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ON THE (p,q)-POLY-KOROBOV POLYNOMIALS AND RELATED POLYNOMIALS†

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ABSTRACT. *D.S. Kim et al.* [9] *considered some identities and relations for Korobov type numbers and polynomials. In this work, we investigate the degenerate Korobov type Changhee polynomials and the (p,q)-poly-Korobov polynomials. We give a generalization of the Korobov type Changhee polynomials and the (p,q) poly-Korobov polynomials. We prove some properties and identities and explicit relations for these polynomials.*

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1. Introduction

A usual, throughout this paper, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{N}_0$ denotes the set of nonnegative integers, $\mathbb{Z}$ denotes the set of integer numbers, $\mathbb{R}$ denotes the set of real numbers. We begin by introducing the following definitions and notations (see also [2]-[17]). It is well known, the Bernoulli polynomials $B_n^{(\alpha)}(x)$ of order α and the Euler polynomials $E_n^{(\alpha)}(x)$ of order α are defined by the following generating functions, respectively;

$$\sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{t^n}{n!} = \left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt},\ |t| < 2\pi \tag{1}$$

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and

$$\sum_{n=0}^{\infty} E_n^{(\alpha)}(x)\frac{t^n}{n!} = \left(\frac{2}{e^t+1}\right)^{\alpha} e^{xt},\ |t| < \pi. \tag{2}$$

For $x = 0$, $B_n^{(\alpha)}(0) = B_n^{(\alpha)}$ and $E_n^{(\alpha)}(0) = E_n^{(\alpha)}$ are called the Bernoulli numbers $B_n^{(\alpha)}$ of order α and the Euler numbers $E_n^{(\alpha)}$ of order α.

Generating function for the Stirling numbers of the second kind in ([8], [9], [10]) are given by

$$\frac{(e^t-1)^k}{k!} = \sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!}. \tag{3}$$

The polylogarithm function $Li_k(z)$ in ([2], [4], [5]) is defined

$$Li_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k},\ k \in \mathbb{Z},\ k > 1. \tag{4}$$

This function is convergent for $|z| < 1$, when $k = 1$

$$Li_1(z) = -\log(1-z). \tag{5}$$

The multi-logarithm [6] is defined by

$$Li_{k_1,\cdots,k_n}(z) = \sum_{0<m_1<\cdots<m_n} \frac{z^{m_n}}{m_1^{k_1}\cdots m_n^{k_n}},\ k_i \geq 1,\ |z| < 1. \tag{6}$$

From (6), the following equation can be obtain easily

$$Li_{\underbrace{1,\cdots,1}_{n\ times}}(z) = \frac{1}{n!}\left(-\log\left(1-z\right)\right)^n. \tag{7}$$

Kim *et al.* in [7] defined the poly-Bernoulli polynomials as

$$\sum_{n=0}^{\infty} \mathcal{B}_n^{(k)}(x)\frac{t^n}{n!} = \frac{Li_k(1-e^{-t})}{e^t-1}e^{xt}. \tag{8}$$

For $k = 1$, we have $\mathcal{B}_n^{(1)}(x) = B_n(x)$.

Hamahata in [4] defined the poly-Euler polynomials as

$$\frac{2Li_k(1-e^{-t})}{t\left(e^t+1\right)}e^{xt} = \sum_{n=0}^{\infty} \mathcal{E}_n^{(k)}(x)\frac{t^n}{n!} \tag{9}$$

when $k = 1$, $\mathcal{E}_n^{(1)}(x) = E_n(x)$.

