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A RESEARCH ON THE SPECIAL FUNCTIONS BY USING q-TRIGONOMETRIC FUNCTIONS

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Abstract. In this paper, we introduce the concepts of q-cosine tangent polynomials and q-sine tangent polynomials. From these polynomials, we find some identities and properties by using q-numbers and q-trigonometric functions.

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1. Introduction

In analytic number theory, some properties and identities for Bernoulli, Euler and Genocchi polynomials are usefully utilized (see [1, 2, 14]). According to appearance of various extended versions of these polynomials, many mathematicians have studied and discovered several research results on these polynomials by using traditional theory and new techniques.

In 2013, Ryoo introduced the concept of tangent polynomials and developed several properties of these polynomials (see [10]). Also, tangent numbers are closely related to Euler and Genocchi numbers.

Definition 1.1. Tangent numbers T_n and tangent polynomials $T_n(x)$ are defined by means of generating functions as follows:

$$\sum_{n=0}^{\infty} T_n \frac{t^n}{n!} = \frac{2}{e^{2t}+1} = 2\sum_{m=0}^{\infty}(-1)^m e^{2mt}$$

and

$$\sum_{n=0}^{\infty} T_n(x) \frac{t^n}{n!} = \frac{2}{e^{2t}+1}e^{tx} = 2\sum_{m=0}^{\infty}(-1)^m e^{(2m+x)t}.$$

From Definition 1.1, we find the following theorems (see [4, 10, 13]).

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Theorem 1.2. *Let m and n be positive integers. If m is odd, then the following hold.*

$$(i) \quad T_n(x) = (-1)^n T_n(2-x).$$

$$(ii) \quad T_n(x) = m^n \sum_{i=0}^{m-1} (-1)^i T_n\left(\frac{2i+x}{m}\right).$$

Theorem 1.3. *For any positive integer n, we have*

$$T_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} T_k(x) y^{n-k}.$$

Recently, mathematicians studied Bernoulli and Euler polynomials by combining with trigonometric functions (see [7]). In [9], we investigated the extended version of tangent polynomials and some properties on these polynomials.

Definition 1.4. Let x and y be real numbers. Then the cosine tangent polynomials and the sine tangent polynomials are defined in terms of generating functions as follows:

$$\sum_{n=0}^{\infty} {}_C\mathcal{T}_n(x,y)\frac{t^n}{n!} = \frac{2}{e^{2t}+1} e^{tx} cos ty$$

and

$$\sum_{n=0}^{\infty} {}_S\mathcal{T}_n(x,y)\frac{t^n}{n!} = \frac{2}{e^{2t}+1} e^{tx} sin ty.$$

Theorem 1.5. *Let k be a nonnegative integer and suppose that $e^{2t} \neq -1$. Then the following hold.*

$$(i) \quad \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} {}_C\mathcal{T}_k(x,y) + {}_C\mathcal{T}_n(x,y) = 2C_n(x,y).$$

$$(ii) \quad \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} {}_S\mathcal{T}_k(x,y) + {}_S\mathcal{T}_n(x,y) = 2S_n(x,y).$$

Theorem 1.6. *Let x and y be real numbers. Then we have*

$$(i) \quad {}_C\mathcal{T}_n(2+x,y) + {}_C\mathcal{T}_n(x,y) = 2C_n(x,y).$$

$$(ii) \quad {}_S\mathcal{T}_n(2+x,y) + {}_S\mathcal{T}_n(x,y) = 2S_n(x,y).$$

We next review some definitions related to q-numbers. (For more, the readers can refer to [3, 8, 11, 12].) For any $n \in \mathbb{N}$, the q-number is defined as follows:

$$[n]_q = \frac{1-q^n}{1-q},$$

where $q \neq 1$.

Definition 1.7. The Gaussian binomial coefficients are defined by

$$\begin{bmatrix} m \\ r \end{bmatrix}_q = \begin{cases} 0 & \text{if } r > m \\ \frac{(1-q^m)(1-q^{m-1})\cdots(1-q^{m-r+1})}{(1-q)(1-q^2)\cdots(1-q^r)} & \text{if } r \leq m, \end{cases}$$

where m and r are nonnegative integers.

