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A RECENT EXTENSION OF THE WEIGHTED MEAN SUMMABILITY OF INFINITE SERIES

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Abstract. We obtain a new matrix generalization result dealing with weighted mean summability of infinite series by using a new general class of power increasing sequences obtained by Sulaiman [9]. This theorem also includes some new and known results dealing with some basic summability methods.

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1. Introduction

By (t_n) we denote the nth $(C,1)$ mean of the sequence (na_n). The series $\sum a_n$ is said to be summable $|C,1|_k$, $k \geq 1$, if (see [3])

$$\sum_{n=1}^{\infty} \frac{1}{n}|t_n|^k < \infty. \tag{1}$$

Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, \quad i \geq 1). \tag{2}$$

The sequence-to-sequence transformation

$$w_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v, \tag{3}$$

defines the sequence (w_n) of the weighted arithmetic mean or simply the $\left(\bar{N}, p_n\right)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see

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[4]). A series $\sum a_n$ with partial sums (s_n) is said to be summable $|\bar{N}, p_n|_k$, $k \geq 1$, if (see [2])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |w_n - w_{n-1}|^k < \infty. \tag{4}$$

If we take $p_n = 1$ for all n, then $|\bar{N}, p_n|_k$ summability is the same as $|C, 1|_k$ summability.

A positive sequence (b_n) is said to be almost increasing if there exists a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$ (see [1]). A positive sequence $a = (a_n)$ is said to be a quasi-β-power increasing if there exists a constant $K = K(\beta, a) \geq 1$ such that

$$Kn^{\beta} a_n \geq m^{\beta} a_m \tag{5}$$

holds for $n \geq m$ (see [5]). It should be noted that every almost increasing sequence is a quasi-β-power increasing sequence for any nonnegative β, but the converse need not be true as can be seen by taking $a_n = n^{-\beta}$.

Let $\sum a_n$ be a given series with partial sums (s_n). Let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix with nonzero diagonal entries. Then A defines a sequence-to-sequence transformation, mapping of the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \quad n = 0, 1, ... \tag{6}$$

A series $\sum a_n$ is said to be summable $|A, p_n|_k$, $k \geq 1$, if (see [8])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |A_n(s) - A_{n-1}(s)|^k < \infty. \tag{7}$$

In the special case, if we take $p_n = 1$ for all n, then $|A, p_n|_k$ summability reduces to $|A|_k$ summability (see [7]).

If we put $a_{nv} = \frac{p_v}{P_n}$, then $|A, p_n|_k$ summability reduces to $\left|\bar{N}, p_n\right|_k$ summability. If we take $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all n, then $|A, p_n|_k$ summability reduces to $|C, 1|_k$ summability.

2. Known Result

In [9], Sulaiman proved the following result dealing with $|\bar{N}, p_n|_k$ summability.

Theorem 2.1 ([9])**.** *If the sequence (X_n) is a quasi-β-power increasing sequence $0 < \beta < 1$, (λ_n) is a sequence of constants both satisfying conditions*

$$\sum_{n=1}^{m} \frac{1}{n} P_n = O(P_m), \tag{8}$$

$$\lambda_n \to 0 \quad as \quad n \to \infty, \tag{9}$$

$$\sum_{n=1}^{\infty} nX_n(\beta)|\Delta|\Delta\lambda_n|| < \infty, \tag{10}$$

and

$$\sum_{n=1}^{m} \frac{1}{n(n^{\beta}X_n)^{k-1}}|t_n|^k = O(m^{\beta}X_m), \tag{11}$$

$$\sum_{n=1}^{m} \frac{p_n}{P_n}\frac{1}{(n^{\beta}X_n)^{k-1}}|t_n|^k = O(m^{\beta}X_m). \tag{12}$$

Then the series $\sum_{n=1}^{\infty} a_n\lambda_n$ *is summable* $|\bar{N}, p_n|_k$, $k \geq 1$.

3. Main Result

The aim of this paper is to generalize Theorem 2.1 for $|A, p_n|_k$ summability method.

