J. Appl. Math. & Informatics Vol. **39***(2021), No. 1 - 2, pp. 215 - 222*
https://doi.org/10.14317/jami.2021.215

CARA UTILITY AND OPTIMAL RETIREMENT†

JONGSUNG CHOI AND HO-SEOK LEE*

Abstract. We explore an optimal consumption/portfolio and retirement problem with a CARA utility function of consumption. The relevant Bellman equation for the value function is transformed into a linear equation and the optimal strategies are obtained explicitly.

AMS Mathematics Subject Classification : 91G10, 91G80.
Key words and phrases : CARA utility, portfolio selection, retirement, subsistence consumption, Bellman equation

1. Introduction

In the area of mathematical finance, optimal stochastic controls with stopping time are actively applied to optimal consumption/portfolio and retirement decision problems. [1] is a pioneering work that casts consumption/portfolio and retirement problem into a trade off framework between labor income and disutility while working. A smooth pasting condition at the retirement wealth level for the value function and the candidate policies are suggested with verification. [4] imposes a subsistence consumption constraint to the problem of [1] with a CRRA(constant relative risk aversion) utility function. [5] also employs a CRRA utility function and extends the restrictions on consumption: impose a upper bound as well as a lower bound.

Recently, [6] considers a CARA(constant absolute risk aversion) utility and considers subsistence consumption. [3] also concentrates on a CARA utility and considers nonnegative wealth constraint as well as subsistence consumption. These works impose subsistence consumption constraints and derive closed form solutions to the optimization problem. Existing literature implicitly assumes the consumption rate is nonnegative and defines consumption process accordingly. If

Received December 29, 2020. Revised January 16, 2021. Accepted January 20, 2021.
*Corresponding author.
†This work was supported by the National Resea rch Foundation of Korea Grant funded by the Korean Government (NRF- 2019R1F1A1060853) and by the Research Grant of Kwangwoon University in 2020.

$u(c)$ is the utiltity function for consumption, for the case where $\lim_{c\downarrow 0} u'(c) = \infty$, the candidate optimal consumption rate derived from first order condition is automatically nonnegative (see [2]).

On the other hand, for the case of the CARA utility which satisfies $\lim_{c\downarrow 0} u'(c) < \infty$, the natural condition $c \geq 0$ binds, so we should carefully consider consumption strategy even though a subsistence consumption constraint is not given explicitly. This paper considers the CARA utility function and an optimal consumption/portfolio and retirement problem based on the tradeoff between labor income and disutility while working.

2. Model

We will be working on a Black-Scholes market model. Let us introduce a standard Brownian motion $(B_t)_{t\geq 0}$ for a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The first constituent of the financial market is a risky free asset, i.e., a money market accout that earns a constant rate of return r (the risky free interest rate) per annum. The second one is a risky asset, a stock, whose price follows log normal distribution: if we denote by S_t the price of a sharehold of the stock at time t , S_t evolves according to the following diffusion equation

$$dS_t/S_t = \mu dt + \sigma dB_t,$$

where μ $(> r)$ and σ are constants.

We define the portfolio process $\boldsymbol{\pi} \triangleq (\pi_t)_{t\geq 0}$: if $\boldsymbol{\pi} \triangleq (\pi_t)_{t\geq 0}$ is $\mathbb{F}$- adapted and satisfies

$$\int_0^t \pi_t^2 dt < \infty \text{ for all } t \geq 0 \text{ a.s.}$$

then we call $\boldsymbol{\pi} \triangleq (\pi_t)_{t\geq 0}$ the portfolio process and it is the amount of money invested in the risky asset. In addition, we call $\mathbf{c} \triangleq (c_t)_{t\geq 0}$ the (nonnegative) consumption rate process if it is $\mathbb{F}$- adapted and satisfies

$$\int_0^t c_t dt < \infty, \ c_t \geq 0, \ \text{for all } t \geq 0 \text{ a.s.}$$

We assume an infinitely lived indivudual who is a wage earner with a constant labor income rate $y > 0$. Therefore, the wealth level X_t follows the diffusion equation

$$\begin{aligned} dX_t &= \{r(X_t - \pi_t) - c_t + y\}\, dt + \pi_t \frac{dS_t}{S_t} \\ &= \{rX_t + (\mu - r)\pi_t - c_t + y\}\, dt + \sigma \pi_t dB_t, \ t < \tau, \end{aligned}$$

where τ is the time of discretionary retirement. On the other hand, the individual's wealth process follows

$$dX_t = \{rX_t + (\mu - r)\pi_t - c_t\}\, dt + \sigma \pi_t dB_t, \ t \geq \tau.$$