For $z = 1$, multi-logarithm function is closely related to multiple zeta values as

$$Li_{k_1,\cdots,k_n}(1) = \zeta\left(k_1,\cdots,k_n\right),\ k_i \geq 1,\ k_n \geq 2.$$

The special values of the multi-logarithm function (see detail in [5], [6]) are following as

$$Li_1(z) = -\log(1-z),\ Li_{1,1}(z) = \frac{1}{2!}\left(-\log\left(1-z\right)\right)^2,\ \cdots$$

$$Li_{\underbrace{1, \cdots, 1}_{n\ times}}(z) = \frac{1}{n!}\left(-\log\left(1-z\right)\right)^{n}. \tag{10}$$

D. S. Kim *et al.* in [10] defined the Changhee polynomials and the first kind Korobov polynomials the following generating functions, respectively,

$$\sum_{n=0}^{\infty} Ch_n\left(x\right)\frac{t^n}{n!} = \frac{2}{t+2}\left(1+t\right)^{x} \tag{11}$$

and

$$\sum_{n=0}^{\infty} K_n\left(x \mid \lambda\right)\frac{t^n}{n!} = \frac{\lambda t}{\left(1+t\right)^{\lambda}-1}\left(1+t\right)^{x}. \tag{12}$$

When $x = 0$, $Ch_n\left(0\right) = Ch_n$ and $K_n\left(0 \mid \lambda\right) = K_n\left(\lambda\right)$ are called the Changhee numbers and the Korobov numbers, respectively.

The Korobov-type Changhee polynomials in [10] are defined the following generating function as

$$\sum_{n=0}^{\infty} Ch_n\left(x \mid \lambda\right)\frac{t^n}{n!} = \frac{2}{\left(1+t\right)^{\lambda}+1}\left(1+t\right)^{x} \tag{13}$$

when $x = 0$, $Ch_n\left(0 \mid \lambda\right) = Ch_n\left(\lambda\right)$ are called the Korobov-type Changhee numbers. Note that

$$\lim_{\lambda\to 1} Ch_n\left(x \mid \lambda\right) = Ch_n\left(x\right) \text{ and } \lim_{\lambda\to 0} Ch_n\left(x \mid \lambda\right) = \left(x\right)_n$$

where

$$\left(x\right)_n = x(x-1)\left(x-2\right)\cdots\left(x-n+1\right). \tag{14}$$

For $\lambda \in \mathbb{R}$, Carlitz [3] introduced the degenerate Bernoulli polynomials by means of the following generating function:

$$\frac{t}{\left(1+\lambda t\right)^{1/\lambda}-1}\left(1+\lambda t\right)^{x/\lambda} = \sum_{n=0}^{\infty} \mathfrak{B}_n\left(x \mid \lambda\right)\frac{t^n}{n!} \tag{15}$$

so that

$$\mathfrak{B}_n\left(x \mid \lambda\right) = \sum_{m=0}^{n}\binom{n}{m}\mathfrak{B}_m\left(\lambda\right)\left(\frac{x}{\lambda}\right)_{n-m}.$$

From (5), we note that

$$\sum_{n=0}^{\infty}\lim_{\lambda\to 0}\mathfrak{B}_n\left(x \mid \lambda\right)\frac{t^n}{n!} = \lim_{\lambda\to 0}\frac{t}{\left(1+\lambda t\right)^{1/\lambda}-1}\left(1+\lambda t\right)^{x/\lambda}$$

$$= \frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n\left(x\right)\frac{t^n}{n!}$$

where $B_n\left(x\right)$ are the Bernoulli polynomials.

2. Degenerate Korobov-type Changhee Polynomials

In this section, we will give some relations and identities for the Changhee polynomials and the Korobov-type Changhee polynomials. Further, we define the degenerate Korobov-type Changhee polynomials and prove some relationships for these polynomials.

