*J. Appl. Math. & Informatics Vol.* **39***(2021), No. 1 - 2, pp. 197 - 214*
https://doi.org/10.14317/jami.2021.197

# ANALYSIS OF MALARIA DYNAMICS USING ITS FRACTIONAL ORDER MATHEMATICAL MODEL

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Abstract. In this paper, we have studied dynamics of fractional order mathematical model of malaria transmission for two groups of human population say semi-immune and non-immune along with growing stages of mosquito vector. The present fractional order mathematical model is the extension of integer order mathematical model proposed by Ousmane Koutou et al. For this study, Atangana-Baleanu fractional order derivative in Caputo sense has been implemented. In the view of memory effect of fractional derivative, this model has been found more realistic than integer order model of malaria and helps to understand dynamical behaviour of malaria epidemic in depth. We have analysed the proposed model for two precisely defined set of parameters and initial value conditions. The uniqueness and existence of present model has been proved by Lipschitz conditions and fixed point theorem. Generalised Euler method is used to analyse numerical results. It is observed that this model is more dynamic as we have considered all classes of human population and mosquito vector to analyse the dynamics of malaria.

AMS Mathematics Subject Classification: 34A34, 34B60, 65L05, 92B05.
*Key words and phrases* :Atangana-Baleanu fractional order derivative in Caputo sense [ABCD], Atangana-Baleanu fractional order integral in Caputo sense [ABCI], Fractional order mathematical model of malaria [FOMMM], Generalised Euler method [GEM].

## 1. Introduction

The concept of fractional calculus emerged through the consequences of theory of calculus in seventeenth century by Isaac Newton, a well known British scientist, as well as Gottfried Leibnitz, a self-taught German mathematician. Fractional calculus deals with the definitions of classical calculus in the form of generalised fractional order [1]. Most of the scientists like Riemann, Caputo, Liouville, Grunwald Letnikov etc. defined fractional order derivatives and integrals

Received May 29, 2020. Revised October 27, 2020. Accepted December 29, 2020.
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in their respective forms [2]. Scientists and researchers prefer to apply fractional calculus in the field of science and engineering like chemistry and physics [3], economics and finance [4], image processing [5], biology and biotechnology [6], signal processing [7] and control systems [8].

In the view of memory property, existence of non-local operator and excellent factuality, fractional calculus has played significant role in the formation of more dynamic and efficient mathematical models. Scientists and researchers have applied fractional operators to define mathematical models for different epidemics considering various cases and constraints. T. J. Anastasio et al. [9] have described dynamics of fractional order mathematical model of brain stem vestibulo-oculomotor neurons. Ozalp et al. [10] have explained fractional order SEIR model with vertical transmission using reproductive number. Fractional order tuberculosis infection model including the impact of diabetes and resistant strains has been analysed by N. H. Sweilam et al. [11]. Carl Pinto M. A. et al. [12] demonstrated fractional mathematical model for malaria transmission control strategy. Pawar D. D. et al. [13] have proposed fractional order mathematical model for tuberculosis with two line treatment and analysed it thoroughly by applying generalised Euler method successfully. Kumar Devendra et al. [14] have proposed the mathematical model with non-integer order with consideration of vaccination, anti malarial drug and control strategy for mosquitoes by spraying. Ousmane Koutou et al. [15] have designed mathematical model of malaria transmission taking into account the immature stages of vectors following the non-immune and semi-immune types of human population. Recently, researchers and scientists have formulated fractional order mathematical models using Atangana-Baleanu derivative which satisfies Lipschitz condition. Khan I. et al. [16] have applied Atangana-Baleanu fractional derivative to formulate mathematical model of human blood flow in nano fluids. Khan M. A. et al. [17] have proposed a fractional order mathematical model for tuberculosis with relapse via Atangana–Baleanu derivative in Caputo sense. Uçar S. et al [18] have analysed basic SEIRA model by transforming it into fractional order with Atangana-Baleanu derivative.

Our present model is a novel extension of integer order mathematical model of malaria transmission dynamics proposed by Ousmane Koutou et al. [15] to fractional order mathematical model in the form of time dependent system of fractional order differential equations by applying Atangana-Baleanu derivative.

## 2. Preliminaries

**2.1. Fractional calculus - brief summary.** In this section, we have presented some basic definitions of fractional derivatives and integrations, demonstrated by various scientists. The nomenclature of definitions has been given the name of the respective scientist. In view of the role of Mittag-Leffler function in the definition of Atangana-Baleanu fractional order derivative, it has been defined.

**Definition 2.1.** [1], [2] The Mittag-Leffler function of one parameter is denoted by $E_\alpha(z)$ and defined as

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} \qquad \alpha \in \mathbb{C}, \;\; Re(\alpha) > 0. \tag{1}$$

**Definition 2.2.** Atangana-Baleanu fractional order derivative in Caputo sense [11]
Let $g : [a, b] \to \mathbb{R}$ be a bounded and continuous function then Atangana-Baleanu fractional derivative in Caputo sense of order $0 < \alpha \leq 1$ is defined as

$$ {}_{a}^{ABC}D_t^\alpha g(t) = \frac{M(\alpha)}{(1-\alpha)} \int_a^t E_\alpha\left(-\alpha \,\frac{(t-q)^\alpha}{(1-\alpha)}\right) g'(q)dq \tag{2}$$

where $\Gamma(.)$ is the gamma function and $M(\alpha) = 1 - \alpha + \frac{\alpha}{\Gamma(\alpha)}$ is normalisation function.

**Definition 2.3.** Atangana-Baleanu fractional order integral
Let $g : [a, b] \to \mathbb{R}$ be a bounded and continuous function. The corresponding fractional integral concerning to Atangana-Baleanu fractional order derivative of order $0 < \alpha \leq 1$ is defined as in [11] is

$$ {}_{a}^{ABC}I_t^\alpha g(t) = \frac{(1-\alpha)}{M(\alpha)} g(t) + \frac{\alpha}{M(\alpha)\Gamma(\alpha)} \int_a^t (t-q)^{\alpha-1} g(q)dq \tag{3}$$

where $M(\alpha) = 1 - \alpha + \frac{\alpha}{\Gamma(\alpha)}$ is normalisation function.

