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A LOWER ESTIMATE FOR THE FIRST DIRICHLET EIGENVALUE ON COMPACT MANIFOLDS

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Abstract. We prove a lower estimate of Neumann eigenvalues on compact manifolds with the condition that the Ricci curvature is bounded below. We improved the earlier results.

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1. Introduction

Let M be a compact Riemannian manifold of dimension n with the boundary ∂M. We consider the solution of the equation, called the Dirichlet eigenvalue problem,

$$\begin{aligned} \Delta u + \lambda u = 0 \qquad & \text{on } M, \\ u = 0 \qquad & \text{on } \partial M. \end{aligned}$$

The aim of this paper is to give a lower bound estimate of the first Dirichlet eigenvalue on compact manifold. The first main result in this area was due to Lichnerowicz and Obata in [5] and [8]. In their work, they assumed that the compact manifold is of dimension n without its boundary, of which RIcci curvature is greater than a positive constant $K > 0$ and estimated the first eigenvalue λ below as followings ;

$$\lambda \geqslant nK. \tag{1}$$

It is remarkable that this constant is sharp.

In [13], S. T. Yau showed that one can estimate the first eigenvalue from below by a lower bound of the volume, an upper bound of the diameter and a lower bound of the Ricci curvature. Furthermore, in [3], P. Li demonstrated to drop the dependency of the volume in the lower estimate of the first eigenvalues

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for general manifolds. P. Li's method depends on a gradient estimate of the first eigenfunction corresponding to λ_1. The following estimate

$$\lambda \geqslant \frac{\pi^2}{2d^2} \tag{2}$$

comes under the condition of the nonnegative Ricci curvature. In [3], he also proved the next estimate (3) with the Ricci curvature condition $\mathrm{Ric}(M) \geqslant -(n-1)K$,

$$\lambda \geqslant \frac{1}{2(n-1)d^2} \exp(-1 - \sqrt{1 + 4(n-1)^2 d^2 K}), \tag{3}$$

where d is the diameter of M and $K > 0$. P. Li [3] also extended these estimates to compact manifolds with boundary. For the Dirichlet boundary value problem, the estimate also depends on the lower bound of the mean curvature of the boundary. For the Neumann boundary problem, he had to assume that the second fundamental form of the boundary is positive semidefinite. However in the Dirichlet boundary problem, the diameter d can be replaced by the radius of the largest geodesic ball inscribed into the manifold.

In [14], under the condition of the nonnegative Ricci curvature, Zhong and Yang proved the sharper estimate

$$\lambda \geqslant \frac{\pi^2}{d^2} \tag{4}$$

than (2) with a judicious test function. Despite the fact that the curvature condition in (3) implies the nonnegativity of the curvature, the inequality (3) does not imply (2). In this paper, we prove a new lower estimate of λ, which improve the result of (3).

2. Lower Estimates of the first Neumann Eigenvalues

Suppose u is a nonconstant eigenfunction corresponding to the first Neumann eigenvalue $\lambda_1 > 0$. Then u has the following property

(*) For any real $\alpha \neq 0$, αu is also a first eigenfunction corresponding to λ_1. By property (*), we can get an eigenfunction u such that

$$1 = \sup u > \inf u = -k \geqslant -1.$$

Theorem 2.1. *suppose M is an n-dimensional compact Riemannian manifold with boundary ∂M and the Ricci curvature* $\mathrm{Ric}(M)$ *of M is bounded below by a nonpositive constant,* $\mathrm{Ric}(M) \geqslant -(n-1)K$. *Assume that ∂M has nonnegative mean curvature with respect to the outward normal, that is convex. Let d denote the inscribed radius of M, i.e. the radius the biggest ball than can be fitted into M.*

Then the first eigenvalue λ with the Dirichlet boundary condition satisfies the inequality

$$\lambda \geqslant \frac{3\pi^2}{16d^2}(1+\sqrt{1+4(n-1)d^2K})\exp(-1-\sqrt{1+4(n-1)d^2K}).$$

In order to prove Theorem 2.1, we find a kind of gradient estimate for u, which is proved in Lemma 2.2.