From (13), we have following relations easily

$$Ch_n\left(x \mid \lambda\right) = \sum_{m=0}^{n} \binom{n}{m} Ch_m\left(\lambda\right)\left(x\right)_{n-m},$$

$$Ch_n\left(x+y \mid \lambda\right) = \sum_{m=0}^{n} \binom{n}{m} Ch_m\left(x \mid \lambda\right)\left(y\right)_{n-m}$$

and

$$Ch_n^{(\alpha+\beta)}\left(x \mid \lambda\right) = \sum_{m=0}^{n} \binom{n}{m} Ch_m^{(\alpha)}\left(x \mid \lambda\right) Ch_{n-m}^{(\beta)}\left(\lambda\right).$$

Theorem 2.1. *The following relation holds true:*

$$\sum_{n=0}^{j} Ch_n(x \mid \lambda) S_2(j,n) = E_j\left(\frac{x}{\lambda}\right)\lambda^j. \tag{16}$$

Proof. By replacing t by $e^{-t}-1$ in (13), we get

$$\sum_{n=0}^{\infty} Ch_n(x \mid \lambda)\frac{\left(e^{-t}-1\right)^n}{n!} = \frac{2}{e^{-t\lambda}+1}e^{-tx}$$

$$\sum_{n=0}^{\infty} Ch_n(x \mid \lambda)\sum_{j=0}^{\infty} S_2(j,n)\left(-1\right)^j\frac{t^j}{j!} = \sum_{j=0}^{\infty} E_j\left(\frac{x}{\lambda}\right)\left(-1\right)^j\frac{t^j}{j!}.$$

From here, we get (16). □

From $\lim\limits_{\lambda\to 0}\left(1+\lambda t\right)^{1/\lambda} = e^{xt}$. We consider the degenerate function of t which are given by

$$t = \lim_{\lambda\to 0}\log\left(1+\lambda t\right)^{1/\lambda}$$

$\frac{\log(1+\lambda t)}{\lambda}$ is called the degenerate function of t. Now we consider the degenerate Korobov-type Changhee polynomials the following generating function as

$$\sum_{n=0}^{\infty} Ch_{n,\lambda}(x)\frac{t^n}{n!} = \frac{2}{\left(1+\frac{1}{\lambda}\log\left(1+\lambda t\right)\right)^{\lambda}+1}\left(1+\frac{1}{\lambda}\log\left(1+\lambda t\right)\right)^{x} \tag{17}$$

where $\lambda \in \mathbb{R}$. For $x=0$, $Ch_{n,\lambda}(0) := Ch_{n,\lambda}$ is degenerate Korobov-type Changhee numbers.

Theorem 2.2. *The following relation holds true:*

$$\sum_{k=0}^{\infty}\sum_{p=0}^{n} Ch_{k,\lambda}(x)p!\frac{l^{p}\lambda^{p-k}}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}S_2(n,p)=E_n\left(\frac{x}{\lambda}\right)\lambda^{n}. \tag{18}$$

Proposition 2.3. *From (3) and by using t by $\frac{e^{\lambda(e^t-1)}-1}{\lambda}$ in (17), we get*

$$\sum_{k=0}^{\infty} Ch_{k,\lambda}(x)\frac{1}{k!}\lambda^{-k}\left(e^{\lambda(e^t-1)}-1\right)^{k}=\frac{2}{e^{t\lambda}+1}e^{xt}$$

$$\sum_{k=0}^{\infty} Ch_{k,\lambda}(x)\frac{1}{k!}\lambda^{-k}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}e^{\lambda l(e^t-1)}=\sum_{n=0}^{\infty}E_n\left(\frac{x}{\lambda}\right)\lambda^{n}\frac{t^n}{n!}$$

$$\sum_{k=0}^{\infty} Ch_{k,\lambda}(x)\frac{\lambda^{-k}}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\sum_{p=0}^{\infty}(\lambda l)^{p}\frac{(e^t-1)^{p}}{p!}=\sum_{n=0}^{\infty}E_n\left(\frac{x}{\lambda}\right)\lambda^{n}\frac{t^n}{n!}$$

$$\sum_{k=0}^{\infty} Ch_{k,\lambda}(x)\frac{\lambda^{-k}}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\sum_{p=0}^{\infty}(\lambda l)^{p}\sum_{n=0}^{\infty}S_2(n,p)\frac{t^n}{n!}=\sum_{n=0}^{\infty}E_n\left(\frac{x}{\lambda}\right)\lambda^{n}\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ both sides, we have (18).