**Theorem 2.4.** *Let $g : [a, b] \to \mathbb{R}$ be a bounded and continuous function then the following results hold as in* [11]

$$\| {}_{a}^{ABC}D_t^\alpha g(t) \| \leq \frac{M(\alpha)}{(1-\alpha)} \|g(t)\|, \qquad where \;\; \|g(t)\| = max_{a \leq t \leq b} \mid g(t) \mid,$$

*Further, the Atangana-Baleanu derivative fulfil the Lipschitz condition* [11]

$$\| {}_{a}^{ABC}D_t^\alpha g_1(t) - {}_{a}^{ABC}D_t^\alpha g_2(t)\| \leq L \,\|g_1(t) - g_2(t)\|$$

where $0 < \alpha \leq 1$ is the order of fractional derivative.

**2.2. Analysis of generalised Euler method [GEM].** In this section, we are presenting extension of generalised Euler method as explained in [10], [13] for a system of fractional order n number of linear and non-linear differential equations as

$$ {}_{a}D_t^\alpha y_i(t) = f_i(t, y_1(t),\; y_2(t),\; y_3(t), \ldots, y_n(t)) \quad 0 < \alpha \leq 1, \quad t > 0 \tag{4}$$

with the initial conditions $y_i(0) = y_{i_0},\; for\; i = 1, 2, 3, . . .n$
We have to find the solution in finite interval $[0,\; a]$. Assume that $y_i(t),\; D_a^\alpha y_i(t), \ldots$ for all $i's$ are continuous on $[0,\; a]$

The general formula for generalised Euler method (GEM)
for $t_{j+1} = t_j + h \; for \; all \; j = 0, \; 1 \; 2 \; 3 \ldots k$ such that $h$ is sufficiently small is

$$y_i(t_{j+1}) = y_i(t_j) + f_i(t, y_1(t_j), \; y_2(t_j), \; y_3(t_j), \ldots, y_n(t_j)) \frac{h^\alpha}{\Gamma(\alpha+1)} \quad (5)$$

$for \;\; all \;\; i = 1, 2, \ldots n.$

## 3. Fractional order model formation

In this section, we have proposed fractional order mathematical model considering two groups of human population, growth stages of mosquito vector and family of classes of mature mosquito vector. This model is the extension of integer order model of malaria transmission dynamics proposed by Ousmane Koutou et al. [15] along with the addition of susceptible class of mosquito vector. The fractional derivative is defined in Atangana-Baleanu fractional order derivative in Caputo sense. In order to define the revised model, we have considered human population in two different groups as non-immune and semi-immune with reference to variance in immunity. There are lot of medical parameters which play vital role to differentiate human population in various classes such as HIV/ AIDS infection, lack of awareness about malaria prevention parameters, age group of individual class etc.

We denote $N_h(t)$, the total number of individuals in human population and considered to be constant such that $N_h(t) = S_n(t) + E_n(t) + I_n(t) + S_s(t) + E_s(t) + I_s(t) + R_s(t)$ ,where $S_n(t), \; E_n(t), \; I_n(t), \; S_s(t), \; E_s(t), \; I_s(t), \; R_s(t)$ denote the class of non-immune susceptible, non-immune latent, non-immune infected and semi-immune susceptible, semi-immune latent, semi-immune infected and semi-immune recovered classes of human individual respectively. The mosquito vector has been classified into various stages as $E(t), \; L(t), \; P(t), \; A(t)$ denoting stages of eggs, larvae, pupae and adult female at the moment t respectively and then it extends to class of susceptible $S_m(t)$, latent $E_m(t)$ and infected $I_m(t)$ mosquitoes. The total number of individuals in adult female mosquito population $A(t)$ is given by $A(t) = S_m(t) + E_m(t) + I_m(t)$. In this model, we have considered that infected classes of both groups further extend to semi-immune recovered class of human population.

The rate at which constant recruitment of individual human including birth is taken as $\Lambda_h$. Rate of individual human natural death is $f_h$ at various stages assumed to be constant. $p$ be the probability of recruitment to be non-immune susceptible human individual and $1-p$ be the probability of recruitment to semi-immune susceptible human individual. The rate of transfer from semi-immune latent to semi-immune infected human individual is $\nu_s$, rate of transfer from semi-immune infected to semi-immune recovered is $\delta_s$. Rate of loss of immunity there by shifting to semi-immune susceptible from recovered class is $\beta_s$. The rate of transfer from non-immune latent to non-immune infected and rate of transfer from non-immune infected to recovered class is taken as $\nu_n$ and $\delta_n$ respectively. The rate of mortality from malaria of non-immune and semi-immune infected classes are denoted by $\gamma_n$ and $\gamma_s$ respectively. The rate of infection of non-immune susceptible class and semi-immune susceptible class of human population by malarial parasite say $k_n$ and $k_s$ respectively are

$$k_n = c_{mn} n_s \frac{I_m}{N_h}$$

$$k_s = c_{ms} n_s \frac{I_m}{N_h}$$

where $n_s$ be the average number of bites per mosquito per unit time. $c_{mn}$ and $c_{ms}$ are probabilities that an infected mosquito bite lead to infection in non-immune susceptible human class and semi-immune susceptible class respectively.
The rate of infection to susceptible mosquitoes from infected human say $k_m$ is

$$k_m = c_{sm} n_s \frac{I_s}{N_h} + c_{nm} n_s \frac{I_n}{N_h} + \bar{c}_{sm}\, n_s \frac{R_s}{N_h}$$

where $c_{nm}$ is probability of susceptible female mosquito bite to an infected non-immune human individual and transfers the infection to susceptible female mosquito. $c_{sm}$ is the probability that susceptible female mosquito bite to semi-immune infected human transfers infection to susceptible female mosquito.
$\bar{c}_{sm}$ is the probability that a bite from susceptible mosquito to semi-immune recovered class transfers infection to the mosquito.
In mosquito population growth cycle, we have taken four main stages as egg, larvae, pupae and adult with the rate of transfer from eggs to larvae, larvae to pupae and from pupae stage to adult female mosquitoes are $s_E$, $s_L$ and $s_P$ respectively. The natural death rate at the stage egg, larvae and pupae female are $d_E$, $d_L$ and $d_P$ respectively. The transfer parameter b is considered as the intrinsic egg laying rate. We have assumed that while selecting suitable site for laying eggs, the adult mosquitoes ensures complete growth of all developmental stages egg, larvae, pupae and finally adults and thereby $K_E$, $K_L$, $K_P$ denote available sites for egg, larvae and pupae respectively.
We assume that the natural mortality rate $f_m$ of mosquito are being considered as constant. We have considered that
$0 < \nu_n \leq k_n$, $0 < \nu_s \leq k_s$ and $0 < \nu_m \leq k_m$, where $\nu_m$ defines the rate of passage from latent class to infected class of mosquito vector.
The fractional order system of differential equations of malaria is hence proposed as follows

