if } i \equiv 2 \ (mod\ 6) \text{ and } 1 \leq i \leq n-1 \\ a & \text{if } i \equiv 3 \ (mod\ 6) \text{ and } 1 \leq i \leq n-1 \\ f & \text{if } i \equiv 4 \ (mod\ 6) \text{ and } 1 \leq i \leq n-1 \\ b & \text{if } i \equiv 5 \ (mod\ 6) \text{ and } 1 \leq i \leq n-1 \\ c & \text{if } i \equiv 0 \ (mod\ 6) \text{ and } 1 \leq i \leq n-1 \ . \end{cases}$$

Nature of n	$v_g(a)$	$v_g(b)$	$v_g(c)$	$v_g(d)$	$v_g(e)$	$v_g(f)$
$6k+1$ $(k \geq 0)$	$2k+1$	$2k$	$2k$	$2k$	$2k$	$2k$
$6k+2$ $(k \geq 0)$	$2k+1$	$2k+1$	$2k$	$2k+1$	$2k$	$2k$
$6k+3$ $(k \geq 0)$	$2k+1$	$2k+1$	$2k$	$2k+1$	$2k+1$	$2k+1$
$6k+4$ $(k \geq 0)$	$2k+2$	$2k+1$	$2k+1$	$2k+1$	$2k+1$	$2k+1$
$6k+5$ $(k \geq 0)$	$2k+2$	$2k+1$	$2k+1$	$2k+1$	$2k+2$	$2k+2$
$6k$ $(k \geq 1)$	$2k$	$2k$	$2k$	$2k$	$2k$	$2k$

Table 2

Nature of n	$e_g(0)$	$e_g(1)$
$6k+1$ $(k \geq 0)$	$12k$	$12k-1$
$6k+2$ $(k \geq 0)$	$12k+1$	$12k+2$
$6k+3$ $(k \geq 0)$	$12k+4$	$12k+3$
$6k+4$ $(k \geq 0)$	$12k+6$	$12k+5$
$6k+5$ $(k \geq 0)$	$12k+8$	$12k+7$
$6k$ $(k \geq 1)$	$12k-2$	$12k-3$

TABLE 3

From TABLE 2 and TABLE 3, it is easy to verify that $|v_g(i) - v_g(j)| \leq 1$ for $i, j \in S_3$ and $|e_g(0) - e_g(1)| \leq 1$. Therefore g is a group S_3 cordial remainder labeling. Hence $T(P_n)$ is a group S_3 cordial remainder graph for every n. □

Example 2.10. A group S_3 cordial remainder labeling of $T(P_6)$ is given in FIGURE 4.

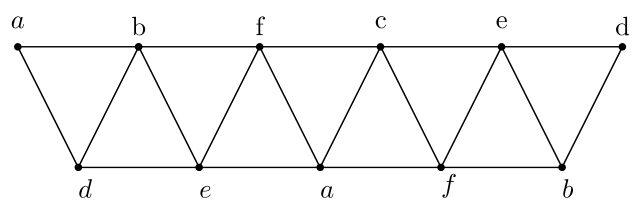

FIGURE 4

Theorem 2.11. *The total graph of cycle $T(C_n)$ is a group S_3 cordial remainder graph for $n \geq 3$.*

Proof. Let $V(T(C_n)) = \{v_i, u_i : 1 \leq i \leq n\}$ and $E(T(C_n)) = \{v_i v_{i+1}, u_i u_{i+1} : 1 \leq i \leq n-1\} \bigcup \{v_i u_i : 1 \leq i \leq n\} \bigcup \{v_i u_{i-1} : 2 \leq i \leq n\} \bigcup \{v_n v_1, u_n u_1, v_1 u_n\}$. Then $T(C_n)$ is of order $2n$ and size $4n$. Define $g : V(T(C_n)) \to S_3$ is as follows:
Case 1. $n = 3$.

$$g(v_i) = \begin{cases} a & \text{if } i = 1 \\ b & \text{if } i = 2 \\ f & \text{if } i = 3\ ; \end{cases} \qquad g(u_i) = \begin{cases} d & \text{if } i = 1 \\ e & \text{if } i = 2 \\ c & \text{if } i = 3\ . \end{cases}$$