The individual enjoys utility from consumption but goes through disutility while working. We assume that the individual's utility of consumption is given by a CARA(constant absolute risk aversion) function, i.e.,

$$u(c) = -\frac{e^{-\alpha c}}{\alpha}, \tag{1}$$

where $\alpha > 0$ is the coefficient of absolute risk aversion. Thus we have the instantaneous utility function as follows

$$\begin{cases} -\dfrac{e^{-\alpha c_t}}{\alpha} - l, & 0 \le t < \tau, \\ -\dfrac{e^{-\alpha c_t}}{\alpha}, & t \ge \tau, \end{cases}$$

where l is the disutility from work. After retirement, the individual (call retiree) intends to maximize the expected discounted utility i.e., to find the following value function

$$v_p(x) := \max_{\mathbf{c},\ \boldsymbol{\pi}} \mathbb{E}\left[\int_0^\infty e^{-\beta t}\frac{e^{-\alpha c_t}}{\alpha}dt\right], \tag{2}$$

where β is the subjective discount rate. Before retirement, the individual's objective to solve the maximiation problem

$$V(x) := \max_{\mathbf{c},\ \boldsymbol{\pi},\ \tau} \mathbb{E}\left[\int_0^\tau e^{-\beta t}\left(\frac{e^{-\alpha c_t}}{\alpha} - l\right)dt + e^{-\beta\tau}v_p(X_\tau)\right]. \tag{3}$$

3. The optimization problems

Our choice of utility function (1) may leads us to some difficulties due to the fact that $\lim_{c\downarrow 0} u'(c) < \infty$ [2]. Contrary to the cases where $\lim_{c\downarrow 0} u'(c) = \infty$, for example CRRA utility functions, in our optimization problems the nonnegative constraint $c_t \ge 0$ binds. The retiree's value function $v_p(x)$ is can be found from [6], which studies consumption/portfolio selection problem with a consumption constraint and CARA utility: $v_p(x)$ is the special case that R (the minimum consumption rate) of [6] set to be 0.

Definition 3.1. We define the market price of risk $\theta := (\mu - r)/\sigma$. Let m_1 and m_2 be the roots to the following equation

$$rm^2 - \left(r + \beta + \frac{1}{2}\theta^2\right)m + \beta = 0, \tag{4}$$

such that $0 < m_1 < 1$ and $m_2 > 1$, and n_- and n_+ be those of the equation

$$\frac{1}{2\alpha}\theta^2 n^2 + \left(r - \beta - \frac{1}{2}\theta^2\right) - r\alpha, \tag{5}$$

with $n_- < 0$ and $n_+ > \gamma$.

The value function from [6] is given by

$$v_p(x) = \begin{cases} C_p x^{m_1} - \dfrac{1}{\alpha\beta}, & 0 < x \le \tilde{x}_p, \\ \dfrac{1}{\beta}\left(r + \dfrac{\theta^2}{2\alpha} n_-\right) D_p e^{(n_- - \alpha)\mathcal{Z}_p(x)} - \dfrac{1}{r\alpha} e^{-\alpha \mathcal{Z}_p(x)}, & x > \tilde{x}_p, \end{cases}$$

where

$$D_p = -\frac{rm_1 - \beta - \theta^2/2}{r^2\alpha(1 - ((1-m_1)/\alpha)n_-)},$$
$$\tilde{x}_p = \frac{1-m_1}{\alpha}\left(n_- D_p + \frac{1}{r}\right),$$
$$C_p = \frac{1}{m_1}\tilde{x}_p^{1-m_1},$$

and $\mathcal{Z}_p$ is the inverse function of $\mathcal{X}_p$ which is defined as follows:

$$\mathcal{X}_p(z) = D_p e^{n_- z} + \frac{1}{r}z + \frac{1}{r^2\alpha}\left(r - \beta - \frac{1}{2}\theta^2\right).$$

With retiree's value function v_p, we are to solve the optimization problem of the individual who has an option to retire.

Problem 3.2.

$$V(x) := \max_{\mathbf{c},\ \boldsymbol{\pi},\tau} \mathbb{E}\left[\int_0^\tau e^{-\beta t}\left\{-\frac{e^{-\alpha c_t}}{\alpha} - 1\right\} dt + e^{-\beta\tau} v_p(X_\tau)\right]. \tag{6}$$

Due to the constraint $c \ge 0$, there exists a threshold wealth level such that

$$c = 0, \text{ for } -\frac{y}{r} < x \le \tilde{x}, \tag{7}$$
$$c > 0, \text{ for } \tilde{x} < x < \bar{x}, \tag{8}$$
$$\tag{9}$$

where $\bar{x}$ is the retirement wealth level.