Lemma 2.2. *Assume the same conditions as in Theorem 2.1. Let $\beta > 1$ be a constant. There is $x_0 \in M$ such that*

$$\frac{8u^2|\nabla u|^2}{(\beta^2-u^2)^2}\bigg|_{x_0} + \frac{2|\nabla u|^2}{\beta^2-u^2}\bigg|_{x_0} \leqslant \frac{\beta^2+1}{\beta^2-1}\lambda + (n-1)K.$$

Proof. Define a nonnegative map $G : M \to \mathbb{R}$ such that

$$G(x) = \frac{|\nabla u|^2}{(\beta^2-u^2)^2}.$$

Since M is compact, we apply the maximum principle to our map $G(x)$. Then there is a point $x_0 \in M$ such that

$$G(x_0) = \sup_M G(x).$$

We claim that

$$\nabla G(x_0) = 0. \tag{5}$$

If $x_0 \in M - \partial M$, (5) is obviously true. Suppose that $x_0 \in \partial M$. Let ν be the unit outward normal vector at $x_0 \in \partial M$. Take a local orthonormal frame $\{e_1, \cdots, e_n\}$ near $x_0 \in \partial M$ with $\nu = e_n$. Since $e_1, \cdots, e_{n-1}$ is an orthonormal frmae of ∂M and $G(x_0)$ is the maximum, we have

$$e_i(G(x_0)) = G_i(x_0) = 0 \quad \text{for } i \le n-1$$

and

$$\nu(G)(x_0) = G_n(x_0) \ge 0. \tag{6}$$

Then

$$\begin{aligned}\nu(G)(x_0) &= e_n(G)(x_0)\\ &= G_n(x_0) = \Big[\frac{\sum 2u_iu_{in}}{(\beta^2-u^2)^2} + 4|\nabla u|^2\cdot\frac{uu_n}{(\beta^2-u^2)^3}\Big]\bigg|_{x_0}.\end{aligned} \tag{7}$$

By (6) and (7), we get

$$0 \leqslant \frac{\nu(G)}{2G}(x_0) = \Big[\frac{\sum u_iu_{in}}{|\nabla u|^2} + \frac{2uu_n}{(\beta^2-u^2)}\Big]\bigg|_{x_0} = \frac{\sum u_iu_{in}}{|\nabla u|^2}\bigg|_{x_0} \tag{8}$$

Since u is an Dirichlet eigenfunction, that is, $u(x) = 0$ for any $x \in \partial M$, we have that $e_i(u) = u_i = 0$ on ∂M for $i \leq n-1$. Since $\{e_1, e_2, \cdots, e_{n-1}\}$ is the local frame of ∂M about $x_0 \in \partial M$. At $x_0 \in \partial M$,

$$\begin{aligned} & \sum u_i u_{in} = u_n u_{nn} = u_n(\Delta u - \sum_{i=1}^{n-1} u_{ii}) \\ = \quad & -\lambda u - \sum_{i=1}^{n-1} u_{ii} = -u_n \sum_{i=1}^{n-1} u_{ii} \\ = \quad & -u_n \sum_{i=1}^{n-1} (e_i(e_i u) - (\nabla_{e_i} e_i)u)(x_0) = u_n \sum_{i=1}^{n-1} (\nabla_{e_i} e_i) u x_0 \\ = \quad & u_n \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} < \nabla_{e_i} e_i, e_j > u_j x_0 = u_n^2 \sum_{i=1}^{n-1} < \nabla_{e_i} e_i, e_n > x_0 \\ = \quad & -u_n^2 \sum_{i=1}^{n-1} h_{jj} = -u_n^2 m(x_0), \end{aligned}$$

where (h_{ij}) is the second fundamental form of ∂M with respect to the outward normal e_n and m is the mean curvature of ∂M with respect to e_n. Since the boundary ∂M of M has a nonnegative mean curvaturel, we have

$$\sum u_i u_{in} = -u_n^2 m(x_0) \leq 0. \tag{9}$$

By (7) and (9) , we get that

$$0 \leq \frac{\nu(G)}{2G}(x_0) = \frac{\sum u_i u_{in}}{|\nabla u|^2} = -\frac{-u_n^2 m(x_0)}{|\nabla u|^2} \leq 0. \tag{10}$$

We havet $\nu(G)(x_0) = G_n(x_0) = 0$ and $\nabla G(x_0) = 0$.
Therefore in all cases off $x_0 \in M - \partial M$ or $x_0 \in \partial M$, (5) holds, that is, $\Delta G(x_0) = 0$. By (5) and the maximum principle, we have

$$\nabla G(x_0) = 0 \qquad \text{and} \quad \Delta G(x_0) \leq 0 \tag{11}$$

Thus we get that for all $j = 1, \cdots, n$,

$$0 = G_j(x_0) = \Big[\frac{2\sum_i u_i u_{ij}}{(\beta^2 - u^2)^2} + \frac{4|\nabla u|^2 u u_j}{(\beta^2 - u^2)^3}\Big]\Bigg|_{x_0}, \tag{12}$$

and

$$\begin{aligned} 0 \quad \geqslant \quad & \Delta G(x_0) = \sum G_{jj}(x_0) \\ = \quad & 2\frac{\sum(u_{ij}^2 + u_i u_{ijj})}{(\beta^2 - u^2)^2} + 8\frac{\sum u_i u_{ij} u_j u}{(\beta^2 - u^2)^3} + 4\frac{|\nabla u|^4}{(\beta^2 - u^2)^3} \\ & +4\frac{|\nabla u|^2 u u_{jj}}{(\beta^2 - u^2)} + 24\frac{|\nabla u|^4 u^2}{(\beta^2 - u^2)^4}. \end{aligned} \tag{13}$$

