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A NOTE ON GENERALIZED SKEW DERIVATIONS ON MULTILINEAR POLYNOMIALS

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Abstract. Let R be a prime ring, Q_r be the right Martindale quotient ring and C be the extended centroid of R. If $\mathcal{G}$ be a nonzero generalized skew derivation of R and $f(x_1, x_2, \cdots, x_n)$ be a multilinear polynomial over C such that $\big(\mathcal{G}(f(x_1, x_2, \cdots, x_n)) - f(x_1, x_2, \cdots, x_n)\big) \in C$ for all $x_1, x_2, \cdots, x_n \in R$, then either $f(x_1, x_2, \cdots, x_n)$ is central valued on R or R satisfies the standard identity $s_4(x_1, x_2, x_3, x_4)$.

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1. Introduction

Let R be a prime ring with center $Z(R)$. Recall that a ring R is prime if for any $a, b \in R$, $aRb = (0)$ implies $a = 0$ or $b = 0$. The standard identity s_4 in four variables is defined as follows:

$$s_4 = \sum (-1)^{\sigma} X_{\sigma(1)} X_{\sigma(2)} X_{\sigma(3)} X_{\sigma(4)}$$

where $(-1)^{\sigma}$ is the sign of a permutation σ of the symmetric group of degree 4. Let Q_r be the right Martindale quotient ring of R, Q be the two-sided Martindale quotient ring of R and $C = Z(Q) = Z(Q_r)$ be the center of Q and Q_r; where C is called the extended centroid of R and this is a field when R is a prime ring. It should be remarked that Q is a centrally closed prime C-algebra. For the definitions and related properties of these objects, we refer to [3].

It is well known that automorphisms, derivations and skew derivations of R can be extended for Q and Q_r. Chang [7] extended the definition of generalized skew derivation to the right Martindale quotient ring Q_r of R as follows: the additive mapping $\mathcal{G} : Q_r \to Q_r$ is generalized skew derivation if $\mathcal{G}(xy) = \mathcal{G}(x)y+$

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$\alpha(x)d(y)$ for all $x, y \in Q$, where d is an associated skew derivation of $\mathcal{G}$ and α is an associated automorphism of $\mathcal{G}$. Moreover, there exists $\mathcal{G}(1) = a \in Q_r$ such that $\mathcal{G}(x) = ax + d(x)$ for all $x \in R$. Furthermore, if $\mathcal{G}(1) \in Q$, then $\mathcal{G}$ can be extended to Q. For fixed elements a and b of R, the mapping $\mathcal{G} : R \to R$ define as $\mathcal{G}(x) = ax - \sigma(x)b$ for all $x \in R$ is a generalized skew derivation of R. A generalized skew derivation of this form is called an inner generalized skew derivation. We will adopt the following notation

$$f(x_1, x_2, \cdots, x_n) = x_1 \cdots x_n + \sum_{\sigma \in S_n, \sigma \neq id} \alpha_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)}$$

for some $\alpha_\sigma \in C$. The polynomial $f(x_1, x_2, \cdots, x_n) \in C\langle x_1, \cdots, x_n \rangle$ is said to be central valued on R if $f(x_1, \cdots, x_n) \in Z(R)$ for all $x_1, x_2, \cdots, x_n \in R$. The polynomial $f(x_1, x_2, \cdots, x_n) \in C\langle x_1, \cdots, x_n \rangle$ is called non central if it is not central valued on R (or equivalently on the central closure CR of R).

In [5], Bergen proved that if σ is an automorphism of R such that $(\sigma(x) - x)^m = 0$ for all $x \in R$, where m is a fixed positive integer, then $\sigma = 1$. Later, Bell and Daif [4] proved some results which have the same flavour when the automorphism was replaced by a nonzero derivation d. They showed that if R is a semiprime ring with a nonzero ideal I such that $d([x, y]) - [x, y] = 0$ for all $x, y \in I$, then I is central. Moreover, Hongan [14] proved that if R is a 2-torsion free semiprime ring and I is a nonzero ideal of R, then I is central if and only if $d([x, y]) - [x, y] \in Z(R)$ for all $x, y \in I$. The similar identities have been investigated by many researchers from various point of view, e.g., see [1][20][22] and reference therein. It is natural to investigate the situation when $f(x_1, x_2, \cdots, x_n)$ is a multilinear polynomial and $\big(d(f(x_1, x_2, \cdots, x_n)) - f(x_1, x_2, \cdots, x_n)\big) \in Z(R)$ is a differential identity for some ideal I of R. In the present paper, our aim is to analyse what will happen in the case, when $(\mathcal{G}(f(x_1, x_2, \cdots, x_n)) - f(x_1, x_2, \cdots, x_n)) \in C$, for all $x_1, x_2, \cdots, x_n \in R$, where $\mathcal{G}$ is a generalized skew derivation associated with automorphisms α of R. More precisely, our motive is to prove the following result.