3. On The (p,q)-Poly-Korobov Polynomials and Related Polynomials

In this section, we consider and investigate the (p,q)-poly-Korobov polynomials and the (p,q)-poly-Korobov-type Changhee polynomials. Also, we give some relations and identities for these polynomials.

Definition 3.1. We define the (p,q)-poly-Korobov polynomials and the (p,q)-poly-Korobov-type Changhee polynomials as the following generating functions, respectively:

$$\sum_{n=0}^{\infty} K_{n,p,q}^{(k)}(x\mid\lambda)\frac{t^n}{n!}=\frac{\lambda Li_{k,p,q}(1-e^{-t})}{(t+1)^{\lambda}-1}(1+t)^{x} \tag{19}$$

and

$$\sum_{n=0}^{\infty} Ch_{n,p,q}^{(k)}(x\mid\lambda)\frac{t^n}{n!}=\frac{2Li_{k,p,q}(1-e^{-t})}{t\left((1+t)^{\lambda}+1\right)}(1+t)^{x} \tag{20}$$

where p, q real numbers such that $0<q<p\leq 1$ and the polylogarithm function is defined as

$$Li_{k,p,q}(t)=\sum_{n=1}^{\infty}\frac{t^n}{[n]_{p,q}^{k}}. \tag{21}$$

The polynomials $K_{n,p,q}^{(k)}(0\mid\lambda):=K_{n,p,q}^{(k)}(\lambda)$ are called the (p,q)-poly-Korobov numbers and the polynomials $Ch_{n,p,q}^{(k)}(0\mid\lambda):=Ch_{n,p,q}^{(k)}(\lambda)$ are called the (p,q)-poly-Korobov-type Changhee numbers.

The polynomial $[n]_{p,q}=\frac{p^n-q^n}{p-q}$ is the n-th (p,q) integer [11].

The first values of the (p,q)-polylogarithm function for $k \leq 0$,

$$\begin{aligned} Li_{0,p,q}(t) &= \frac{x}{1-x},\ Li_{-1,p,q}(t) = \frac{x}{(1-px)(1-qx)}, \\ Li_{-2,p,q}(t) &= \frac{x(1+pqx)}{(1-p^2x)(1-q^2x)(1-pqx)},\ \cdots. \end{aligned}$$

The (p,q)-polylogarithm function for $k \leq 0$ is a rational function. For k is a nonnegative integer

$$Li_{-k,p,q}(t) = \sum_{n=1}^{\infty} \frac{x^n}{[n]_{p,q}^{-k}} = \frac{1}{(p-q)^k} \sum_{l=0}^{k} (-1)^{k-l} \binom{k}{l} \frac{p^l q^{k-l} x}{1-p^l q^{k-l} x}.$$

For $n=3$ in (8), we get

$$Li_{1,1,1}(t) = \frac{1}{3!} \left(-\log(1-t)\right)^3. \tag{22}$$

From (19) and (22), for $k=p=q=1$, we write as

$$\sum_{n=0}^{\infty} K_{n,1,1}^{(1)}(x \mid \lambda) \frac{t^n}{n!} = \frac{\lambda Li_{1,1,1}(1-e^{-t})}{(t+1)^{\lambda}-1} (1+t)^x$$

$$= \frac{t^2}{3!} \frac{\lambda t}{(t+1)^{\lambda}-1} (1+t)^x = \frac{1}{3!} \sum_{n=0}^{\infty} (n-1) n K_{n-2}(x \mid \lambda) \frac{t^n}{n!}.$$

Comparing the coefficients, we have

$$K_{n,1,1}^{(1)}(x \mid \lambda) = \frac{1}{3!} n(n-1) K_{n-2}(x \mid \lambda).$$

Similarly, from (20) and (22), for $k=p=q=1$, we have

$$Ch_{n,1,1}^{(1)}(x \mid \lambda) = \frac{2}{3!} n(n-1) Ch_{n-2}(x \mid \lambda).$$