Given a normal matrix $A = (a_{nv})$, we associate two lower semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, ... \tag{13}$$

and

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, ... \tag{14}$$

It is known that

$$A_n(s) = \sum_{v=0}^{n} a_{nv}s_v = \sum_{v=0}^{n} \bar{a}_{nv}a_v \tag{15}$$

and

$$\bar{\Delta}A_n(s) = \sum_{v=0}^{n} \hat{a}_{nv}a_v. \tag{16}$$

Let ω be the class of all matrices $A = (a_{nv})$ satisfying

$$A \quad \text{is a positive normal matrix,} \tag{17}$$

$$\bar{a}_{n0} = 1, \quad n = 0, 1, ... \tag{18}$$

$$a_{n-1,v} \geq a_{nv}, \quad n \geq v + 1. \tag{19}$$

Theorem 3.1. *Let* $A \in \omega$ *satisfying*

$$a_{nn} = O\left(\frac{p_n}{P_n}\right) \tag{20}$$

$$\sum_{v=1}^{n-1} \frac{1}{v}|\hat{a}_{nv}| = O(a_{nn}). \tag{21}$$

Let (X_n) be a quasi-β-power increasing sequence $0<\beta<1$. If the sequences (λ_n), and (X_n) satisfy all the conditions of Theorem 2.1, then the series

$$\sum_{n=1}^{\infty} a_n \lambda_n$$

is summable $|A, p_n|_k$, $k \geq 1$.

The following lemmas are required to prove our theorem.

Lemma 3.2 ([9]). *Let (X_n) be a quasi-β-power increasing sequence such that the conditions* (9) *and* (10) *of Theorem 2.1 are satisfied. Then*

$$n^{\beta+1} X_n |\Delta\lambda_n| = O(1) \quad as \quad n \to \infty, \tag{22}$$

$$\sum_{n=1}^{\infty} n^{\beta} X_n |\Delta\lambda_n| < \infty, \tag{23}$$

$$n^{\beta} X_n |\lambda_n| = O(1) \quad as \quad n \to \infty, \tag{24}$$

where $X_n(\beta) = n^{\beta} X_n$.

Lemma 3.3 ([6]). *Let $A \in \omega$ and from the condition* (13), (14), (18) *and* (19)*, then*

$$\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| \leq a_{nn}, \tag{25}$$

$$\sum_{n=v+1}^{m+1} |\Delta_v(\hat{a}_{nv})| \leq a_{vv}, \tag{26}$$

and

$$\sum_{n=v+1}^{m+1} |\hat{a}_{n,v+1}| \leq 1. \tag{27}$$

Proof of Theorem 3.1

Proof. Let (V_n) denotes the A-transform of the series $\sum a_n \lambda_n$. Then, by the definition, we have that

$$\bar{\Delta} V_n = \sum_{v=1}^{n} \hat{a}_{nv} a_v \lambda_v = \sum_{v=1}^{n} v a_v v^{-1} \hat{a}_{nv} \lambda_v.$$

Applying Abel's transformation to this sum, we have that

$$\bar{\Delta} V_n = \sum_{v=1}^{n-1} \Delta_v \left(\hat{a}_{nv} \lambda_v v^{-1}\right) \sum_{r=1}^{v} r a_r + \hat{a}_{nn} \lambda_n n^{-1} \sum_{v=1}^{n} v a_v.$$

By the formula for the difference of products of sequences (see [4]) we have

$$\begin{aligned}
&\bar{\Delta}V_n \\
&= \sum_{v=1}^{n-1}(v+1)t_v\left(\frac{1}{v(v+1)}\hat{a}_{nv}\lambda_v + \frac{1}{(v+1)}\Delta_v(\hat{a}_{nv})\lambda_v + \frac{1}{(v+1)}\hat{a}_{n,v+1}\Delta\lambda_v\right) \\
&+ \frac{n+1}{n}a_{nn}\lambda_n t_n
\end{aligned}$$

$$\bar{\Delta}V_n = \sum_{v=1}^{n-1}\hat{a}_{nv}\lambda_v v^{-1}t_v + \sum_{v=1}^{n-1}\Delta_v(\hat{a}_{nv})\lambda_v t_v + \sum_{v=1}^{n-1}\hat{a}_{n,v+1}t_v\Delta\lambda_v + a_{nn}\lambda_n t_n\frac{n+1}{n}$$

$$\bar{\Delta}V_n = V_{n,1} + V_{n,2} + V_{n,3} + V_{n,4}.$$

To complete the proof of Theorem 3.1, it is sufficient to show that

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1} \mid V_{n,r}\mid^k < \infty, \quad \text{for} \quad r = 1,2,3,4. \tag{28}$$