$$\begin{aligned}
{}_a^{ABC}D_t^{\alpha} N_h(t) &= \Lambda_h - f_h N_h(t) - \gamma_n I_n(t) - \gamma_s I_s(t) \qquad (6)\\
{}_a^{ABC}D_t^{\alpha} S_n(t) &= p\Lambda_h - (f_h + k_n)\, S_n(t)\\
{}_a^{ABC}D_t^{\alpha} E_n(t) &= k_n S_n(t) - (f_h + \nu_n)\, E_n(t)\\
{}_a^{ABC}D_t^{\alpha} I_n(t) &= \nu_n\, E(t) - (f_h + \gamma_n + \delta_n)\, I_n(t)\\
{}_a^{ABC}D_t^{\alpha} S_s(t) &= (1-p)\Lambda_h + \beta_s\, R_s(t) - (f_h + k_s)\, S_s(t)\\
{}_a^{ABC}D_t^{\alpha} E_s(t) &= k_s S_s(t) - (f_h + \nu_s) E_s(t)
\end{aligned}$$

$$ {}_a^{ABC}D_t^{\alpha} I_s(t) = \nu_s E_s(t) - (f_h + \gamma_s + \delta_s)\, I_s(t) $$
$$ {}_a^{ABC}D_t^{\alpha} R_s(t) = \delta_n I_n(t) + \delta_s I_s(t) - (f_h + \beta_s)\, R_s(t) $$
$$ {}_a^{ABC}D_t^{\alpha} S_m(t) = s_P P(t) - (f_m + k_m) S_m(t) $$
$$ {}_a^{ABC}D_t^{\alpha} E_m(t) = k_m S_m(t) - (f_m + \nu_m) E_m(t) $$
$$ {}_a^{ABC}D_t^{\alpha} I_m(t) = \nu_m E_m(t) - f_m I_m(t) $$
$$ {}_a^{ABC}D_t^{\alpha} A(t) = s_p P(t) - f_m A(t) $$
$$ {}_a^{ABC}D_t^{\alpha} E(t) = bA(t)\left(1 - \frac{E(t)}{K_E}\right) - (s_E + d_E)\, E(t) $$
$$ {}_a^{ABC}D_t^{\alpha} L(t) = s_E E(t)\left(1 - \frac{L(t)}{K_L}\right) - (s_L + d_L)\, L(t) $$
$$ {}_a^{ABC}D_t^{\alpha} P(t) = s_L L(t)\left(1 - \frac{P(t)}{K_P}\right) - (s_P + d_P)\, P(t) $$

where $0 < \alpha \leq 1$ is the order of fractional derivative and we obtain integer order model at $\alpha = 1$. Every class of human population and mosquito vector has respective initial condition.

**3.1. Mathematical analysis of model.** The system of mathematical model has described the maturation cycle of mosquito vector and the dynamics of malaria transmission by introducing two sets of classes of human population and one set of classes of mosquito vector as defined above.
Further, according to the model, we have defined two sets of vectors as

$$ \triangle = \{ (S_s, E_s, I_s, R_s, S_n, E_n, I_n, S_m, E_m, I_m) \in \mathbb{R}_+^{10} : $$
$$ S_s + E_s + I_s + R_s + S_n + E_n + I_n \leq \frac{\Lambda_h}{f_h} \quad and \quad S_m + E_m + I_m \leq \frac{s_p K_p}{f_m} \} $$

$$ \Theta = \{ (E, L, P, A) \in \mathbb{R}_+^4 : E(t) \leq K_E,\, L(t) \leq K_L,\, P(t) \leq K_P,\, A(t) \leq \frac{s_p K_p}{fm} \} $$

The above sets $\triangle$ and $\Theta$ prove the dynamics of malaria transmission and maturation cycle of mosquito vector is biologically well defined. The entire model is well defined in $\Gamma = \triangle \times \Theta \subset \mathbb{R}_+^{10} \times \mathbb{R}_+^4$

**3.1.1. Positivity and boundedness of Solution.** In this section, we have analysed property of positivity invariant and boundedness of the solution of mathematical model

**Lemma 3.1.** [15] *The set $\Theta$ is a positive invariant region in the present model. We assume that*
*1. All the class functions are periodic positive functions with the common period $\omega$.*

*2. All the parameters of the model are positive except the disease induced death rate $f_h$ which is assumed to be non-negative.*

**Theorem 3.2.** *For initial conditions $\Phi \in \mathbb{R}_+^{14}$, the mathematical model has unique solution. Further the compact space $\Gamma$ is a positively invariant set which attracts all positive orbits in $\mathbb{R}_+^{14}$*

*Proof.* For all initial conditions $\Phi \in \mathbb{R}_+^{14}$, the function F is locally Lipschitz in X(t). By using Cauchy-Lipschitz theorem [19], present mathematical model has unique local solution.
We have

$$ {}_{s}^{ABC}D_t^{\alpha} N_h(t) = \Lambda_h - f_h N_h(t) - \gamma_n I_n(t) - \gamma_s I_s(t) \leq \Lambda_h - f_h N_h(t) $$
$$ {}_{s}^{ABC}D_t^{\alpha} A(t) = s_p P(t) - f_m A(t) \leq s_p K_p - f_m A(t) $$