Here we have $v_g(a) = v_g(b) = v_g(c) = v_g(d) = v_g(e) = v_g(f) = 1$ and $e_g(0) = e_g(1) = 6$. Therefore $|v_g(i) - v_g(j)| \leq 1$ for $i, j \in S_3$ and $|e_g(0) - e_g(1)| \leq 1$.
Case 2. $n = 4$.

$$g(v_i) = \begin{cases} a & \text{if } i = 1 \\ b & \text{if } i = 2 \\ f & \text{if } i = 3 \\ c & \text{if } i = 4\ ; \end{cases} \qquad g(u_i) = \begin{cases} d & \text{if } i = 1 \\ c & \text{if } i = 2 \\ f & \text{if } i = 3 \\ e & \text{if } i = 4\ . \end{cases}$$

Here we have $v_g(a) = v_g(b) = v_g(d) = v_g(e) = 1, v_g(c) = v_g(f) = 2$ and $e_g(0) = 8, e_g(1) = 8$. Therefore $|v_g(i) - v_g(j)| \leq 1$ for $i, j \in S_3$ and $|e_g(0) - e_g(1)| \leq 1$.
Case 3. $n = 5$.

$$g(v_i) = \begin{cases} a & \text{if } i = 1 \\ f & \text{if } i = 2 \\ b & \text{if } i = 3 \\ d & \text{if } i = 4 \\ e & \text{if } i = 5 \text{ ;} \end{cases} \qquad g(u_i) = \begin{cases} d & \text{if } i = 1 \\ a & \text{if } i = 2 \\ b & \text{if } i = 3 \\ c & \text{if } i = 4 \\ f & \text{if } i = 5 \text{ .} \end{cases}$$

Here we have $v_g(a) = v_g(b) = v_g(d) = v_g(f) = 2, v_g(c) = v_g(e) = 1$ and $e_g(0) = 10, e_g(1) = 10$. Therefore $|v_g(i) - v_g(j)| \leq 1$ for $i, j \in S_3$ and $|e_g(0) - e_g(1)| \leq 1$.
Case 4. $n \geq 6$.
Subcase 4.1. $n \equiv 0 \ (mod\ 6)$.

Let $n = 6k$ and $k \geq 1$.

$$g(v_i) = \begin{cases} a & \text{if } i \equiv 1 \ (mod\ 6) \text{ and } 1 \leq i \leq 6k \\ b & \text{if } i \equiv 2 \ (mod\ 6) \text{ and } 1 \leq i \leq 6k \\ f & \text{if } i \equiv 3 \ (mod\ 6) \text{ and } 1 \leq i \leq 6k \\ c & \text{if } i \equiv 4 \ (mod\ 6) \text{ and } 1 \leq i \leq 6k \\ e & \text{if } i \equiv 5 \ (mod\ 6) \text{ and } 1 \leq i \leq 6k \\ d & \text{if } i \equiv 0 \ (mod\ 6) \text{ and } 1 \leq i \leq 6k \text{ ;} \end{cases}$$

$$g(u_i) = \begin{cases} d & \text{if } i \equiv 1 \ (mod\ 6) \text{ and } 1 \leq i \leq 6k \\ e & \text{if } i \equiv 2 \ (mod\ 6) \text{ and } 1 \leq i \leq 6k \\ a & \text{if } i \equiv 3 \ (mod\ 6) \text{ and } 1 \leq i \leq 6k \\ f & \text{if } i \equiv 4 \ (mod\ 6) \text{ and } 1 \leq i \leq 6k \\ b & \text{if } i \equiv 5 \ (mod\ 6) \text{ and } 1 \leq i \leq 6k \\ c & \text{if } i \equiv 0 \ (mod\ 6) \text{ and } 1 \leq i \leq 6k \text{ .} \end{cases}$$

Here we have $v_g(a) = v_g(b) = v_g(c) = v_g(d) = v_g(e) = v_g(f) = 2k$ and $e_g(0) = e_g(1) = 12k$. Therefore $|v_g(i) - v_g(j)| \leq 1$ for $i, j \in S_3$ and $|e_g(0) - e_g(1)| \leq 1$.
Subcase 4.2. $n \equiv 5 \ (mod\ 6)$.