Theorem 3.2. *The following relations holds true:*

$$K_{n,p,q}^{(k)}(x \mid \lambda) = \sum_{m=0}^{n} \binom{n}{m} (x)_{n-m}\ K_{m,p,q}^{(k)}, \tag{i}$$

$$Ch_{n,p,q}^{(k)}(x \mid \lambda) = \sum_{m=0}^{n} \binom{n}{m} (x)_{n-m}\ Ch_{m,p,q}^{(k)},$$

$$K_{n,p,q}^{(k)}(x+y \mid \lambda) = \sum_{m=0}^{n} \binom{n}{m} K_{m,p,q}^{(k)}(x \mid \lambda)\ (y)_{n-m} \tag{ii}$$

and

$$Ch_{n,p,q}^{(k)}(x+y \mid \lambda) = \sum_{m=0}^{n} \binom{n}{m} Ch_{m,p,q}^{(k)}(x \mid \lambda)\ (y)_{n-m}.$$

Theorem 3.3. *There are the following relationships for the (p, q)-poly-Korobov polynomials and the (p, q)-poly-Korobov-type Changhee polynomials:*

$$\begin{aligned}&K_{n,p,q}^{(k)}(x+\lambda \mid \lambda)-K_{n,p,q}^{(k)}(x \mid \lambda)\\&=\lambda \sum_{r=0}^{m}\binom{m}{r}\sum_{n=0}^{\infty}\frac{(n+1)!\,(-1)^{n+1+r}}{[n+1]_{p,q}^{k}}S_2(r, n+1)(x)_{m-r}\end{aligned} \tag{23}$$

and

$$\begin{aligned}&m\left(Ch_{m-1,p,q}^{(k)}(x+\lambda \mid \lambda)+Ch_{m-1,p,q}^{(k)}(x \mid \lambda)\right)\\&=2\sum_{r=0}^{m}\binom{m}{r}\sum_{n=0}^{\infty}\frac{(n+1)!\,(-1)^{n+1+r}}{[n+1]_{p,q}^{k}}S_2(r, n+1)(x)_{m-r}\,.\end{aligned} \tag{24}$$

Proof. By using (19) and (3), we write as

$$\sum_{n=0}^{\infty}\left(K_{n,p,q}^{(k)}(x+\lambda \mid \lambda)-K_{n,p,q}^{(k)}(x \mid \lambda)\right)\frac{t^n}{n!}=\lambda Li_{k,p,q}\left(1-e^{-t}\right)(1+t)^x$$

$$=\lambda \sum_{n=0}^{\infty}\frac{(n+1)!\,(-1)^{n+1}}{[n+1]_{p,q}^{k}}\frac{\left(e^{-t}-1\right)^{n+1}}{(n+1)!}\sum_{l=0}^{\infty}(x)_l\frac{t^l}{l!}$$

$$=\lambda \sum_{n=0}^{\infty}\frac{(n+1)!\,(-1)^{n+1}}{[n+1]_{p,q}^{k}}\sum_{r=0}^{\infty}S_2(r, n+1)(-1)^r\frac{t^r}{r!}\sum_{l=0}^{\infty}(x)_l\frac{t^l}{l!}.$$

By using the Cauchy product rule and comparing the coefficient both sides, we have (23).

The proof of equation (24) can be make easily, we omit it. □

Corollary 3.4. *From (23) and (24), we have the following relationships between the (p, q)-poly-Korobov polynomials and the (p, q)-poly-Korobov-type Changhee polynomials:*

$$\begin{aligned}&2\left(K_{m,p,q}^{(k)}(x+\lambda \mid \lambda)-K_{m,p,q}^{(k)}(x \mid \lambda)\right)\\&=\lambda m\left(Ch_{m-1,p,q}^{(k)}(x+\lambda \mid \lambda)+Ch_{m-1,p,q}^{(k)}(x \mid \lambda)\right).\end{aligned}$$

Theorem 3.5. *The following relation holds true:*

$$\begin{aligned}&\sum_{r=0}^{m}\binom{m}{r}K_{r,p,q}^{(k)}(x \mid \lambda)(\lambda)_{m-r}-K_{m,p,q}^{(k)}(x \mid \lambda)\\&=\lambda \sum_{r=0}^{m}\binom{m}{r}\sum_{n=0}^{\infty}\frac{(n+1)!\,(-1)^{n+1+r}}{[n+1]_{p,q}^{k}}S_2(r, n+1)(x)_{m-r}\end{aligned} \tag{25}$$

where is $(x)_n = x(x-1)\cdots(x-n+1)$.