It leads to, if $N_h(t) > \frac{\Lambda_h}{f_h}$ and $A(t) > \frac{s_p K_p}{f_m}$ then $\frac{dN_h(t)}{dt} < 0$ and $\frac{dA(t)}{dt} < 0$
By applying the standard comparison theorem, we conclude that
for all $t \geq 0$, $N_h(t) \leq \frac{\Lambda_h}{f_h}$ and $A(t) \leq \frac{s_p K_p}{f_m}$. Both the cases proves that the set $\triangle$ and $\Theta$ are positive invariant with respect to the present mathematical model. By lemma 3.1, the compact set $\Gamma = \triangle \times \Theta$ is also invariant and the solutions of mathematical model are non-negative and bounded. □

**3.1.2. Existence and uniqueness of solution of the model.** In view of crucial role of existence and uniqueness of solution in the analysis of mathematical model of natural phenomenon, we have examined existence and uniqueness of solution of fractional order mathematical model with exponential law by using fixed point theory.
Now we apply fractional integral operator 2.1 to the mathematical model as in equation 6. For further understanding, the mathematical model can be written as

$$ {}_{a}^{ABC}D_t^{\alpha} X(t) = F(t, X(t)) \tag{7} $$

Where $X(t) = (S_s(t), E_s(t), I_s(t), R_s(t), S_n(t), E_n(t), I_n(t), S_m(t), E_m(t), I_m(t), E(t), L(t), P(t), A(t))^T$.
The function $F : \mathbb{R}_+ \times \mathbb{R}_+^{14} \to \mathbb{R}_+^{14}$ is $C^{\infty}(\mathbb{R}_+^{14})$.

**Theorem 3.3** (Existence theorem)**.** *Let $H$ be an open connected set in $\mathbb{R}^2$, assume that the equation 7 satisfies the following conditions*
*i.* ${}_{a}^{ABC}D_t^{\alpha} X(t) = F(t, X(t))$ *is continuous on $\mathcal{D}$ with initial value conditions as given above.*
*ii.* ${}_{a}^{ABC}D_t^{\alpha} X(t) = F(t, X(t))$ *is Lipschitz continuous with respect to y on S with Lipschitz constant* $0 < L < 1$.

$$ \| {}_{a}^{ABC}D_t^{\alpha} X_i(t) - {}_{a}^{ABC}D_t^{\alpha} X_j(t) \| \leq L \, \| F(t, X_i(t)) - F(t, X_j(t)) \| $$

*Let $(t_0, X_i(t_0)) \in \mathcal{D}$ and a and b be positive constants such that*

$$ \mathbf{E} = \{(t, X(t)) : |t - t_0| \leq a, \ \|X(t) - X(t_0)\| \leq b\} $$

**E** *is a subset of* $\mathcal{D}$. *Let* $M = max_{(t,\,X(t))}\,|F(t,\,X(t))|$ *and* $h = min\left(a,\,\frac{b}{M}\right)$, *then all* $X(t)$ *have unique solution in the interval* $|t - t_0| \leq h$

*Proof.* Let **E** be a closed rectangle inside open connected set $\mathcal{D}$ and $F(t,\,X(t))$ defined Atangana-Baleanu differential operator, $F(t,\,X(t))$ satisfies all properties assumed in the theorem 2.4.
Let $E_1$ be a rectangle defined as
$E_1 = \{(t, X(t)) : |t - t_0| \leq h,\ \|X(t) - X(t_0)\| \leq b\}$
We can observe that if $a \leq \frac{b}{M}$ then $h = a$ and $\mathbf{E} = E_1$ and if $\frac{b}{M} \leq a$ then $h = \frac{b}{M}$ and $E_1 \subset \mathbf{E}$
Let's establish Atangana-Baleanu integral of Caputo sense as

$$X(t) - X(t_0) = {}^{ABC}_{t_0}I^{\alpha}_{t} F(t,\,X(t)) \tag{8}$$

By construction of successive approximation, the proof can be established to show that the sequence $\{X_m(t)\}$ converges to $X(t)$ on $[t_0,\,t_0 + h]$ and gives rise to uniqueness and existence of solution of integral (8).
Our first task is to verify that all $X_m(t)$ are well defined and continuously differentiable on $[t_0,\,t_0 + h]$.
Applying induction argument, it is to be noted that $X_m(t)$ are well defined only if $(t, X_{m-1}(t)) \in E_1$ for all $t \in [t_0,\,t_0 + h]$.
This holds trivially for $X(t_0)$.
Assume that, for $m \geq 1$, if $(t, X_{m-1}(t)) \in E_1$ and $\|X_{m-1}(t) - X(t_0)\| \leq b$ holds for all $t \in [t_0,\,t_0 + h]$.
The same statement is true for $X_m$ and it is sufficient to prove induction statement as $E_1 \subset \mathbf{E}$ ,we have

$$\begin{aligned}
\|X_m(t) - X(t_0)\| &= \|{}^{ABC}_{t_0}I^{\alpha}_{t}\ F(t,\,X_{m-1}(t))\| \\
&\leq \|\ F(t,\,X_{m-1}(t))\| \frac{t^{\alpha}}{\Gamma(\alpha - 1)} \\
&\leq M t^{\alpha} < b
\end{aligned} \tag{9}$$

Thus $F(t,\,X(t))$ is well defined and continuous on $[t_0, t_0 + h]$.
Hence the said properties holds for $X_m(t)$ and induction is complete.
For the betterment of further calculations, we express

$$\Omega(t, X(t)) = {}^{ABC}_{t_0}D^{\alpha}_{t}\{X(t)\}$$

By theorem 2.4, we are free to conclude that all the above kernel satisfy Lipschitz conditions, demonstrated as follows for each class

$$\|\Omega(t, X(t)) - \Omega(t, X_1(t))\| \leq b\|X(t) - X_1(t)\| \tag{10}$$

By definition of Atangana-Baleanu fractional order integral

$$X(t) - X(t_0) = \frac{(1-\alpha)}{M(\alpha)}\Omega(t,\,X(t)) + \frac{\alpha}{\Gamma(\alpha)M(\alpha)}\int_{t_0}^{t}\Omega(\psi,\,X(\psi))(t-\psi)^{\alpha-1}d\psi$$

Here, we construct the recursive formula for any positive integer $m$ as

$$X_m(t) = \frac{(1-\alpha)}{M(\alpha)}\Omega(t,\, X_{m-1}(t)) + \frac{\alpha}{\Gamma(\alpha)M(\alpha)}\int_{t_0}^{t}\Omega\,(\psi,\, X_{m-1}(\psi))(t-\psi)^{\alpha-1}d\psi$$