Firstly, using Hölder's inequality, we have

$$\begin{aligned}
&\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1} \mid V_{n,1}\mid^k = \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left|\sum_{v=1}^{n-1}\hat{a}_{nv}\lambda_v t_v v^{-1}\right|^k \\
&\le \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}|\hat{a}_{nv}||\lambda_v|^k|t_v|^k\frac{1}{v}\right)\times\left(\sum_{v=1}^{n-1}\frac{1}{v}|\hat{a}_{nv}|\right)^{k-1} \\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1} a_{nn}^{k-1}\sum_{v=1}^{n-1}|\hat{a}_{nv}||\lambda_v|^k|t_v|^k\frac{1}{v} \\
&= O(1)\sum_{v=1}^{m}\frac{1}{v}\frac{|\lambda_v|(v^\beta X_v|\lambda_v|)^{k-1}|t_v|^k}{(v^\beta X_v)^{k-1}}\sum_{n=v+1}^{m+1}|\hat{a}_{nv}| \\
&= O(1)\sum_{v=1}^{m}\frac{1}{v}\frac{|\lambda_v||t_v|^k}{(v^\beta X_v)^{k-1}} \\
&= O(1)\sum_{v=1}^{m-1}\left(\sum_{r=1}^{v}\frac{1}{r}\frac{|t_r|^k}{(r^\beta X_r)^{k-1}}\right)\Delta|\lambda_v| + O(1)\left(\sum_{v=1}^{m}\frac{1}{v}\frac{|t_v|^k}{(v^\beta X_v)^{k-1}}\right)|\lambda_m| \\
&= O(1)\sum_{v=1}^{m-1}v^\beta X_v|\Delta\lambda_v| + O(1)m^\beta X_m|\lambda_m| = O(1) \quad \text{as} \quad m\to\infty,
\end{aligned}$$

by the hypotheses of the Theorem 3.1, Lemma 3.2, and Lemma 3.3. And using Lemma 3.2, and Lemma 3.3. we have that

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1} \mid V_{n,2}\mid^k = \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left|\sum_{v=1}^{n-1}\Delta_v(\hat{a}_{nv})\lambda_v t_v\right|^k$$

$$
\begin{aligned}
&\leq \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\lambda_v|^k|t_v|^k\right)\times\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\right)^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}a_{nn}^{k-1}\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\lambda_v|^k|t_v|^k\\
&= O(1)\sum_{v=1}^{m}|\lambda_v|^k|t_v|^k\sum_{n=v+1}^{m+1}|\Delta_v(\hat{a}_{nv})|\\
&= O(1)\sum_{v=1}^{m}\frac{p_v}{P_v}|\lambda_v||t_v|^k\frac{(v^\beta X_v|\lambda_v|)^{k-1}}{(v^\beta X_v)^{k-1}}\\
&= O(1)\sum_{v=1}^{m-1}\left(\sum_{r=1}^{v}\frac{p_r}{P_r}\frac{|t_r|^k}{(r^\beta X_r)^{k-1}}\right)\Delta|\lambda_v| + O(1)\left(\sum_{v=1}^{m}\frac{p_v}{P_v}\frac{|t_v|^k}{(v^\beta X_v)^{k-1}}\right)|\lambda_m|\\
&= O(1)\sum_{v=1}^{m-1}v^\beta X_v|\Delta\lambda_v| + O(1)m^\beta X_m|\lambda_m|\\
&= O(1) \quad \text{as} \quad m\to\infty,
\end{aligned}
$$

by the hypotheses of the Theorem 3.1. Also, using Lemma 3.2, and Lemma 3.3. we have that