Let $n = 6k+5$ and $k \geq 1$. Assign the labels to the vertices v_i, u_i for $1 \leq i \leq 6k$ as in Subcase (4.1) and for the remaining vertices assign the following labels:

$$g(v_i) = \begin{cases} a & \text{if } i = 6k+1 \\ f & \text{if } i = 6k+2 \\ b & \text{if } i = 6k+3 \\ d & \text{if } i = 6k+4 \\ e & \text{if } i = 6k+5 \text{ ;} \end{cases} \qquad g(u_i) = \begin{cases} d & \text{if } i = 6k+1 \\ a & \text{if } i = 6k+2 \\ b & \text{if } i = 6k+3 \\ c & \text{if } i = 6k+4 \\ f & \text{if } i = 6k+5 \text{ .} \end{cases}$$

Here we have $v_g(a) = v_g(b) = v_g(d) = v_g(f) = 2k+2, v_g(c) = v_g(e) = 2k+1$ and $e_g(0) = e_g(1) = 12k+10$. Therefore $|v_g(i) - v_g(j)| \leq 1$ for $i, j \in S_3$ and $|e_g(0) - e_g(1)| \leq 1$.
Subcase 4.3. $n \equiv 4 \ (mod\ 6)$.

Let $n = 6k+4$ and $k \geq 1$. Assign the labels to the vertices v_i, u_i for $1 \leq i \leq 6k$ as in Subcase (4.1). Then, we assign the labels to the last four vertices are as follows:

$$g(v_i) = \begin{cases} a & \text{if } i = 6k+1 \\ b & \text{if } i = 6k+2 \\ f & \text{if } i = 6k+3 \\ c & \text{if } i = 6k+4 \ ; \end{cases} \qquad g(u_i) = \begin{cases} d & \text{if } i = 6k+1 \\ c & \text{if } i = 6k+2 \\ f & \text{if } i = 6k+3 \\ e & \text{if } i = 6k+4 \ . \end{cases}$$

Here we have $v_g(a) = v_g(b) = v_g(d) = v_g(e) = 2k+1, v_g(c) = v_g(f) = 2k+2$ and $e_g(0) = e_g(1) = 12k+8$. Therefore $|v_g(i) - v_g(j)| \leq 1$ for $i, j \in S_3$ and $|e_g(0) - e_g(1)| \leq 1$.

Subcase 4.4. $n \equiv 3\ (mod\ 6)$.

Let $n = 6k+3$ and $k \geq 1$. Assign the labels to the vertices v_i, u_i for $1 \leq i \leq 6k$ as in Subcase (4.1). Then, we assign the labels to the last three vertices are as follows:

$$g(v_i) = \begin{cases} a & \text{if } i = 6k+1 \\ b & \text{if } i = 6k+2 \\ f & \text{if } i = 6k+3 \ ; \end{cases} \qquad g(u_i) = \begin{cases} d & \text{if } i = 6k+1 \\ e & \text{if } i = 6k+2 \\ c & \text{if } i = 6k+3 \ . \end{cases}$$

Here we have $v_g(a) = v_g(b) = v_g(c) = v_g(d) = v_g(e) = v_g(f) = 2k+1$ and $e_g(0) = e_g(1) = 12k+6$. Therefore $|v_g(i) - v_g(j)| \leq 1$ for $i, j \in S_3$ and $|e_g(0) - e_g(1)| \leq 1$.

Subcase 4.5. $n \equiv 2\ (mod\ 6)$.

Let $n = 6k+2$ and $k \geq 1$. Assign the labels to the vertices v_i, u_i for $1 \leq i \leq 6k$ as in Subcase (4.1). Then, we assign the labels to the last two vertices are as follows:

$$g(v_i) = \begin{cases} b & \text{if } i = 6k+1 \\ c & \text{if } i = 6k+2 \ ; \end{cases} \qquad g(u_i) = \begin{cases} f & \text{if } i = 6k+1 \\ a & \text{if } i = 6k+2 \ . \end{cases}$$

Here we have $v_g(a) = v_g(b) = v_g(c) = v_g(f) = 2k+1, v_g(d) = v_g(e) = 2k$ and $e_g(0) = e_g(1) = 12k+4$. Therefore $|v_g(i) - v_g(j)| \leq 1$ for $i, j \in S_3$ and $|e_g(0) - e_g(1)| \leq 1$.