Theorem 1.1. *Let R be a prime ring with extended centroid C. If $f(x_1, x_2, \cdots, x_n)$ is a multilinear polynomial over C and $\mathcal{G}$ is a nonzero generalized skew derivation of R such that $\big(\mathcal{G}(f(x_1, x_2, \cdots, x_n)) - f(x_1, x_2, \cdots, x_n)\big) \in C$, for all $x_1, x_2, \cdots, x_n \in R$, then either $f(x_1, x_2, \cdots, x_n)$ is central valued on R or R satisfies the standard identity $s_4(x_1, x_2, x_3, x_4)$.*

To prove our main theorem, we need to recall some more terminology and known results. Let $R = M_s(F)$ be the algebra of $s \times s$ matrices over a field F. Notice that the set $f(R) = \{f(x_1, x_2, \cdots, x_n) : x_1, x_2, \cdots, x_n \in R\}$ is invariant under the action of all inner automorphism of R. If $x = (x_1, x_2, \cdots, x_n) \in R \times R \times \cdots \times R = R^n$, then for any inner automorphism χ of $M_s(F)$, we have $\overline{x} = (\chi(x_1), \cdots, \chi(x_n)) \in R^n$ and $\chi(f(x)) = f(\overline{x}) \in f(R)$. We denote by e_{ij}, the unit matrix having 1 in the (i, j)th-entry and zero elsewhere. Let us recall some results from [17] and [18]. Suppose that $\mathcal{S}$ is a ring with 1 and $e_{ij} \in M_s(\mathcal{S})$

is the unit matrix. For a sequence $v = (H_1, \cdots, H_n)$ in $M_s(\mathcal{S})$, the value of v is defined by the product $|v| = H_1 \cdots H_n$ and v is non-vanishing if $|v| \neq 0$. For a permutation σ for $\{1, 2, \cdots, n\}$, we write $v^\sigma = (H_{\sigma(1)}, \cdots, H_{\sigma(n)})$. We call v is simple if it is of the from $v = (h_1 e_{i1j1}, \cdots h_n e_{injn})$ where $h_i \in \mathcal{S}$. A simple sequence v is called even if for some σ, $|v^\sigma| = pe_{ii} \neq 0$ and odd if for some σ, $|v^\sigma| = pe_{ij} \neq 0$ where $i \neq j$.

Fact 1.1. ([17, Lemma]) Let $\mathcal{S}$ be a F-algebra with 1 and $R = M_s(\mathcal{S})$, $s \geq 2$. If $f(x_1, x_2, \cdots, x_n)$ is a multilinear polynomial over F such that $f(v) = 0$, for all odd simple sequences v, then $f(x_1, x_2, \cdots, x_n)$ is central valued on R.

Fact 1.2. ([18, Lemma 2]) Let $\mathcal{S}$ be a F-algebra with 1 and $R = M_s(\mathcal{S})$, $s \geq 2$. Suppose that $f(x_1, x_2, \cdots, x_n)$ is a multilinear polynomial over F and $v = (H_1, \cdots, H_n)$ is a simple sequence of R. Then, (i) if v is even, then $f(v)$ is a diagonal matrix. (ii) if v is odd, then $f(v) = he_{lt}$ for some $h \in \mathcal{S}$ and $l \neq t$.