$$
\begin{aligned}
&\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\mid V_{n,3}\mid^k = \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left|\sum_{v=1}^{n-1}\hat{a}_{n,v+1}\Delta\lambda_v t_v\right|^k\\
&\leq \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\Delta\lambda_v||t_v|\right)^k\\
&\leq \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|\frac{|\Delta\lambda_v|}{(v^\beta X_v)^{k-1}}|t_v|^k\right)\times\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|v^\beta X_v|\Delta\lambda_v|\right)^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}a_{nn}^{k-1}\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|\frac{|\Delta\lambda_v|}{(v^\beta X_v)^{k-1}}|t_v|^k\right)\times\left(\sum_{v=1}^{n-1}v^\beta X_v|\Delta\lambda_v|\right)^{k-1}\\
&= O(1)\sum_{v=1}^{m}\frac{|\Delta\lambda_v|}{(v^\beta X_v)^{k-1}}|t_v|^k\sum_{n=v+1}^{m+1}|\hat{a}_{n,v+1}|\\
&= O(1)\sum_{v=1}^{m}\frac{v|\Delta\lambda_v|}{(v^\beta X_v)^{k-1}}\frac{|t_v|^k}{v}\\
&= O(1)\sum_{v=1}^{m-1}\left(\sum_{r=1}^{v}\frac{1}{r}\frac{|t_r|^k}{(r^\beta X_r)^{k-1}}\right)\Delta(v|\Delta\lambda_v|) + O(1)\left(\sum_{v=1}^{m}\frac{1}{v}\frac{|t_v|^k}{(v^\beta X_v)^{k-1}}\right)m|\Delta\lambda_m|
\end{aligned}
$$

$$= O(1)\sum_{v=1}^{m-1} v^{\beta} X_v \Delta(v|\Delta\lambda_v|) + O(1)m^{\beta+1}X_m|\Delta\lambda_m|$$
$$= O(1)\sum_{v=1}^{m-1} v^{\beta} X_v |\Delta\lambda_v| + O(1)\sum_{v=1}^{m-1} v^{\beta+1} X_v |\Delta|\Delta\lambda_v|| + O(1)m^{\beta+1}X_m|\Delta\lambda_m|$$
$$= O(1) \quad \text{as} \quad m \to \infty,$$

by the hypotheses of the Theorem 3.1. Finally, we have

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1} | V_{n,4} |^k = O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1} a_{nn}^k |\lambda_n|^k |t_n|^k$$
$$= O(1)\sum_{n=1}^{m} \frac{p_n}{P_n} \frac{|t_n|^k}{(n^{\beta}X_n)^{k-1}} (n^{\beta}X_n|\lambda_n|)^{k-1}|\lambda_n|$$
$$= O(1)\sum_{n=1}^{m} \frac{p_n}{P_n} \frac{|t_n|^k}{(n^{\beta}X_n)^{k-1}} |\lambda_n|$$
$$= O(1)\sum_{n=1}^{m-1} \left(\sum_{v=1}^{n} \frac{p_v}{P_v} \frac{|t_v|^k}{(v^{\beta}X_v)^{k-1}}\right) \Delta|\lambda_n| + O(1)\left(\sum_{n=1}^{m} \frac{p_n}{P_n} \frac{|t_n|^k}{(n^{\beta}X_n)^{k-1}}\right)|\lambda_m|$$
$$= O(1)\sum_{n=1}^{m-1} n^{\beta} X_n |\Delta\lambda_n| + O(1)m^{\beta}X_m|\lambda_m|$$
$$= O(1) \quad \text{as} \quad m \to \infty,$$

by the hypotheses of the Theorem 3.1, and Lemma 3.2. □

This completes the proof of Theorem 3.1.

4. Conclusions

1. If we take $a_{nv} = \frac{p_v}{P_n}$ in Theorem 3.1, then we can return to Theorem 2.1.
2. If we take $p_n = 1$ for all n in Theorem 3.1, then we have a new theorem on $|A|_k$ summability method.
3. If we put $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all n in Theorem 3.1, then we obtain another result concerning $|C,1|_k$ summability method.

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