Subcase 4.6. $n \equiv 1\ (mod\ 6)$.

Let $n = 6k+1$ and $k \geq 1$. Assign the labels to the vertices v_i, u_i for $1 \leq i \leq 6k$ as in Subcase (4.1), except that the vertices v_{6k+1}, u_{6k+1} are labeled by c, f respectively. Here we have $v_g(a) = v_g(b) = v_g(d) = v_g(e) = 2k, v_g(c) = v_g(f) = 2k+1$ and $e_g(0) = e_g(1) = 12k+2$. Therefore $|v_g(i) - v_g(j)| \leq 1$ for $i, j \in S_3$ and $|e_g(0) - e_g(1)| \leq 1$.

Thus g is a group S_3 cordial remainder labeling. Hence $T(C_n)$ is a group S_3 cordial remainder graph for $n \geq 3$. $\square$

Example 2.12. A group S_3 cordial remainder labeling of $T(C_8)$ is given in FIGURE 5.

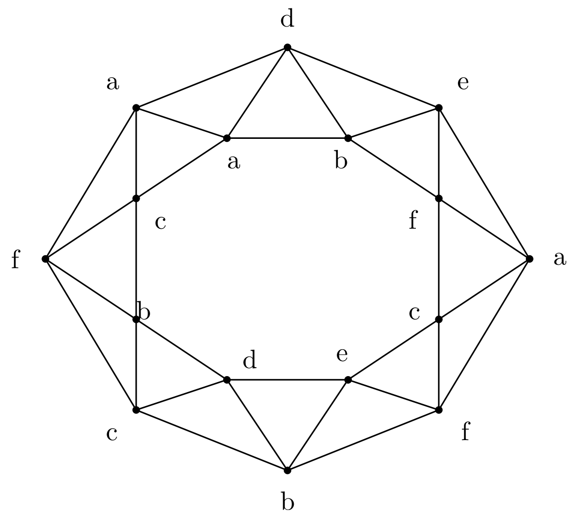

Figure 5

References

1. I. Cahit, *Cordial graphs: A weaker version of graceful and harmonious graphs*, Ars Combin. **23** (1987), 201-207.
2. J. A. Gallian, *A Dyamic Survey of Graph Labeling*, The Electronic J. Combin. **22** (2019), # DS6.
3. F. Harary, *Graph Theory*, Addison-wesley, Reading, Mass, 1972.
4. S. Jenifer Wency, A. Lourdusamy and F. Patrick, *Several result on group S_3 cordial remainder labeling*, AIP Conference Proceedings **2261** (2020), 030035.
5. A. Lourdusamy, S. Jenifer Wency and F. Patrick, *Group S_3 cordial remainder labeling*, International Journal of Recent Technology and Engineering **8** (2019), 8276-8281.
6. A. Lourdusamy, S. Jenifer Wency and F. Patrick, *Group S_3 cordial remainder labeling of subdivision of graphs*, Journal of Applied Mathematics & Informatics, **38** (2020), 221-238.
7. A. Lourdusamy, S. Jenifer Wency and F. Patrick, *Group S_3 cordial remainder labeling for wheel and snake related graphs*, Jordan Journal of Mathematics and Statistics (Accepted).
8. A. Lourdusamy, S. Jenifer Wency and F. Patrick, *Some result on group S_3 cordial remainder graphs*, (Submitted for Publication).
9. A. Lourdusamy and F. Patrick, *Sum divisor cordial labeling for path and cycle related graphs*, Journal of Prime Research in Mathematics **15** 2019, 101-114.
10. A. Rosa, *On certain valuations of the vertices of a graph*, Theory of Graphs(Internat Sympos., Rome, 1966) 349-355.

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