For $r = 0$, the value is 1 since the numerator and the denominator are both empty products. Like the classical binomial coefficients, the Gaussian binomial coefficients are center-symmetric. There are several analogues of the binomial formula and this definition has a number of properties (see [3, 5, 8, 11, 12]).

Definition 1.8. The q-analogues of $(x - a)^n$ and $(x + a)^n$ are defined by

$$(x \ominus a)_q^n = \begin{cases} 1 & \text{if } n = 0 \\ (x-a)(x-qa)\cdots(x-q^{n-1}a) & \text{if } n \geq 1 \end{cases}$$

and

$$(x \oplus a)_q^n = \begin{cases} 1 & \text{if } n = 0 \\ (x+a)(x+qa)\cdots(x+q^{n-1}a) & \text{if } n \geq 1, \end{cases}$$

respectively.

Definition 1.9. Let z be any complex number with $|z| < 1$. Then two forms of q-exponential functions can be expressed as

$$e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} \qquad \text{and} \qquad E_q(z) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{z^n}{[n]_q!}.$$

From Definition 1.9, we note that (1) $e_q(x)e_q(y) = e_q(x + y)$ if $yx = qxy$; (2) $e_q(x)E_q(-x) = 1$; and (3) $e_{q^{-1}}(x) = E_q(x)$.

Definition 1.10. The q-derivative operator of any function f is

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x} \quad (x \neq 0) \quad \text{and} \quad D_q f(0) = f'(0).$$

We can prove that f is differentiable at 0, and it is clear that $D_q x^n = [n]_q x^{n-1}$.

Definition 1.11. In q-calculus, the q-trigonometric functions are given by

$$\sin_q(x) = \frac{e_q(ix) - e_q(-ix)}{2i}, \qquad \mathrm{SIN}_q(x) = \frac{E_q(ix) - E_q(-ix)}{2i}$$

$$\cos_q(x) = \frac{e_q(ix) + e_q(-ix)}{2}, \qquad \mathrm{COS}_q(x) = \frac{E_q(ix) + E_q(-ix)}{2},$$

where $\mathrm{SIN}_q(x) = \sin_{q^{-1}}(x)$ and $\mathrm{COS}_q(x) = \cos_{q^{-1}}(x)$ (see [6]).

The main purpose of this paper is to define the q-cosine tangent polynomials and the q-sine tangent polynomials by using q-exponential functions and Definition 1.8. To find some properties of these polynomials, we use $C_{n,q}(x, y)$ and $S_{n,q}(x, y)$. We obtain some relations among q-tangent polynomials, q-cosine tangent polynomials and q-sine tangent polynomials. In addition, we derive q-partial derivatives of q-cosine tangent polynomials and q-sine tangent polynomials by using q-derivative.

2. Main results

In this section, we define the q-cosine tangent polynomials and the q-sine tangent polynomials by using q-analogues of $(x+a)^n$ and $(x-a)^n$. We also find some properties of these polynomials by applying q-power series of q-trigonometric functions.

Theorem 2.1. *Let x and y be real numbers and let $i=\sqrt{-1}$. Then the following hold.*

$$(i)\quad \sum_{n=0}^{\infty}\frac{T_{n,q}((x\oplus iy)_q)+T_{n,q}((x\ominus iy)_q)}{2}\frac{t^n}{[n]_q!}=\frac{[2]_q}{e_q(2t)+1}e_q(tx)COS_q(ty).$$

$$(ii)\quad \sum_{n=0}^{\infty}\frac{T_{n,q}((x\oplus iy)_q)-T_{n,q}((x\ominus iy)_q)}{2i}\frac{t^n}{[n]_q!}=\frac{[2]_q}{e_q(2t)+1}e_q(tx)SIN_q(ty).$$

Proof. By substituting $(x\oplus iy)_q$ instead of z in q-tangent polynomials, we find

$$\begin{aligned}\sum_{n=0}^{\infty}T_{n,q}((x\oplus iy)_q)\frac{t^n}{[n]_q!}&=\frac{[2]_q}{e_q(2t)+1}e_q(tx)E_q(ity)\\&=\frac{[2]_q}{e_q(2t)+1}e_q(tx)\left(COS_q(ty)+iSIN_q(ty)\right).\end{aligned}\tag{2.1}$$