Proposition 3.6. *From (19), we write as*

$$\sum_{n=0}^{\infty} K_{n,p,q}^{(k)}(x \mid \lambda) \frac{t^n}{n!} \left((t+1)^{\lambda} - 1 \right) = \lambda Li_{k,p,q}\left(1-e^{-t}\right)(1+t)^x$$

$$\sum_{n=0}^{\infty} K_{n,p,q}^{(k)}(x \mid \lambda) \frac{t^n}{n!} \sum_{l=0}^{\infty} (\lambda)_l \frac{t^l}{l!} - \sum_{n=0}^{\infty} K_{n,p,q}^{(k)}(x \mid \lambda) \frac{t^n}{n!}$$
$$= \lambda \sum_{n=0}^{\infty} \frac{(n+1)!\,(-1)^{n+1}}{[n+1]_{p,q}^{k}} \sum_{r=0}^{\infty} S_2(r, n+1)(-1)^r \frac{t^r}{r!} \sum_{l=0}^{\infty} (\lambda)_l \frac{t^l}{l!}.$$

Using Cauchy product rule to every side of these equalities and comparing the coefficients, we have (25).

Corollary 3.7. *From (23) and (25), we have*

$$\sum_{r=0}^{m} \binom{m}{r} K_{r,p,q}^{(k)}(x \mid \lambda)(\lambda)_{m-r} - K_{m,p,q}^{(k)}(x \mid \lambda) = K_{n,p,q}^{(k)}(x+\lambda \mid \lambda) - K_{n,p,q}^{(k)}(x \mid \lambda).$$

Theorem 3.8. *The following relation holds true:*

$$\sum_{r=0}^{m} r \binom{m}{r} Ch_{r-1,p,q}^{(k)}(x \mid \lambda)(x)_{m-r} + m Ch_{m-1,p,q}^{(k)}(x \mid \lambda)$$
$$= 2 \sum_{r=0}^{m} \binom{m}{r} \sum_{n=0}^{\infty} \frac{(n+1)!\,(-1)^{n+1+r}}{[n+1]_{p,q}^{k}} S_2(r, n+1)(x)_{m-r}. \tag{26}$$

Proof. By using (20), we write as

$$\sum_{n=0}^{\infty} Ch_{n,p,q}^{(k)}(x \mid \lambda) \frac{t^{n+1}}{n!} \left((1+t)^{\lambda} + 1 \right) = 2 Li_{k,p,q}\left(1-e^{-t}\right)(1+t)^x$$

$$\sum_{m=0}^{\infty} m Ch_{m-1,p,q}^{(k)}(x \mid \lambda) \frac{t^m}{m!} \sum_{l=0}^{\infty} (\lambda)_l \frac{t^l}{l!} + \sum_{m=0}^{\infty} Ch_{m-1,p,q}^{(k)}(x \mid \lambda) \frac{t^m}{m!}$$
$$= 2 \sum_{n=0}^{\infty} \frac{(n+1)!\,(-1)^{n+1}}{[n+1]_{p,q}^{k}} \sum_{r=0}^{\infty} S_2(r, n+1)(-1)^r \frac{t^r}{r!} \sum_{l=0}^{\infty} (x)_l \frac{t^l}{l!}.$$

Using Cauchy product rule to every side of these equalities and comparing the coefficients, we have (26). □