Remark 1.1. Since $f(x_1, x_2, \cdots, x_n)$ is not central valued on R, by Fact 1.1, there exists an odd simple sequence $r = (x_1, x_2, \cdots, x_n)$ of R such that $f(x) = f(x_1, x_2, \cdots, x_n) \neq 0$. By Fact 1.2, we see that $f(x) = \eta e_{lt}$, where $0 \neq \eta \in F$ and $l \neq t$. As $f(x_1, x_2, \cdots, x_n)$ is a multilinear polynomial and F is a field, we may assume that $\eta = 1$. Now, for distinct i, j, let $\sigma \in S_n$ be such that $\sigma(l) = i$ and $\sigma(t) = j$, and let χ be the automorphism of R defined by

$$\chi(\sum_{m,q} \zeta_{mq} e_{mq}) = \sum_{m,q} \zeta_{mq} e_{\sigma(m)\sigma(t)},$$

then $f(\chi(x)) = f(\chi(x_1), \cdots, \chi(x_n)) = \chi(f(x)) = \eta e_{ij}$.

Fact 1.3. ([13, Lemma 1]) Let F be an infinite field and $s \geq 2$. If $H_1, \ldots, H_k$ are not scalar matrices in $M_s(F)$, then there exists an invertible matrix $B \in M_m(C)$ such that any matrices $BH_1B^{-1}, \ldots, BH_kB^{-1}$ have all nonzero entries.

Fact 1.4. ([12, Theorem 1]) Let R be a prime ring with an automorphism α and an X-outer α-derivation d. Then any generalized polynomial identity of R in the form $\Psi(x_i, d(x_i)) = 0$ yields the generalized polynomial identity $\Psi(x_i, y_i) = 0$ of R for any distinct indeterminates x_i, y_i.

Fact 1.5. ([12, Theorem 1]) Let R be a prime ring with an automorphism α and an X-outer α-derivation d. Then any generalized polynomial identity of R in the form $\Psi(x_i, \alpha(x_i), d(x_i)) = 0$ yields the generalized polynomial identity $\Psi(x_i, y_i, z_i) = 0$ of R for any distinct indeterminates x_i, y_i.

2. Proof of Theorem 1.1

We begin with two propositions which will be used for the proof of our main result.

Proposition 2.1. *Let R be a prime ring with extended centroid C and $f(x_1, x_2, \cdots, x_n)$ be a multilinear polynomial over C. If* $\big(bf(x_1, x_2, \cdots, x_n) - f(x_1, x_2, \cdots,$

$x_n)c - f(x_1, x_2, \cdots, x_n)\big) \in C$, for any $a, b, x_1, x_2, \cdots, x_n \in R$, then either $f(x_1, x_2, \cdots, x_n)$ is central valued on R or $b, c \in C$ and R satisfies the standard identity s_4.

Proof. Suppose neither $f(x_1, x_2, \cdots, x_n)$ is not central valued on R nor $b, c \notin C$. In both the cases, $\big(bf(x_1, x_2, \cdots, x_n) - f(x_1, x_2, \cdots, x_n)c - f(x_1, x_2, \cdots, x_n)\big) \in C$ is a non trivial generalized polynomial identity for R. By [9, Theorem 2], $[\big(bf(x_1, x_2, \cdots, x_n) - f(x_1, x_2, \cdots, x_n)c - f(x_1, x_2, \cdots, x_n)\big), y] = 0$ is also an identity for RC. By Martindale's theorem [19], RC is a primitive ring with nonzero socle. Thus, there exists a vector space V over a division ring D such that RC is a dense of D-linear transformations over V.

If $dim_D V = \infty$, then by [24, Lemma 2], RC satisfies the following generalized identity $[(bx - xc - x), y] = 0$. Suppose there exists $v \in V$ such that $\{v, vb\}$ is linearly D-independent. By density of RC, there exists $w \in V$ such that $\{v, vb, w\}$ is linearly D-independent and $x_0, y_0 \in RC$ such that $vx_0 = 0$, $v(bx_0) = w$, $vy_0 = 0$, $wy_0 = v$. This leads to the contradiction $0 = v[(bx_0 - x_0c - x_0), y_0] = v \neq 0$. Thus $\{v, vb\}$ is linearly D-dependent for all $v \in V$, which implies that $b \in C$. From this, RC satisfies $[-xc - x, y] = 0$. As above, suppose that there exists $v \in V$ such that $\{v, vc\}$ is linearly D-independent. Then, there exists $w \in V$ such that $\{v, vc, w\}$ is linearly D-independent and there exist $x_0, y_0 \in RC$ such that $vx_0 = v$, $vy_0 = 0$, $v(cy_0) = -v$ This implies that $0 = v[-x_0c - x_0, y_0] = v \neq 0$, a contradiction. Also, in this case we conclude that $\{v, vc\}$ is linearly D-dependent for all $v \in V$, and so $c \in C$.