From (12), we get that all $j = 1, \cdots, n$,

$$\sum_i u_i u_{ij} \;=\; -\frac{2|\nabla u|^2 u u_j}{\beta^2 - u^2}. \tag{14}$$

Substituting (14) into (13), we get that at x_0

$$\begin{aligned} 0 \quad \geqslant \quad & \frac{\sum (u_{ij}^2 + u_i u_{ijj})}{(\beta^2 - u^2)^2} + 4\frac{u^2|\nabla u|^4}{(\beta^2 - u^2)^4} + \frac{2|\nabla u|^4}{(\beta^2 - u^2)^3} \\ & -2\frac{\lambda|\nabla u|^2 u^2}{(\beta^2 - u^2)^3}. \end{aligned} \tag{15}$$

By rotating the orthonormal frame we can take the orthonormal frame $\{e_0, \cdots, e_n\}$ about x_0 such that

$$e_1 = \frac{\nabla u}{|\nabla u|} \neq 0.$$

Then $e_1(u) = u_1 = |\nabla u|$ and $u_2 = \cdots = u_n = 0$ at x_0. We have that for all $j = 1, \cdots, n$

$$0 = \frac{G_j}{2G}(x_0) = \Big[\frac{\sum u_i u_{ij}}{|\nabla u|^2} + \frac{2uu_j}{(\beta^2 - u^2)}\Big]\Big|_{x_0}$$

and

$$\frac{u_{11}}{|\nabla u|^2} = -\frac{2u}{(\beta^2 - u^2)}. \tag{16}$$

And we have

$$u_{jij} = u_{jji} + \sum_l u_l R_{ljj1}$$

$$\text{and} \quad u_{j1j} = u_{jj1} + u_1 R_{1jj1}.$$

Then we get that

$$\begin{aligned} \sum_{i,j} \big[u_{ij}^2 + u_i u_{ijj}\big] \qquad & = \textstyle\sum_{i,j} \big[u_{ij}^2\big] + \sum_j \big[u_1 u_{jj1} + u_1^2 R_{1jj1}\big] \\ & \geqslant \textstyle\sum_{i,j} \big[u_{ij}^2\big] - \lambda|\nabla u|^2 - |\nabla u|^2 (n-1)K. \end{aligned} \tag{17}$$

It follows from (15), (16) and (17) that

$$\begin{aligned} 0 \quad \geqslant \quad & \frac{\sum u_{ij}^2}{(\beta^2 - u^2)^2} - \frac{\lambda|\nabla u|^2}{(\beta^2 - u^2)^2} - \frac{|\nabla u|^2(n-1)K}{(\beta^2 - u^2)^2} \\ & + \frac{4u^2|\nabla u|^4}{(\beta^2 - u^2)^4} + \frac{2|\nabla u|^4}{(\beta^2 - u^2)^3} - \frac{2\lambda|\nabla u|^2 u^2}{(\beta^2 - u^2)^2} \\ \geqslant \quad & \frac{\sum_{j\geqslant 1} u_{jj}^2}{(\beta^2 - u^2)^2} - \frac{\lambda|\nabla u|^2}{(\beta^2 - u^2)^2} - \frac{(n-1)K|\nabla u|^2}{(\beta^2 - u^2)^2} \\ & + \frac{4u^2|\nabla u|^4}{(\beta^2 - u^2)^4} + \frac{2|\nabla u|^4}{(\beta^2 - u^2)^3} - \frac{2\lambda|\nabla u|^2 u^2}{(\beta^2 - u^2)^2}. \end{aligned}$$

$$\begin{aligned}
&\geqslant \frac{u_{11}^2}{(\beta^2-u^2)^2}-\frac{\lambda|\nabla u|^2}{(\beta^2-u^2)^2}-\frac{(n-1)K|\nabla u|^2}{(\beta^2-u^2)^2}\\
&\quad+\frac{2|\nabla u|^4}{(\beta^2-u^2)^3}-\frac{2\lambda|\nabla u|^2u^2}{(\beta^2-u^2)^2}.\\
&\geqslant \frac{8u^2|\nabla u|^4}{(\beta^2-u^2)^4}-\frac{\lambda|\nabla u|^2}{(\beta^2-u^2)^2}-\frac{(n-1)K|\nabla u|^2}{(\beta^2-u^2)^2}\\
&\quad+\frac{2|\nabla u|^4}{(\beta^2-u^2)^3}-\frac{2\lambda|\nabla u|^2u^2}{(\beta^2-u^2)^2}.
\end{aligned} \tag{18}$$