Corollary 3.9. *From (24) and (26), we have*

$$\sum_{r=0}^{m} r \binom{m}{r} Ch_{r-1,p,q}^{(k)}(x \mid \lambda)(x)_{m-r} + m Ch_{m-1,p,q}^{(k)}(x \mid \lambda)$$
$$= m \left(Ch_{m-1,p,q}^{(k)}(x+\lambda \mid \lambda) + Ch_{m-1,p,q}^{(k)}(x \mid \lambda) \right).$$

Theorem 3.10. *There is the following relationships between the (p, q)-poly-Korobov polynomials and the Bernoulli polynomials*

$$\sum_{n=0}^{m} K_{n,p,q}^{(k)}(x \mid \lambda) S_2(m,n)(-1)^m = \sum_{l=0}^{m} \binom{m}{l} B_{m-l}\left(\frac{x}{\lambda}\right)(-\lambda)^{m-l} \frac{(-1)^l l!}{[l+1]_{p,q}^k}. \tag{27}$$

Proof. By replacing t by $e^{-t}-1$ in (19), we get

$$\begin{aligned}\sum_{n=0}^{\infty} K_{n,p,q}^{(k)}(x \mid \lambda) \frac{(e^{-t}-1)^n}{n!} &= \frac{\lambda e^{-tx}}{e^{-t\lambda}-1} Li_{k,p,q}(-t) \\ &= -\frac{1}{t}\frac{(-\lambda t)}{e^{-t\lambda}-1} e^{-t\lambda\left(\frac{x}{\lambda}\right)} Li_{k,p,q}(-t) \\ &= -\frac{1}{t}\sum_{m=0}^{\infty} B_m\left(\frac{x}{\lambda}\right)(-\lambda)^m \frac{t^m}{m!}\sum_{l=0}^{\infty}\frac{(-1)^{l+1} l!}{[l+1]_{p,q}^k}\frac{t^{l+1}}{l!}\end{aligned}$$

$$\sum_{n=0}^{\infty} K_{n,p,q}^{(k)}(x \mid \lambda)\sum_{m=n}^{\infty} S_2(m,n)(-1)^m \frac{t^m}{m!} = \sum_{r=0}^{\infty} B_r\left(\frac{x}{\lambda}\right)(-\lambda)^r \frac{t^r}{r!}\sum_{l=0}^{\infty}\frac{(-1)^l l!}{[l+1]_{p,q}^k}\frac{t^l}{l!}.$$

Using Cauchy product rule and comparing both sides of these equation, we have (27). ☐

Theorem 3.11. *There is the following relationships between the (p, q)-poly-Korobov-type Changhee polynomials and the Euler polynomials:*

$$\sum_{n=0}^{\infty} nCh_{n-1,p,q}^{(k)}(x \mid \lambda) S_2(r,n)(-1)^r = r\sum_{l=0}^{r-1}\binom{r-1}{l} E_{r-1-l}\left(\frac{x}{\lambda}\right)\frac{\lambda^{r-l+1} l!}{[l+1]_{p,q}^k}. \tag{28}$$

Proof. By replacing t by $e^{-t}-1$ in (20), we get

$$\sum_{n=0}^{\infty} Ch_{n,p,q}^{(k)}(x \mid \lambda)\frac{(e^{-t}-1)^n}{n!} = \frac{2e^{-tx}}{(e^{-t}-1)(e^{-t\lambda}+1)} Li_{k,p,q}(-t)$$

$$\sum_{n=0}^{\infty} Ch_{n,p,q}^{(k)}(x \mid \lambda)\frac{(e^{-t}-1)^{n+1}}{n!} = \sum_{n=0}^{\infty} E_n\left(\frac{x}{\lambda}\right)(-\lambda)^n \frac{t^n}{n!}\sum_{n=0}^{\infty}\frac{(-t)^{n+1}}{[n+1]_{p,q}^k}$$

$$\begin{aligned}&\sum_{n=0}^{\infty}(n+1) Ch_{n,p,q}^{(k)}(x \mid \lambda)\sum_{r=0}^{\infty} S_2(r,n+1)(-1)^r \frac{t^r}{r!} \\ = \quad &\sum_{r=0}^{\infty}\left(r\sum_{l=0}^{r-1}\binom{r-1}{l} E_{r-1-l}\left(\frac{x}{\lambda}\right)(-\lambda)^{r-1-l}\frac{(-1)^{r-1} l!}{[l+1]_{p,q}^k}\right)\frac{t^r}{r!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in both sides, we have (28). ☐