Now, consider that $dim_D V$ is finite dimensional. In this case, RC is a simple ring which satisfies a non trivial generalized polynomial identity. By [23, Theorem 2.3.29] $RC \subseteq M_s(F)$, for a suitable field F. Moreover, $M_s(F)$ satisfy the same generalized polynomial identity as RC. Hence $\big(bf(x_1, x_2, \cdots, x_n) - f(x_1, x_2, \cdots, x_n)c - f(x_1, x_2, \cdots, x_n)\big) \in Z(M_s(F))$ for all $x_1, x_2, \cdots, x_n \in M_s(F)$. Let $s \geq 2$, otherwise we have noting to prove. Suppose that R does not satisfy s_4. Since $f(x_1, x_2, \cdots, x_n)$ is not central, by [18], there exist $u_1, \ldots, u_n \in M_s(F)$ and $\gamma \in F - \{0\}$ such that $f(u_1, \ldots, u_n) = \gamma e_{kl}$, with $k \neq l$. Moreover, as the set $\{f(x_1, \ldots, x_n) : x_1, \ldots, x_n \in M_s(F)\}$ is invariant under the action of all F-automorphisms of $M_s(F)$, then for any $i \neq j$ there exist $x_1, \ldots, x_n \in M_s(F)$ such that $f(x_1, \ldots, x_n) = e_{ij}$. Moreover, $(be_{ij} - e_{ij}c - e_{ij})$ has rank at most 2, that is $(be_{ij} - e_{ij}c - e_{ij}) = 0$. Right multiplying by e_{ij}, we obtain $0 = (e_{ij}c)e_{ij}$. It follows that the (j, i)-entry of the matrix c is zero, for all $i \neq j$ and this means that c is diagonal, that is $c = \sum_t p_t c_{tt}$ with $p_t \in F$. If χ is a F-automorphism of $M_s(F)$, then the same conclusion holds for $\chi(c)$ as $\big(\chi(b)f(x_1, x_2, \cdots, x_n) - f(x_1, x_2, \cdots, x_n)\chi(c) - f(x_1, x_2, \cdots, x_n)\big) \in Z(M_s(F))$. Now suppose that $i \neq j$ and $\chi(x) = (1 + e_{ij})x(1 - e_{ij})$. Since $\chi(c) = (1 + e_{ij})c(1 - e_{ij})$, then c must be diagonal with $c_{ii} = c_{jj}$ and hence c is central element. Similarly we can show that b is central in $M_s(F)$. Therefore, in any case we get the conclusion that both a and b are central elements of R. This completes the proof. □

Proposition 2.2. *Let R be a prime ring and $f(x_1, x_2, \cdots, x_n)$ be a multilinear polynomial over C. If $\mathcal{G}$ is the generalized inner skew derivation associated with automorphism α of R such that $\big(\mathcal{G}(f(x)) - f(x)\big) \in C$ for all $x = (x_1, x_2, \cdots, x_n) \in R$, then either R satisfies s_4 or $f(x)$ is central valued on R.*

Proof. Since $\mathcal{G}$ is an inner generalized skew derivation induced by elements $b, c \in R$ and $\alpha \in Aut(R)$, i.e., $\mathcal{G}(x) = bx - \alpha(x)c$ for all $x \in R$, we see that R satisfies $\big(bf(x_1, x_2, \cdots, x_n) - \alpha(x_1, x_2, \cdots, x_n)c - f(x_1, x_2, \cdots, x_n)\big) \in Z(R)$. Firstly, we suppose that α is an X-inner automorphism of R,i.e., there exists an element $q \in Q$ such that $\alpha(x) = qxq^{-1}$ for all $x \in R$. Moreover, it is easy to see that the generalized polynomial $\Psi(x_1, x_2, \cdots, x_n) = \big(bf(x_1, x_2, \cdots, x_n) - qf(x_1, x_2, \cdots, x_n)q^{-1}c - f(x_1, x_2, \cdots, x_n)\big)$ is a generalized polynomial identity for R.