Since $|\nabla u|(x_0)\neq 0$, (18) is a quadratic equation with respect to $|\nabla u|^2$. Then we have that at x_0

$$\frac{8u^2|\nabla u|^2}{(\beta^2-u^2)^2}+2\frac{|\nabla u|^2}{(\beta^2-u^2)}\leqslant \frac{\beta^2+u^2}{\beta^2-u^2}\lambda+\,(n-1)K.$$

Since $\sup_M u(x)=1$, our Lemma holds, that is,

$$\frac{8u^2|\nabla u|^2}{(\beta^2-u^2)^2}+\frac{2|\nabla u|^2}{(\beta^2-u^2)}\leqslant \frac{\beta^2-u^2}{\beta^2-1}\lambda+\,(n-1)K.$$

□

The inequality in Lemma 2.2 occurs at a point x_0. Lemma 2.3 shows that the inequality holds globally on M, which is a kind of gradient estimate of u.

Lemma 2.3. *For any $x\in M$,*

$$\frac{2u^2|\nabla u|^2}{(\beta^2-u^2)^2}(x)+\frac{2|\nabla u|^2}{(\beta^2-u^2)}(x)\leqslant \frac{\beta^2+u^2}{\beta^2-1}\lambda+\,(n-1)K. \tag{19}$$

Proof. Let $x\in M$ be any point.

$$\frac{2u^2|\nabla u|^2}{(\beta^2-u^2)^2}(x)+\frac{2|\nabla u|^2}{(\beta^2-u^2)}(x)\leqslant \frac{2\beta^2|\nabla u|^2}{(\beta^2-u^2)^2}(x)\leqslant \frac{2\beta^2|\nabla u|^2}{(\beta^2-u^2)^2}(x_0), \tag{20}$$

where the inequality comes from $G(x_0)=\sup_M G(x)$. by Lemma 2.2, we have that

$$\begin{aligned}
\frac{2u^2|\nabla u|^2}{(\beta^2-u^2)^2}(x)+\frac{2|\nabla u|^2}{(\beta^2-u^2)}(x) &\leqslant \frac{2\beta^2|\nabla u|^2}{(\beta^2-u^2)^2}(x_0)\\
&\leqslant \frac{2u^2|\nabla u|^2}{(\beta^2-u^2)^2}(x_0)+\frac{2|\nabla u|^2}{(\beta^2-u^2)}(x_0)\\
&\leqslant \frac{2\beta^2}{\beta^2-1}\lambda+\,(n-1)K.
\end{aligned}$$

□

Proof of Theorem2.1. If $x_1\in M$ is the point where u achieves its supremum and γ is the shortest geodesic joining x_1 and N, then γ has length at most d.

Now we assume that $1 < \beta \le \sqrt{2}$. Then γ has length at most d that is the inscribed radius of M. Let $\gamma(t_0) = x_0$. Then we have

$$\begin{aligned}
&\int_\gamma \Big(\frac{u}{(\beta^2-u^2)} + \frac{1}{\sqrt{\beta^2-u^2}}\Big) du \\
&= \int_0^{t_0} \frac{u\langle \nabla u, \gamma'(t)\rangle}{\beta^2-u^2} dt + \int_0^{t_0} \frac{\langle \nabla u\,,\, \gamma'(t)\rangle}{\sqrt{\beta^2-u^2}} dt \\
&\le \int_0^{t_0} \Big[\frac{u|\nabla u|}{\beta^2-u^2} + \frac{|\nabla u|}{\sqrt{\beta^2-u^2}}\Big]|\gamma'(t)|dt \\
&\le \int_0^{t_0} \Big[\frac{2u^2|\nabla u|^2}{(\beta^2-u^2)^2} + \frac{2|\nabla u|^2}{\beta^2-u^2}\Big]^{\frac{1}{2}}|\gamma'(t)|dt \\
&\le \Big[\frac{2\beta^2}{\beta^2-1}\lambda + (n-1)K\Big]^{\frac{1}{2}} \int_0^{t_0} |\gamma'(t)|dt \\
&\le d\Big[\frac{\beta^2+1}{\beta^2-1}\lambda + (n-1)K\Big]^{\frac{1}{2}}.
\end{aligned} \tag{21}$$