For $-\frac{y}{r} < x \le \tilde{x}$, the Bellman equation for $V(x)$ is given by

$$\max_\pi \left[\{rx + y + \pi(\mu - r)\} V'(x) + \frac{1}{2}\sigma^2\pi^2 V''(x) - \beta V(x) - \frac{1}{\alpha} - l\right] = 0. \tag{10}$$

For $\tilde{x} < x \le \bar{x}$, the Bellman equation for $V(x)$ is given by

$$\max_{c,\ \pi}\left[\{rx - c + y + \pi(\mu - r)\} V'(x) + \frac{1}{2}\sigma^2\pi^2 V''(x) - \beta V(x) - \frac{e^{-\alpha c}}{\alpha} - l\right] = 0. \tag{11}$$

4. Solution

To guarantee the exsistence of the solution to the optimization problem, we assume the following inequality.

Assumption 4.1. *We assume that*

$$y > l.$$

Theorem 4.2. *The value function V(x) is given by*

$$V(x) = \begin{cases} C\left(x+\frac{y}{r}\right)^{m_1} - \frac{1}{\alpha\beta} - \frac{l}{\beta}, & -\frac{y}{r} < x \le \tilde{x}, \\ \frac{r+\frac{\theta^2}{2\alpha}n_+}{\beta}A_+ e^{(n_+-\alpha)\mathcal{Z}(x)} + \frac{r+\frac{\theta^2}{2\alpha}n_-}{\beta}A_- e^{(n_--\alpha)\mathcal{Z}(x)} & \\ -\frac{1}{r\alpha}e^{-\alpha\mathcal{Z}(x)} - \frac{l}{\beta}, & \tilde{x} < x < \bar{x}, \end{cases} \tag{12}$$

where

$$\tilde{x} = A_+ + A_- + \frac{1}{r^2\alpha}\left(r-\beta-\frac{1}{2}\theta^2\right) - \frac{y}{r}, \tag{13}$$

$$\bar{x} = \mathcal{X}(\bar{c}), \tag{14}$$

$$\bar{c} = \frac{1}{\alpha}\log\frac{y}{l}, \tag{15}$$

$$A_+ = \frac{n_-}{n_- - n_+}\frac{y}{r}e^{-n_+\bar{c}}, \tag{16}$$

$$A_- = D_p + \frac{n_+}{n_+ - n_-}\frac{y}{r}e^{-n_-\bar{c}}, \tag{17}$$

$$C = \frac{1}{m_1}\left(\tilde{x}+\frac{y}{r}\right)^{1-m_1}, \tag{18}$$

and $\mathcal{Z}$ is the inverse function of $\mathcal{X}$ which is defined as follows:

$$\mathcal{X}(z) = A_+e^{n_+z} + A_-e^{n_-z} + \frac{1}{r}z + \frac{1}{r^2\alpha}\left(r-\beta-\frac{1}{2}\theta^2\right) - \frac{y}{r}.$$

Proof. For $-\frac{y}{r} < x \le \tilde{x}$, the first order condition for the value function is given by

$$\pi = -\frac{\theta}{\sigma}\frac{V'(x)}{V''(x)}. \tag{19}$$

Thus, HJB equation (10) can be rewritten as

$$\beta V(x) = (rx+y)V'(x) - \frac{1}{2}\theta^2\frac{\left(V'(x)\right)^2}{V''(x)} - \frac{1}{\alpha} - l. \tag{20}$$

We try a homonenous solution of the form

$$\left(x+\frac{y}{r}\right)^m$$

and obtain (4). We have a particular solution $-\frac{1}{\alpha\beta}-\frac{l}{\beta}$. Therefore, we have

$$V(x)=C\left(x+\frac{y}{r}\right)^{m_1}-\frac{1}{\alpha\beta}-\frac{l}{\beta},\ -\frac{y}{r}<x\le\tilde{x},$$

for a constant C.

For $\tilde{x}<x<\bar{x}$, we have the first order conditions for (11)

$$\begin{cases} c &= -\dfrac{1}{\alpha}\log V'(x), \\[2ex] \pi &= -\dfrac{\theta}{\sigma}\dfrac{V'(x)}{V''(x)}. \end{cases} \tag{21}$$

Therefore, (11) becomes

$$\beta V(x)=(rx+y)V'(x)-\frac{1}{2}\theta^2\frac{(V'(x))^2}{V''(x)}+\frac{1}{\alpha}V'(x)(\log V'(x)-1)-l. \tag{22}$$

As in [2], we write $x=X(c)$ and denote bye $C(\cdot)$ as the inverse function of $X(\cdot)$ such that $c=C(x),\ x=X(C(x))$. Then we can rewrite the first order condition (21) as follows

$$\begin{cases} V'(x) &= e^{-\alpha c}, \\[2ex] V''(x) &= -\alpha e^{-\alpha C(x)}C'(x)=-\alpha\dfrac{e^{-\alpha c}}{X'(c)}. \end{cases} \tag{23}$$