If $\{1, q^{-1}c\}$ is C-linearly independent, then $\phi(x_1, x_2, \cdots, x_n)$ is a non-trivial generalized polynomial identity for R. It follows from [9] that, $\phi(x_1, x_2, \cdots, x_n)$ is a non-trivial generalized polynomial identity for Q. By the well-known Martindale's theorem [19], we say that Q is a primitive ring having nonzero socle with the field C which is associated division ring R of Q. By Jacobson's theorem [15, p. 75], Q is isomorphic to a dense subring of the ring of linear transformations of a vector space V over C, containing some nonzero linear transformations of finite rank. Assume that $dim_C V = \infty$. By Lemma 2 of [24], the set $f(R) = \{f(r_1, r_2, \cdots, r_n) | r_i \in R\}$ is dense in R. Since $\psi(x_1, x_2, \cdots, x_n) = 0$ is a generalized polynomial identity of R, then Q satisfies the generalized polynomial identity $[br - qrq^{-1}c - r, s]$. In particular for $r = 1$, $[b - c - 1, s]$ is an identity for Q, i.e., $b - c - 1 \in C$, say $b = 1 + c + \tau$ for some $\tau \in C$. Thus Q satisfies $[(c+\tau)r - qrq^{-1}, s]$. Once again for $s = r$, it follows that Q satisfies $[cr - qrq^{-1}c, r]$. Since R cannot satisfy any polynomial identity ($dim_C V = \infty$) and by [8, Lemma 3.2], we have $q^{-1}c \in C$, which leads to the contradiction. On the other hand, if $dim_C V = k \geq 2$ is a finite positive integer, then $Q \cong M_k(C)$. We may assume that $q^{-1}c$ is not a scalar matrix. Otherwise in view of Proposition 2.1, we have done. By Fact 1.4, there exists some invertible matrix $B \in M_k(C)$ such that each matrix $B(q^{-1}c)B^{-1}$, BqB^{-1} has all entries nonzero. Denote by $\phi(x) = BxB^{-1}$, the inner automorphism induced by B. Here e_{hl} denotes the usual unit matrix with 1 in (h, l)-entry and zero elsewhere. Since the set $\{f(r_1, r_2, \cdots, r_n) : r_1, r_2, \cdots, r_n \in M_s(C)\}$ is invariant under the action of all inner automorphisms of R, we have $\phi(b)r - \phi(q)r\phi(q^{-1}c) - r$ for all $r \in f(R)$. Let us write $\phi(q) = \sum_{hl} q_{hl}e_{hl}$ and $\phi(q^{-1}c) = \sum_{hl} p_{hl}e_{hl}$ for $0 \neq q_{hl}, p_{hl} \in C$. Since $e_{ij} \in f(R)$ for all $i \neq j$, for any $i \neq j$ we have $[\phi(b)e_{ij} - \phi(q)e_{ij}\phi(q^{-1}c) - e_{ij}, e_{ij}] = 0$.Therefore, we can easily see that $q_{ji}p_{ji} = 0$, which is a contradiction.

In case, if $\{1, q^{-1}c\}$ is C-linearly dependent, i.e., $q^{-1}c \in C$, then we have done by Proposition 2.1. So we may assume that α is X-outer. Since R and Q satisfy the same generalized polynomial identities with automorphisms [10], then

$\phi(x_1, x_2, \cdots, x_n) = [bf(x_1, x_2, \cdots, x_n) - \alpha(f(x_1, x_2, \cdots, x_n))c - f(x_1, x_2, \cdots, x_n), s]$ is also satisfied by Q. Moreover Q is centrally closed prime C-algebra. In view of Proposition 2.1, we may assume that $c \neq 0$. In this case, by [11, Main Theorem], $\psi(x_1, x_2, \cdots, x_n)$ is a non trivial generalized identity for R and Q. By [16, Theorem 1], we deduce that RC has nonzero socle and Q is primitive. Since α is an outer automorphism, then any $(x_i)^\alpha$-word degree in $\psi(x_1, x_2, \cdots, x_n)$ is equal to 1, by [11, Theorem 3], Q satisfies the identity $[bf(x_1, x_2, \cdots, x_n) - f^\alpha(y_1, y_2, \cdots, y_n)c - f(x_1, x_2, \cdots, x_n), s]$ where $f^\alpha(x_1, x_2, \cdots, x_n)$ is the polynomial obtained from $f(x_1, x_2, \cdots, x_n)$ by replacing each coefficients μ with $\alpha(\mu)$. In particular, Q (and so also R) satisfies the generalized polynomial identity $[bf(x_1, x_2, \cdots, x_n) - f(x_1, x_2, \cdots, x_n)c - f(x_1, x_2, \cdots, x_n), s]$. In view of Proposition 2.1, we obtain the required conclusions. □