Since we assume $\beta \le \sqrt{2}$, we have that

$$\frac{1}{2}\ln\frac{\beta^2}{\beta^2-1} + \sin^{-1}\frac{1}{\beta} \le \int_\gamma \Big(\frac{u}{(\beta^2-u^2)} + \frac{1}{\sqrt{\beta^2-u^2}}\Big) du. \tag{22}$$

By (21) and (22),

$$\frac{1}{2}\ln\frac{\beta^2}{\beta^2-1} + \sin^{-1}\frac{1}{\beta} \le\, d\,\Big(\frac{2\beta^2}{\beta^2-1}\lambda + (n-1)K\Big)^{\frac{1}{2}}.$$

And we have

$$\lambda \geqslant \frac{\beta^2-1}{2\beta^2}\,\Big(\frac{1}{4d^2}\Big(\ln\frac{\beta^2}{\beta^2-1} + 2\sin^{-1}\frac{1}{\beta}\Big)^2 - (n-1)K\Big)$$

Take $\alpha_0 > 1$ such that $\sin^{-1}\frac{1}{\alpha_0} = \frac{19\pi}{60} < 1$.

$$\lambda \geqslant \frac{\beta^2-1}{2\beta^2}\frac{1}{4d^2}\,\Big(\Big(\ln\frac{\beta^2}{\beta^2-1} + \frac{19\pi}{60}\Big)^2 - 4(n-1)d^2K\Big)$$

Let $t = \frac{\beta^2-1}{\beta^2}$. Then $0 < t < 1$ and $\ln t < 0$. Define a function f by

$$f(t) = \frac{t}{8d^2}\,\Big(\Big(-\ln t + \frac{19\pi}{60}\Big)^2 - 4(n-1)Kd^2\Big)$$

for any $0 < t < 1$. Now in order to find $0 < t_0 \leqslant 1$ such that $f(t_0) = \sup_{0<t<1} f(t)$, we differentiate the function $f(t)$ and then

$$f'(t) = \frac{1}{8d^2}\,\Big((\ln t)^2 - 2(\frac{19\pi}{60} - 1)\ln t + \frac{19\pi}{30} + \frac{361\pi^2}{3600} - 4(n-1)Kd^2\Big).$$

Let $\ln t_0 = -1 - \sqrt{1 + 4d^2(n-1)K} + \frac{19\pi}{60}$. Then t_0 is a root of $f(t)$, that is, $f(t_0) = 0$. Then

$$\ln t_0 = -1 - \sqrt{1 + 4K(n-1)d^2} + \frac{19\pi}{60} \leqslant 0,$$

and $f(t)$ attains its maximum at t_0 and

$$f(t_0) = \frac{1}{8d^2}\exp(\frac{19\pi}{60})(1 + \sqrt{1 + 4(n-1)d^2K})\exp(-1 - \sqrt{1 + 4(n-1)d^2K}).$$

Since $\exp x \geqslant 1.8 * x^2$, we have that

$$\lambda \geqslant \frac{3\pi^2}{16d^2}(1 + \sqrt{1 + 4(n-1)d^2K})\exp(-1 - \sqrt{1 + 4(n-1)d^2K}).$$

□

References

1. I. Chavel, *Eigenvalues in Riemannian Geometry*, Academic Press, Orlando, FL, 1984.
2. J. Cheeger, *A lower bound for the smallest eigenvalue of the Laplacian*, Princeton Univ. Press, Princeton, NJ, 1970.
3. P. Li and S.T. Yau, *Estimates of eigenvalues of a compact Riemannian manifold*, Proc. Sympos. Pure Math. Amer. Math. Soc. Providence **36** (1980), 205-240.
4. A. Lichnerowicz *Geometric des groups de transformations*, Dunod, Paris, 1958.
5. R. Schon, and S.T. Yau, *Lectures on Differential Geometry, Conference Precedings and Lecture Notes in Geometry and Topology*, International Press, 1994.
6. D. Yang, *Lower bound estimates on the first eigenvalue for compact manifolds with positive Ricci curvature*, Pacific J. Math. **190** (1999), 383-398.
7. S. Yau, *Isoperimetric constants and the first eigenvalue of a compact Riemannian manifold*, Ann. Scient. Ec. Norm. Sup. **4** (1985), 487-507.
8. J.-Q. Zhong and H.C. Yang, *On the estimate of the first eigenvalue of a compact Riemannian manifold*, Sci. Sinica A **27** (1984), 1265-1273.

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