By substituting $(x\ominus iy)_q$ instead of z in q-tangent polynomials, we also find

$$\begin{aligned}\sum_{n=0}^{\infty}T_{n,q}((x\ominus iy)_q)\frac{t^n}{[n]_q!}&=\frac{[2]_q}{e_q(2t)+1}e_q(tx)E_q(-ity)\\&=\frac{[2]_q}{e_q(2t)+1}e_q(tx)\left(COS_q(ty)-iSIN_q(ty)\right).\end{aligned}\tag{2.2}$$

Now, (i) comes from the sum of (2.1) and (2.2), and (ii) can be obtained by subtracting (2.1) by (2.2). □

Definition 2.2. Let x and y be real numbers. Then the q-cosine tangent polynomials and the q-sine tangent polynomials are defined in terms of generating functions as follows:

$$\sum_{n=0}^{\infty}{}_CT_{n,q}(x,y)\frac{t^n}{[n]_q!}=\frac{[2]_q}{e_q(2t)+1}e_q(tx)COS_q(ty)$$

and

$$\sum_{n=0}^{\infty}{}_ST_{n,q}(x,y)\frac{t^n}{[n]_q!}=\frac{[2]_q}{e_q(2t)+1}e_q(tx)SIN_q(ty).$$

By Theorem 2.1 and Definition 2.2, we obtain

Corollary 2.3.

$$(i) \quad {}_CT_{n,q}(x,y)=\frac{T_{n,q}((x\oplus iy)_q)+T_{n,q}((x\ominus iy)_q)}{2}.$$

$$(ii) \quad {}_ST_{n,q}(x,y)=\frac{T_{n,q}((x\oplus iy)_q)-T_{n,q}((x\ominus iy)_q)}{2i}.$$

Here, we remark that $C_{n,q}(x,y)$ and $S_{n,q}(x,y)$ were considered in [7, 9]. From [7], we can know that $C_{n,q}(x,y)$ and $S_{n,q}(x,y)$ are very useful polynomials to find identities of polynomials and relations of another polynomials. Recall that

$$\sum_{n=0}^{\infty}C_{n,q}(x,y)\frac{t^n}{[n]_q!}=e_q(tx)COS_q(ty) \text{ and } \sum_{n=0}^{\infty}S_{n,q}(x,y)\frac{t^n}{[n]_q!}=e_q(tx)SIN_q(ty).$$

Theorem 2.4. *Let $T_{n,q}(x,y)$ denote the q-tangent polynomials. For $|q|<1$, the following relations hold.*

$$(i) \quad {}_CT_{n,q}(x,y)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q T_{k,q}C_{n-k,q}(x,y),$$

$$(ii) \quad {}_ST_{n,q}(x,y)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q T_{k,q}S_{n-k,q}(x,y),$$

Proof. (i) From the definition of q-cosine tangent polynomials, we derive a relation between q-tangent polynomials and q-cosine tangent polynomials as (2.3):

$$\begin{aligned}\sum_{n=0}^{\infty}{}_CT_{n,q}(x,y)\frac{t^n}{[n]_q!}&=\sum_{n=0}^{\infty}T_{n,q}\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}C_{n,q}(x,y)\frac{t^n}{[n]_q!}\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q T_{k,q}C_{n-k,q}(x,y)\right)\frac{t^n}{[n]_q!}\end{aligned}\tag{2.3}$$

By comparing the both sides of (2.3), we get the required result.

(ii) By using the Cauchy product in the definition of q-sine tangent polynomials, we obtain the required relation. □

By setting $q\to 1$ in Theorem 2.4, we obtain

Corollary 2.5.

$$(i) \quad {}_CT_n(x,y)=\sum_{k=0}^{n}\binom{n}{k}T_kC_{n-k}(x,y),$$

$$(ii) \quad {}_ST_n(x,y)=\sum_{k=0}^{n}\binom{n}{k}T_kS_{n-k}(x,y).$$

In q-calculus, we note that

$$COS_q(x)=\sum_{n=0}^{\infty}(-1)^nq^{(2n-1)n}\frac{x^{2n}}{[2n]_q!} \text{ and } SIN_q(x)=\sum_{n=0}^{\infty}(-1)^nq^{(2n+1)n}\frac{x^{2n+1}}{[2n+1]_q!}.\tag{2.4}$$

For a real number x, $[x]$ denotes the greatest integer not exceeding x.