Theorem 3.12. *The following relation holds true:*

$$2\left\{\sum_{m=0}^{n}\binom{n}{m}K_{m,p,q}^{(k)}\left(x\mid\lambda\right)\left(\lambda\right)_{n-m}-K_{n,p,q}^{(k)}\left(x\mid\lambda\right)\right\}$$
$$=\lambda n\left\{\sum_{m=0}^{n-1}\binom{n-1}{m}Ch_{m,p,q}^{(k)}\left(x\mid\lambda\right)\left(\lambda\right)_{n-m}+Ch_{n-1,p,q}^{(k)}\left(x\mid\lambda\right)\right\}. \qquad (29)$$

Proof. From (19) and (20), we write as

$$2\sum_{n=0}^{\infty}K_{n,p,q}^{(k)}\left(x\mid\lambda\right)\frac{t^{n}}{n!}\left(\left(t+1\right)^{\lambda}-1\right)=\lambda\sum_{n=0}^{\infty}Ch_{n,p,q}^{(k)}\left(x\mid\lambda\right)\frac{t^{n}}{n!}\left(\left(t+1\right)^{\lambda}+1\right)t.$$

After making the mathematical operation for these equation, we have (29). □

Corollary 3.13. *From (19) and (20), we have the following relationships between the (p,q)-poly-Korobov polynomials and the Korobov polynomials, the (p,q)-poly-Korobov-type Changhee polynomials and the Korobov-type Changhee polynomials, respectively,*

$$nK_{n,p,q}^{(k)}\left(x\mid\lambda\right)=\sum_{r=0}^{n}\binom{n}{r}K_{n-r}\left(x\mid\lambda\right)\sum_{l=0}^{\infty}\frac{\left(-1\right)^{l+r}\left(l+1\right)!}{\left[l+1\right]_{p,q}^{k}}S_{2}(r,l+1)$$

and

$$nCh_{n-1,p,q}^{(k)}\left(x\mid\lambda\right)=\sum_{r=0}^{n}\binom{n}{r}Ch_{n-r}\left(x\mid\lambda\right)\sum_{l=0}^{\infty}\frac{\left(-1\right)^{l+r}\left(l+1\right)!}{\left[l+1\right]_{p,q}^{k}}S_{2}(r,l+1).$$

4. Conclusion

The important subjects of the Analytic number theory are the Bernoulli polynomials and Euler polynomials. Srivastava [15], Srivastava et al. in ([16], [17]) introduced and investigated some basic properties of these numbers and polynomials. They proved some theorems and recurrences relations for these polynomials. Carlitz [3] introduced degenerate Bernoulli polynomials. Bayad et al. in [2], Hamahata [4], Imatomi et al. [5], Kim et al. ([6], [7]) considered and investigated poly-Bernoulli and poly-Euler polynomials. Kim et al. ([9], [10]) and Kruchinin [12] introduced Korobov polynomials. Kim et al. [10] considered the Korobov type polynomials associated with p-adic integrals on $\mathbb{Z}_p$. Komatsu et al. [11] introduced and investigated the (p,q)-analogue of poly-Euler polynomials.

In this work, we define the degenerate Korobov-type Changhee polynomials. We give some relations between the Euler polynomials and the degenerate Korobov-type Changhee polynomials. Further, we consider the (p,q)-poly-Korobov polynomials and the (p,q)-poly-Korobov type Changhee polynomials. We give some recurrence relations and identities for the degenerate Korobov-type Changhee polynomials and the (p,q)-poly-Korobov-type Changhee polynomials.

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