Proof of Theorem 1.1 For any skew derivation d of R, we have generalized skew derivation $\mathcal{G}$ of the form $\mathcal{G}(x) = bx + d(x)$ for all $x \in R$ and $b \in Q_r$. Let us put $f(x_1, x_2, \cdots, x_n) = \sum_{\sigma \in S_n} \mu_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)}$, where $\mu_\sigma \in C$. By [12, Theorem], we know that R and Q_r satisfy the same generalized polynomial identities with a single skew derivation. Thus Q_r satisfies $\Psi(x_1, x_2, \cdots, x_n, d(x_1), \cdots, d(x_n)) = bf(x_1, x_2, \cdots, x_n) + d(f(x_1, x_2, \cdots, x_n)) - f(x_1, x_2, \cdots, x_n) \in C$. Since $\mathcal{G}$ is X-inner, then d is X-inner, i.e., there exist $c \in Q_r$ and $\alpha \in Aut(Q_r)$ such that $d(x) = cx - \alpha(x)c$ for all $x \in R$. Hence $\mathcal{G}(x) = (b + c)x - \alpha(x)c$ and we conclude by Proposition 2.2.

Now, we assume that d is X-outer and $\alpha \in Aut(Q_r)$ is associated automorphism of d. Further we assume that $f(x_1, x_2, \cdots, x_n)$ is not central valued on R. We denote by $f^d(x_1, x_2, \cdots, x_n)$ as the polynomial obtained from $f(x_1, x_2, \cdots, x_n)$ by replacing each coefficient μ_σ with $d(\mu_\sigma)$. It should be remarked that

$$\begin{aligned} d(\mu_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)}) =& d(\mu_\sigma) x_{\sigma(1)} \cdots x_{\sigma(n)} \\ &+ \alpha(\mu_\sigma) \sum_{j=0}^{n-1} \alpha(x_{\sigma(1)} \cdots x_{\sigma(j)}) d(x_{\sigma(j+1)}) x_{\sigma(j+2)} \cdots x_{\sigma(n)}. \end{aligned}$$

So, we have

$$\begin{aligned} &\mathcal{G}(f(x_1, x_2, \cdots, x_n)) - f(x_1, x_2, \cdots, x_n) \\ &= af(x_1, x_2, \cdots, x_n) + f^d(x_1, x_2, \cdots, x_n) \\ &\quad + \sum_{\sigma \in S_n} \alpha(\mu_\sigma) \\ &\quad \times \sum_{j=0}^{n-1} \alpha(x_{\sigma(1)} \cdots x_{\sigma(j)}) d(x_{\sigma(j+1)}) x_{\sigma(j+2)} \cdots x_{\sigma(n)} - f(x_1, x_2, \cdots, x_n) \in C. \end{aligned}$$

Since Q_r satisfies $\Psi(x_1, x_2, \cdots, x_n, d(x_1), \cdots, d(x_n))$, by Chuang [10, Theorem 1] it follows that R satisfies $\Psi(x_1, x_2, \cdots, x_n, y_1, \cdots, y_n)$, i.e.,

$$\begin{aligned} af(x_1,&x_2, \cdots, x_n) + f^d(x_1, x_2, \cdots, x_n) \\ &+ \sum_{\sigma \in S_n} \alpha(\mu_\sigma) \\ &\times \sum_{j=0}^{n-1} \alpha(x_{\sigma(1)} \cdots x_{\sigma(j)}) y_{\sigma(j+1)} x_{\sigma(j+2)} \cdots x_{\sigma(n)} - f(x_1, x_2, \cdots, x_n) \in C. \end{aligned}$$