Theorem 2.6. *Let $T_{n,q}(x)$ be the q-tangent polynomials. Let $|q|<1$ and let n be a nonnegative integer. Then the following hold.*

$$(i)\quad {}_CT_{n,q}(x,y)=\sum_{k=0}^{[\frac{n}{2}]}\begin{bmatrix} n\\ 2k\end{bmatrix}_q(-1)^kq^{(2k-1)k}T_{n-2k,q}(x)y^{2k},$$

$$(ii)\quad {}_ST_{n,q}(x,y)=\sum_{k=0}^{[\frac{n-1}{2}]}\begin{bmatrix} n\\ 2k+1\end{bmatrix}_q(-1)^kq^{(2k+1)k}T_{n-(2k+1),q}(x)y^{2k+1},$$

Proof. (i) By using the power series of $COS_q(x)$ in the generating function of q-cosine tangent polynomials, we derive

$$\begin{aligned}\sum_{n=0}^{\infty}{}_CT_{n,q}(x,y)\frac{t^n}{[n]_q!}&=\sum_{n=0}^{\infty}{}_CT_{n,q}(x)\frac{t^n}{[n]_q!}(-1)^nq^{(2n-1)n}y^{2n}\frac{t^{2n}}{[n]_q!}\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(-1)^kq^{(2k-1)k}T_{n-k,q}(x)y^{2k}\right)\frac{t^{n+k}}{[n-k]_q![2k]_q!}\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix} n+k\\ 2k\end{bmatrix}_q(-1)^kq^{(2k-1)k}T_{n-k,q}(x)y^{2k}\right)\frac{t^{n+k}}{[n+k]_q!}.\end{aligned}\tag{2.5}$$

From (2.5), we can complete the proof.

(ii) By using the right equation of (2.4) for the generating function of q-sine tangent polynomials, we find the desired result. □

By putting $y=1$ in Theorem 2.6, we obtain

Corollary 2.7.

$$(i)\quad {}_CT_{n,q}(x,1)=\sum_{k=0}^{[\frac{n}{2}]}\begin{bmatrix} n\\ 2k\end{bmatrix}_q(-1)^kq^{(2k-1)k}T_{n-2k,q}(x).$$

$$(ii)\quad {}_ST_{n,q}(x,1)=\sum_{k=0}^{[\frac{n-1}{2}]}\begin{bmatrix} n\\ 2k+1\end{bmatrix}_q(-1)^kq^{(2k+1)k}T_{n-(2k+1),q}(x).$$

Theorem 2.8. *Suppose that $e_q(2t)\neq -1$. Then we have*

$$(i)\quad [2]_qC_{n,q}(x,y)=\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q2^{n-k}{}_CT_{k,q}(x,y)+{}_CT_{n,q}(x,y),$$

$$(ii)\quad [2]_qS_{n,q}(x,y)=\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q2^{n-k}{}_ST_{k,q}(x,y)+{}_ST_{n,q}(x,y).$$

Proof. (i) If $e_q(2t) \neq -1$, then we can multiply $e_q(2t)+1$ in the q-cosine tangent polynomials as

$$\sum_{n=0}^{\infty} {}_CT_{n,q}(x,y)\frac{t^n}{[n]_q!}(e_q(t)+1) = [2]_q e_q(tx)COS_q(tx) = [2]_q \sum_{n=0}^{\infty} C_{n,q}(x,y)\frac{t^n}{[n]_q!}. \tag{2.6}$$

The left-hand side in (2.6) can be expressed as

$$\sum_{n=0}^{\infty} {}_CT_{n,q}(x,y)\frac{t^n}{[n]_q!}(e_q(2t)+1) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q 2^{n-k}{}_CT_{k,q}(x,y) + {}_CT_{n,q}(x,y)\right)\frac{t^n}{[n]_q!}. \tag{2.7}$$

From (2.6) and (2.7), we finish the proof.