We express the difference between the successive terms by using recursive formula as mentioned above

$$\omega_m(t) = X_m(t) - X_{m-1}(t) = \frac{(1-\alpha)}{M(\alpha)}\{\Omega(t,\, X_{m-1}(t)) - \Omega(t,\, X_{m-2}(t))\}$$
$$+ \frac{\alpha}{\Gamma(\alpha)M(\alpha)}\int_{t_0}^{t}\{\Omega\,(\psi,\, X_{m-1}(\psi)) - \Omega\,(\psi,\, X_{m-2}(\psi))\}(t-\psi)^{\alpha-1}d\psi$$

It is worth to observe the summations giving the $m^{th}$ terms of classes.

$$X_m(t) = \sum_{j=1}^{m}\omega_j(t)$$

By theorem 2.4, as all the above kernels $\omega_m$ satisfy Lipschitz condition,

$$\begin{aligned}\therefore\ \|\omega_m(t)\| &= \|X_m(t) - X_{m-1}(t)\| \\ &\leq \frac{(1-\alpha)}{M(\alpha)}\|\Omega(t,\, X_{m-1}(t)) - \Omega\,(t,\, X_{m-2}(t))\| \\ &+ \frac{\alpha}{\Gamma(\alpha)M(\alpha)}\int_{t_0}^{t}\|\Omega(\psi,\, X_{m-1}(\psi)) - \Omega(\psi,\, X_{m-2}(\psi))\| \\ &\qquad (t-\psi)^{\alpha-1}d\psi\end{aligned}$$

$$\begin{aligned}\therefore\ \|\omega_m(t)\| \ &\leq\ \frac{(1-\alpha)}{M(\alpha)}b\|X_{m-1}(t) - X_{m-2}(t)\| \\ &+ \frac{1}{\Gamma(\alpha)M(\alpha)}b\|X_{m-1}(t) - X_{m-2}(t)\|(t-t_0)^{\alpha}\end{aligned}$$

Consequently, we can deduce that

$$\|\omega_m(t)\| \leq \frac{(1-\alpha)}{M(\alpha)}b\|\omega_{m-1}(t)\| + \frac{1}{\Gamma(\alpha)M(\alpha)}b\|\omega_{m-1}(t)\|(t-t_0)^{\alpha} \tag{11}$$

Taking all these inequality into account, we conclude that we obtain the existence of the solution of the mathematical model. □

**Theorem 3.4.** *The mathematical model involving Atangana-Baleanu Caputo sense fractional order derivative 3.1.2 has a solution if there exist $t_0$ such that*

$$\frac{(1-\alpha)}{M(\alpha)}b + \frac{1}{\Gamma(\alpha)M(\alpha)}b(t-t_0)^{\alpha} < 1$$

*Proof.* As we know that all class functions are bounded 11 and utilizing recursive algorithm, we get

$$\|\omega_m(t)\| \leq \|X_m(t_0)\|\left[\frac{(1-\alpha)}{M(\alpha)}b + \frac{1}{\Gamma(\alpha)M(\alpha)}b(t-t_0)^{\alpha}\right]^m$$

Hence, the solution of mathematical model is continuous and exists.
Now, by theorem 2.4, the system of Atangana-Baleanu fractional differential equation with initial conditions satisfies Lipschitz condition.
Let's assume the condition such that,

$$X(t) - X(t_0) = \omega_{m-1}(t) - A_m(t)$$

Thus, we have

$$\begin{aligned}\|A_m(t)\| &= \|\frac{(1-\alpha)}{M(\alpha)}(\Omega(t, X(t)) - \Omega(t, X_{m-1}(t)) \\ &\quad + \frac{\alpha}{\Gamma(\alpha)M(\alpha)}\int_{t_0}^{t}(\Omega(\psi, X(\psi)) - \Omega(\psi, X_{m-1}(\psi))(t-\psi)^{\alpha-1}d\psi\| \\ &\leq \frac{(1-\alpha)}{\Gamma(\alpha)M(\alpha)}b\|X(t) - X_{m-1}(t)\| \\ &\qquad + \frac{1}{\Gamma(\alpha)M(\alpha)}b\|X(t) - X_{m-1}(t)\|(t-t_0)^{\alpha}\end{aligned}$$

By using this process recursive formula,we have

$$\|A_m(t)\| \leq \left(\frac{(1-\alpha)}{M(\alpha)} + \frac{1}{\Gamma(\alpha)M(\alpha)}(t-t_0)^{\alpha}\right)^{m+1} b^{m+1}a$$

Then, at particular point $t_0$

$$\|A_m(t)\| \leq \left(\frac{(1-\alpha)}{M(\alpha)} + \frac{1}{\Gamma(\alpha)M(\alpha)}(t-t_0)^{\alpha}\right)^{m+1} b^{m+1}a \tag{12}$$

Taking the limit of equation 12, as m tends to $\infty$, we get

$$\|A_m\| \to 0$$

This completes the proof of existence through the convergence of the sequence $\{X_m(t)\}$.
Now, let's prove the uniqueness of the solution of the present mathematical model. Let's assume that $X^*(t)$ be the solution of the proposed fractional order model.

$$\begin{aligned}X(t) - X^*(t) = &\frac{(1-\alpha)}{M(\alpha)}(\Omega(t, X(t)) - \Omega(t, X^*(t))) \\ &+ \frac{\alpha}{\Gamma(\alpha)M(\alpha)}\int_{t_0}^{t}(\Omega(\psi, X(\psi)) - \Omega(\psi, X^*(\psi)))(t-\psi)^{\alpha-1}d\psi\end{aligned}$$

$$\begin{aligned}\|X(t) - X^*(t)\| \leq &\frac{(1-\alpha)}{M(\alpha)}\|\Omega(t, X(t)) - \Omega(t, X^*(t))\| \\ &+ \frac{\alpha}{\Gamma(\alpha)M(\alpha)}\int_{t_0}^{t}\|\Omega(\psi, X(\psi)) - \Omega(\psi, X^*(\psi))\|(t-\psi)^{\alpha-1}d\psi\end{aligned}$$