Now, suppose α is X-inner, say $\alpha(x) = qxq^{-1}$ for all $x \in R$ and for some invertible $q \in Q$. Then by the hypothesis and by Fact 1.4, we have

$$\begin{aligned} af(x_1,&x_2, \cdots, x_n) + f^d(x_1, x_2, \cdots, x_n) \\ &+ \sum_{\sigma \in S_n} q\mu_\sigma q^{-1} \\ &\times \sum_{j=0}^{n-1} q x_{\sigma(1)} \cdots x_{\sigma(j)} q^{-1} y_{\sigma(j+1)} x_{\sigma(j+2)} \cdots x_{\sigma(n)} - f(x_1, x_2, \cdots, x_n) \in C, \end{aligned}$$

for all $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n \in R$. In particular, by taking $y_1 = y_2 = \cdots = y_n = 0$, we see that $af(x_1, x_2, \cdots, x_n) + f^d(x_1, x_2, \cdots, x_n) - f(x_1, x_2, \cdots, x_n) \in C$ for all $x_1, x_2, \cdots, x_n \in R$. Therefore

$$q \sum_{\sigma \in S_n} \sum_{j=1}^{n} x_{\sigma(1)} \cdots x_{\sigma(j-1)} q^{-1} y_{\sigma(j)} x_{\sigma(j+1)} \cdots x_{\sigma(n)} \in C,$$

for all $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n \in R$, and hence for all $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n \in Q$, as R and Q satisfy the same generalized polynomial identities. Replacing $y_\sigma(j)$ by $q[z, x_{\sigma(i)}]$ in the last identity, we obtain $q[z, f(x_1, x_2, \cdots, x_n)] \in C$, for all $x_1, x_2, \cdots, x_n \in Q$. First we see that, if $q[z, f(x_1, x_2, \cdots, x_n)] = 0$ for all $x_1, x_2, \cdots, x_n \in Q$, then $f(x_1, x_2, \cdots, x_n)$ is clearly central valued on R as $q \in Q$ is invertible. On the other hand, if $q[z, f(p_1, p_2, \cdots, p_n)] \neq 0$ for some $p_1, p_2, \cdots, p_n \in Q$, then by [6, Theorem 2], we conclude that R satisfies s_4, the standard identity in four variables.

Finally, we assume that α is X-outer. Then by the hypothesis and by Fact 1.5, we have

$$\begin{aligned} af(x_1,&x_2, \cdots, x_n) + f^d(x_1, x_2, \cdots, x_n) \\ &+ \sum_{\sigma \in S_n} \alpha(\mu_\sigma) \\ &\times \sum_{j=0}^{n-1} z_{\sigma(1)} \cdots z_{\sigma(j)} y_{\sigma(j+1)} x_{\sigma(j+2)} \cdots x_{\sigma(n)} - f(x_1, x_2, \cdots, x_n) \in C, \end{aligned}$$

for all $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n, z_1, z_2, \cdots, z_n \in R$. In particular, by taking $y_1 = y_2 = \cdots = y_n = 0$, we see that $af(x_1, x_2, \cdots, x_n) + f^d(x_1, x_2, \cdots, x_n) - f(x_1, x_2, \cdots, x_n) \in C$ for all $x_1, x_2, \cdots, x_n \in R$. Therefore

$$\sum_{\sigma \in S_n} \alpha(\mu_\sigma) \sum_{j=1}^{n} z_{\sigma(1)} \cdots z_{\sigma(j-1)} y_{\sigma(j)} x_{\sigma(j+1)} \cdots x_{\sigma(n)} \in C,$$

for all $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n, z_1, z_2, \cdots, z_n \in R$. Replacing $z_{\sigma(j)}$ by $x_{\sigma(j)}$ and $y_{\sigma(j)}$ by $[z, x_{\sigma(j)}]$, we obtain

$$[z, f^d(x_1, x_2, \cdots, x_n)] \in C,$$

for all $x_1, x_2, \cdots, x_n, z \in R$. Therefore $f^d(x_1, x_2, \cdots, x_n)$ is central valued on R or by [2, Theorem 2] R satisfies s_4. If the first possibility holds then clearly $f(x_1, x_2, \cdots, x_n)$ is central valued on R, a contradiction. Therefore R must satisfy s_4, the standard identity in four variables. This completes the proof.

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