Plugging (23) into (22) to obtain

$$\beta V(X(c))=r(X(c)+y)e^{-\alpha c}+\frac{1}{2\alpha}\theta^2e^{-\alpha c}X''(c)-\frac{1}{\alpha}(\alpha c+1)e^{-\alpha c}-l. \tag{24}$$

We differentiate (24) with respect to c and have

$$\frac{1}{2\gamma}\theta^2X''(c)+\left(r-\beta-\frac{1}{2}\theta^2\right)X'(c)-\alpha rX(c)+\alpha c-\alpha y=0. \tag{25}$$

If we try a homogeneous solution form e^{nc} to the equation (25), we have (5). A particular solution to (25) can be found as

$$\frac{1}{r}c+\frac{1}{r^2\alpha}\left(r-\beta-\frac{1}{2}\theta^2\right)-\frac{y}{r}.$$

Therefore, the general solution to the linear equation (25) is given by

$$X(c)=A_+e^{n_+c}+A_-e^{n_-c}+\frac{1}{r}c+\frac{1}{r^2\alpha}\left(r-\beta-\frac{1}{2}\theta^2\right)-\frac{y}{r},$$

for some constants A_+ and A_-. The value function $V(x)$ is obtained from (24):

$$V(x)=\frac{r+\frac{\theta^2}{2\alpha}n_+}{\beta}A_+e^{(n_+-\alpha)c}+\frac{r+\frac{\theta^2}{2\alpha}n_-}{\beta}A_-e^{(n_--\alpha)c}-\frac{1}{r\alpha}e^{-\alpha c}-\frac{l}{\beta},\ \tilde{x}<x<\bar{x}.$$

Due to the wealth thresholds $\tilde{x}$ and $\bar{x}$, there exists the corresponding consumption threshold $\tilde{c} = 0$ and $\bar{c}$. Then we have

$$\tilde{x} = X(\tilde{c}),$$

and

$$\bar{x} = X(\bar{c}) = X(\bar{c}).$$

We apply C^2 conditions at $x = \tilde{x}$ and $x = \bar{x}$ to obtain coefficients and wealth thresholds. Then we have the expressions from (13) to (18). $\square$

We obtain the optimal policies from the value function (12) and the first order conditions (19) and (21).

Theorem 4.3. *The optimal consumption/portfolio policies* $(\mathbf{c}^*, \boldsymbol{\pi}^*)$ *are given by*

$$c_t^* = \begin{cases} 0, \ -\dfrac{y}{r} < X_t \leq \tilde{x}, \\ \\ \mathcal{Z}(X_t), \ \tilde{x} < X_t < \bar{x}, \end{cases}$$

$$\pi_t^* = \begin{cases} \dfrac{1}{1-m_1}\left(X_t + \dfrac{y}{r}\right), \ -\dfrac{y}{r} < X_t \leq \tilde{x}, \\ \\ \dfrac{\theta}{\sigma\alpha} X'(\mathcal{Z}(X_t)), \ \tilde{x} < X_t < \bar{x}, \end{cases}$$

and the optimal retirement time τ^* *is given by*

$$\tau^* = \inf\{X_t \geq \bar{x}\}.$$

References

1. K.J. Choi and G. Shim, *Disutility, Optimal Retirement, and Portfolio Selection*, Mathematical finance **16** (2006), 443-467.
2. I. Karatzas, J.P. Lehoczky, S.P. Sethi, and S.E. Shreve, *Explicit solution of a general consumption/investment problem*, Mathematics of Operations Research **11** (1986), 261-294.
3. J.Y. Kim and Y.H. Shin, *Optimal consumption and portfolio selection with negative wealth constraints, subsistence consumption constraints, and CARA utility*, Journal of the Korean Statistical Society **47** (2018), 509-519.
4. H.S. Lee and Y.H. Shin, *An optimal consumption, investment and voluntary retirement choice problem with disutility and subsistence consumption constraints: A dynamic programming approach*, Journal of Mathematical Analysis and Applications **428** (2018), 762-771.
5. K.H. Roh and Y.H. Shin, *Optimal consumption and portfolio selection with lower and upper bounds on consumption*, Advances in Difference Equations **2020** (2020), 1-11.
6. G. Shim and Y.H. Shin, *Portfolio Selection with Subsistence Consumption Constraints and CARA Utility*, Mathematical Problems in Engineering **2014** (2014), 1-6.

Jongsung Choi is currently an associate professor at Kwangwoon University. His research interests include computational finance.

Ingenium College, Kwangwoon University, Seoul 01897, Korea.

e-mail: jeschoi@kw.ac.kr

Ho-Seok Lee received Ph.D. from KAIST. Since 2016 he has been at Kwangwoon University. His research interests include continuous time finance and portfolio selection.

Department of Mathematics, Kwangwoon University, Seoul 01897, Korea.
e-mail: hoseoklee@kw.ac.kr