(ii) By using the power series of q-sine function, we find the desired result. □

By setting $q \to 1$ in Theorem 2.8, we have

Corollary 2.9. *Let ${}_CT_n(x,y)$ be the cosine tangent polynomials and let ${}_ST_n(x,y)$ be the sine tangent polynomials. Then the following hold.*

$$(i) \quad 2C_n(x,y) = \sum_{k=0}^{n}\binom{n}{k}2^{n-k}{}_CT_k(x,y) + {}_CT_n(x,y),$$

$$(ii) \quad 2S_n(x,y) = \sum_{k=0}^{n}\binom{n}{k}2^{n-k}{}_ST_k(x,y) + {}_ST_n(x,y),$$

From Theorem 2.8, the following result holds.

Corollary 2.10.

$$\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q 2^{n-k}\left({}_CT_{k,q}(x,y) - {}_ST_{k,q}(x,y)\right) = \frac{(1+q^2)C_{n,q}(x,y) + {}_ST_{n,q}(x,y)}{1+q} - \frac{(1+q)^2 S_{n,q}(x,y) + {}_CT_{n,q}(x,y)}{1+q}.$$

Theorem 2.11. *For $0 < q < 1$ and real numbers x and y, we obtain*

$$(i) \quad \frac{\partial}{\partial x}{}_CT_{n,q}(x,y) = [n]_q {}_CT_{n-1,q}(x,y), \quad \frac{\partial}{\partial y}{}_CT_{n,q}(x,y) = -[n]_q {}_ST_{n-1,q}(x,qy).$$

$$(ii) \quad \frac{\partial}{\partial x}{}_ST_{n,q}(x,y) = [n]_q {}_ST_{n-1,q}(x,y), \quad \frac{\partial}{\partial y}{}_ST_{n,q}(x,y) = [n]_q {}_CT_{n-1,q}(x,qy).$$

Proof. (i) Let x be any real number. Then we find the q-partial derivative of q-cosine tangent polynomials. By using the q-derivative of the q-cosine function, we have

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{\partial}{\partial x}{}_CT_{n,q}(x,y)\frac{t^n}{[n]_q!}&=\sum_{n=0}^{\infty}{}_CT_{n,q}(x,y)\frac{t^{n+1}}{[n]_q!}\\&=\sum_{n=0}^{\infty}[n]_q{}_CT_{n-1,q}(x,y)\frac{t^n}{[n]_q!}.\end{aligned}\tag{2.8}$$

From (2.8), we obtain the left result of (i). In a similar way as in the proof of the left result of (i), we find

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{\partial}{\partial y}{}_CT_{n,q}(x,y)\frac{t^n}{[n]_q!}&=-\sum_{n=0}^{\infty}{}_ST_{n,q}(x,qy)\frac{t^{n+1}}{[n]_q!}\\&=-\sum_{n=0}^{\infty}[n]_q{}_ST_{n-1,q}(x,qy)\frac{t^n}{[n]_q!}.\end{aligned}\tag{2.9}$$

Hence we have the right result of (i) from (2.9).

(ii) For any real number x, we find the q-partial derivative for q-sine tangent polynomials by using the q-derivative of the q-sine function as

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{\partial}{\partial x}{}_ST_{n,q}(x,y)\frac{t^n}{[n]_q!}&=\sum_{n=0}^{\infty}{}_ST_{n,q}(x,y)\frac{t^{n+1}}{[n]_q!}\\&=\sum_{n=0}^{\infty}[n]_q{}_ST_{n-1,q}(x,y)\frac{t^n}{[n]_q!}.\end{aligned}\tag{2.10}$$

In addition, for any real number y, we investigate the q-partial derivative for q-sine tangent polynomials by using the q-derivative of the q-sine function as

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{\partial}{\partial y}{}_ST_{n,q}(x,y)\frac{t^n}{[n]_q!}&=\sum_{n=0}^{\infty}{}_CT_{n,q}(x,qy)\frac{t^{n+1}}{[n]_q!}\\&=\sum_{n=0}^{\infty}[n]_q{}_CT_{n-1,q}(x,qy)\frac{t^n}{[n]_q!}.\end{aligned}\tag{2.11}$$

From (2.10) and (2.11), we obtain the required results. □

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