Employing the results present in 10, we get

$$\|X(t) - X_i^*(t)\| \leq \frac{(1-\alpha)}{M(\alpha)} b\|X(t) - X^*(t)\| + \frac{1}{\Gamma(\alpha)M(\alpha)} b\|X(t) - X^*(t)\|(t-t_0)^\alpha$$

$$\|X(t) - X^*(t)\| \left(1 - \frac{(1-\alpha)}{M(\alpha)} b - \frac{1}{\Gamma(\alpha)M(\alpha)} b(t-t_0)^\alpha\right) \leq 0 \quad (13)$$

□

**Theorem 3.5.** *The fractional order model has a unique solution if*

$$\|X(t) - X^*(t)\| \left(1 - \frac{(1-\alpha)}{M(\alpha)} b - \frac{1}{\Gamma(\alpha)M(\alpha)} b(t-t_0)^\alpha\right) > 0 \quad (14)$$

*Proof.* By using equation number 13

$$\|X(t) - X^*(t)\| \left(1 - \frac{(1-\alpha)}{M(\alpha)} b - \frac{1}{\Gamma(\alpha)M(\alpha)} b(t-t_0)^\alpha\right) \leq 0$$

Hence, we can conclude that

$$\|X(t) - X^*(t)\| = 0$$

It implies the proof.

$$X(t) = X^*(t)$$

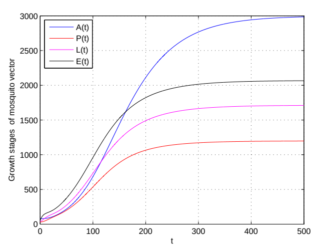
Thus, the present model has unique solution. □

## 4. Numerical analysis and description

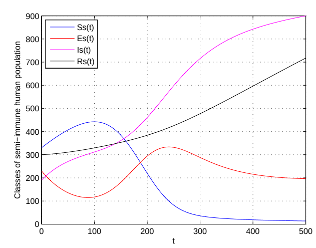
In this section, we have presented and analysed the numerical solution of the model by using generalised Euler method [GEM]. The numerical results are significantly helpful to explore the behaviour of the system of mathematical model and the impact of fractional order of the system consequently. In the first assumption, we have taken suitable values of parameters along with initial values of the classes as given in [15]. Most of the parameters and initial conditions of the present model are fitted by keeping in mind the realistic statistics. The values of parameter are fitted as $p = 0.25$; $c_{mn} = 0.021$; $c_{ms} = 0.012$; $c_{nm} = 0.11$; $c_{sm} = 0.08$; $c_{\bar{m}n} = 0.008$; $\Lambda_h = 10$; $\upsilon_n = 0.10$; $\upsilon_s = 0.06$; $\upsilon_m = 0.083$; $\delta_s = 0.01$; $\delta_n = 0.001$; $f_h = 0.00063$; $f_m = 0.1$; $K_A = 3000$; $K_L = 5000$; $K_P = 4000$; $b = 2$; $s_E = 0.6$; $d_E = 0.3$; $s_L = 0.4$; $d_L = 0.3$; $s_P = 0.25$; $d_P = 0.15$; $\gamma_n = 0.000018$; $\gamma_s = 0.00003$; $\beta_s = 0.0055$; $\nu_s = 0.025$ and the initial values as $E(0) = 50$; $L(0) = 40$; $P(0) = 30$; $A(0) = 80$; $Sn(0) = 600$; $Ss(0) = 330$; $In(0) = 500$; $En(0) = 250$; $Es(1) = 230$; $Is(1) = 190$; $Rs(1) = 300$; $Sm(1) = 80$; $Em(1) = 80$; $Im(1) = 50$.

In figure 1, we have demonstrated numerical analysis of all the classes of human population and mosquito along with maturation stages of mosquito vector for

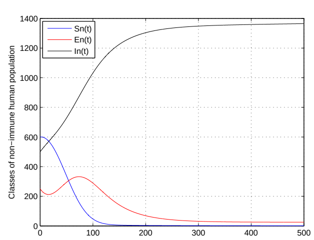
integer order one ($\alpha = 1$). It has been observed that the graphical results match with the previous results which are presented in [15].
Further, figure 2 describes the dynamics of every class by varying fractional order. In figure 2(a), the maturation stages of mosquitoes have been analysed and it has been observed that the change in every stage of growth shows change in behaviour upto certain period and then takes the steady state. It can be emphasised here that the rate of growth at various stages of mosquitoes can be formulated by fractional order instead of varying parameters as in [15]. In natural scenario also, the rate of growth of mosquitoes reaches upto certain stage then proceeds to saturation level. Figure 2(b) deals with dynamics of classes of semi-immune human population for various fractional order ($\alpha$) of differential equations present in the model. The change in fractional order ($\alpha$) shows the variation in rate of susceptible class in downward direction upto certain period which later stabilises. In latent class, the graph indicates more variation than other classes. In this class, growth shows mixed effects with respect to changes in fractional order ($\alpha$) then proceeding to stable position.



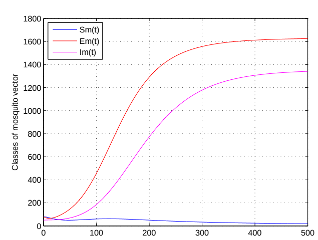

(a) Numerical analysis of growth stages of mosquito vector




(b) Numerical analysis of classes of semi-immune human population

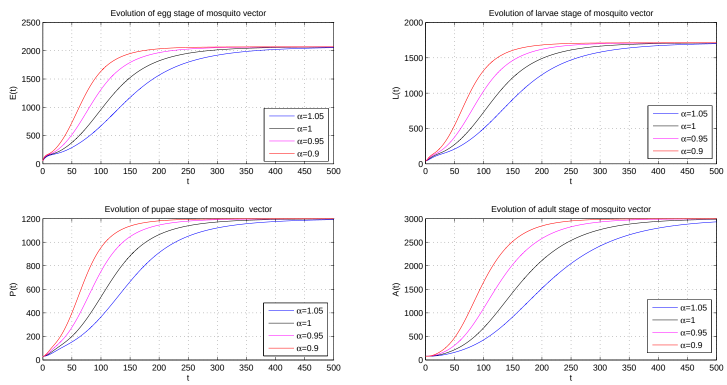



(c) Numerical analysis of classes of non-immune human population

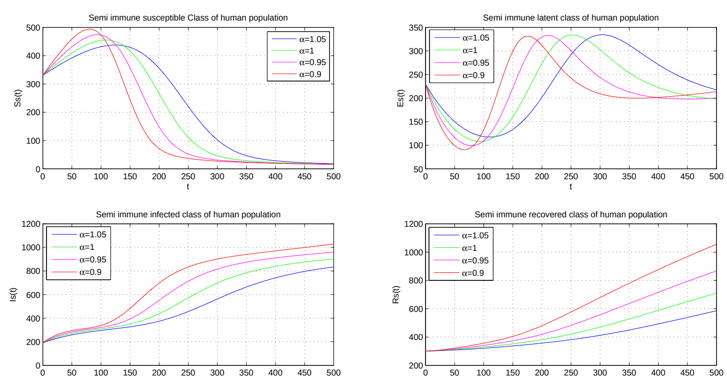



(d) Numerical analysis of mature mosquito vector

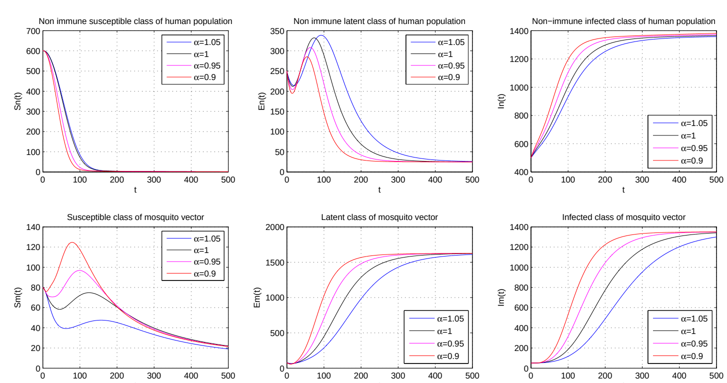
FIGURE 1. Numerical analysis for mathematical model taking values at $\alpha = 1$




(a) Dynamics of growth stages of mosquito vectors by varying $\alpha$



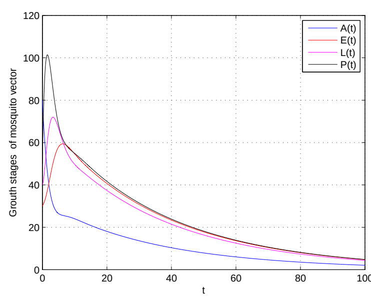

(b) Dynamics of semi-immune human population by varying $\alpha$



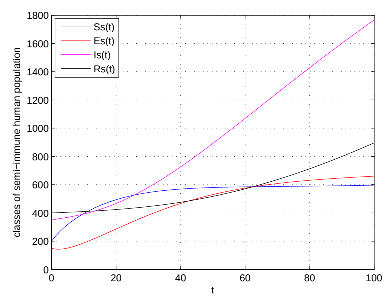

(c) Dynamics of non-immune human population and mosquito vector by varying $\alpha$

FIGURE 2. Numerical analysis for mathematical model taking values at $\alpha = 1.05$, $\alpha = 1$, $\alpha = 0.95$ and $\alpha = 0.9$

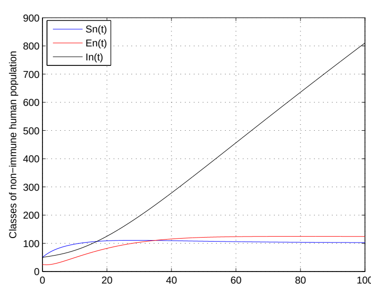
While the growth in semi-immune infected class and semi-immune recovered class of human population shows flat curve proceeding towards the stability. In this class, speed of growth is more for $\alpha = 0.9$ than $\alpha = 1.05$. In figure 2(c), dynamics of non-immune classes of human population and classes of mosquito vectors have been presented in sub-plots. The rate of non-immune susceptible class and non-immune latent class rapidly decreases and take constant position. On the other side, number of non-immune infected human population increases rapidly and then takes steady state. The rate of change of non-immune susceptible and latent human population is directly proportional to fractional order (i.e. $\alpha$) and that of non-immune infected human population is inversely proportional. Lastly, the effect of change in fractional order ($\alpha$)in classes of mosquito vectors have been analysed. The rate of change of every class of mosquito vector is being inversely proportional to the fractional order ($\alpha$) of the respective class.



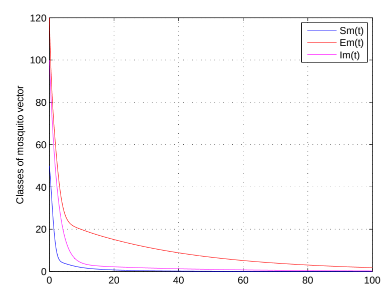

(a) Numerical analysis of growth stages of mosquito vector




(b) Numerical analysis of classes of semi-immune human population

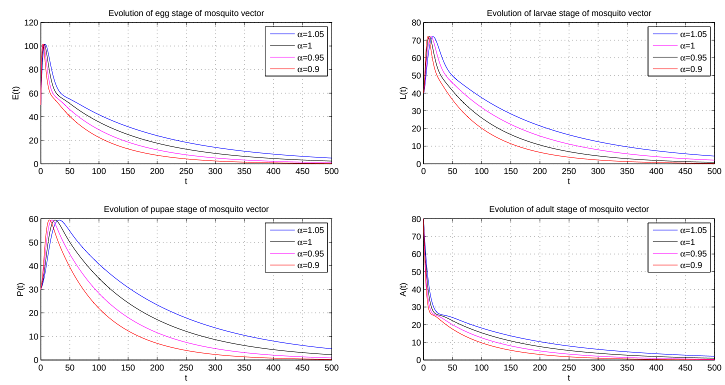



(c) Numerical analysis of classes of non-immune human population

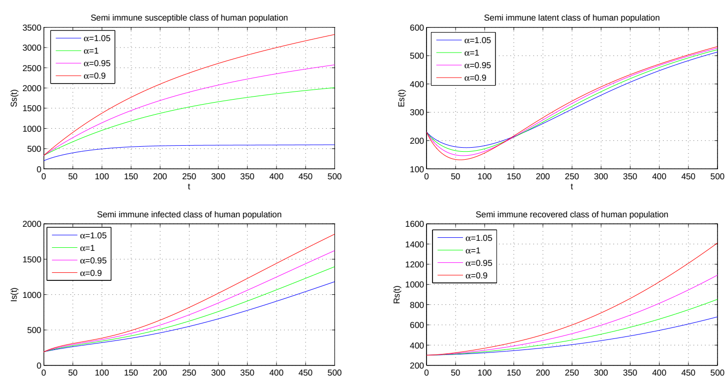



(d) Numerical analysis of mature mosquito vector

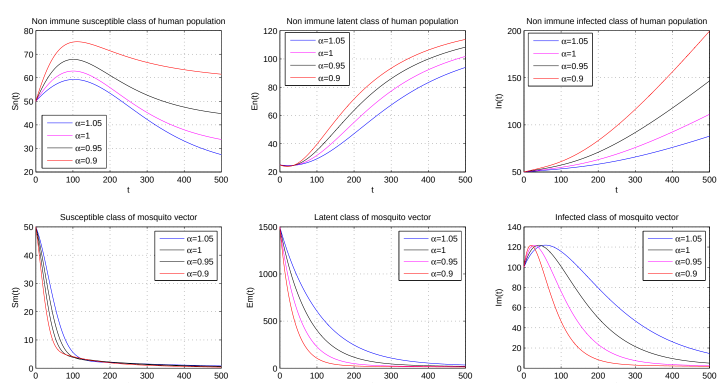
FIGURE 3. Numerical analysis for mathematical model taking values at $\alpha = 1$




(a) Dynamics of growth stages of mosquito vectors by varying $\alpha$




(b) Dynamics of semi-immune human population by varying $\alpha$




(c) Dynamics of non-immune human population and mosquito vector by varying $\alpha$

FIGURE 4. Numerical analysis for mathematical model by varying $\alpha$

In the another set, values of parameters and initial value conditions have been fitted such that another state of dynamics of malaria transmission can be analysed through graphical simulation as given below.
The figures 3 and 4 have been plotted by fitting the set of values of parameters and the initial values of various classes as $p = 0.25$; $c_{mn} = 0.021$; $c_{ms} = 0.012$; $c_{nm} = 0.11$; $c_{sm} = 0.08$; $c_{\bar{mn}} = 0.008$; $\Lambda_h = 50$; $\upsilon_n = 0.10$; $\upsilon_s = 0.06$; $\upsilon_m = 0.083$; $\delta_s = 0.01$; $\delta_n = 0.001$; $f_h = 0.00063$; $f_m = 0.6$; $k_n = 10000$; $K_L = 5000$; $K_P = 4000$; $b = 2$; $s_E = 0.6$; $d_E = 0.3$; $s_L = 0.4$; $d_L = 0.3$; $s_P = 0.25$; $d_P = 0.15$; $\gamma_n = 0.000018$; $\gamma_s = 0.00003$; $\beta_s = 0.0055$; $\nu_s = 0.25$; with initial values $E(0) = 50$; $L(0) = 40$; $P(0) = 30$; $A(0) = 80$; $S_n(0) = 50$; $S_s(0) = 200$; $I_n(0) = 50$; $E_n(0) = 25$; $E_s(0) = 150$; $I_s(0) = 350$; $R_s(0) = 400$; $S_m(0) = 50$; $E_m(0) = 120$; $I_m(0) = 100$.
We have illustrated the dynamics of all the classes of human population and mosquito vector systematically. In this case, figure 3(a) and 3(d) depicts that all classes of mosquito vector decrease sharply and take stable position after some period. In figure 3(b) and 3(c), the graph of infected classes in non-immune and semi-immune classes grows gradually. As time extends, population of mosquito decreases in all growth stages with all classes but infected human population shows consistent growth.
In figure 4, the dynamics of all classes of human population and mosquito vectors have been shown graphically with respect to variation of fractional order (i.e. $\alpha$). In figure 4(a), the graph describes growth of mosquitoes in four stages indicating positive relation with fractional order. In all the stages, population of mosquito vector increases rapidly and then it decreases. In figure 4(b), classes of semi-immune human population has been explained. The rate of growth in semi-immune susceptible, latent, infected and recovered classes are inversely proportional to fractional order (i. e. $\alpha$), while the rate of growth in latent class shows the mixed proportion with fractional order. In figure 4(c), all classes of non-immune human population increases upto certain stage and then comes to constant state. Further the growth rate of all classes of non-immune human population is directly proportional to the fractional order ($\alpha$). In all classes of adult mosquito vector, the rate of growth is directly proportional to fractional order ($\alpha$).

## 5. Conclusion

In our research, fractional order mathematical model of malaria transmission dynamics using Atangana-Baleanu operator has been analysed. In present model, we hypothesized the effects of malaria in two groups of human population on the basis of immunity factor of human population. Considering the impact of mosquito vector on malaria dynamics, we have incorporated various stages of mosquito vector along with family of classes of adult mosquitoes. It has been observed that present model is well defined biologically. The positivity and boundedness of the model has been studied. As Atangana-Baleanu operator satisfies Lipschitz condition, it has been applied successfully to check existence and uniqueness of the solution of present model. Generalised Euler method has been applied successfully to illustrate the dynamics of all the stages of mosquito vectors, classes of human population and classes of mature mosquito vector for various values of fractional order ($\alpha$) using MATLAB. The present mathematical model, a transformation of integer order model on observation yields that the present result for integer order so obtained match with the previous results explained by Ousmane Koutou et al. [15]. This fractional order model gives separate compartment for semi-immune and non-immune groups of populations providing us a broad spectrum to calibrate different degree of infectivity. It also gives us scope to default from treatment. Therefore, the authors have depicted the variation in dynamics of malaria transmission incorporating immunity factor in humans efficiently, which we hope would help in eradication of malaria. The authors hence highlight the utility of fractional order mathematical model for understanding, analysis and interpretation of dynamics of the disease.

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