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SOME PROPERTIES OF DEGENERATE CARLITZ-TYPE TWISTED q-EULER NUMBERS AND POLYNOMIALS†

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Abstract. In this paper, we define degenerate Carlitz-type twisted q-Euler numbers and polynomials by generalizing the degenerate Euler numbers and polynomials, Carlitz's type degenerate q-Euler numbers and polynomials. We also give some interesting properties, explicit formulas, symmetric properties, a connection with degenerate Carlitz-type twisted q-Euler numbers and polynomials.

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1. Introduction

Mathematicians have been working in the fields of the Euler numbers and polynomials, Bernoulli numbers and polynomials, tangent numbers and polynomials, and Stirling numbers(see [1-9, 10, 11, 13, 18, 19, 20]). In recent years, we have been studied some properties and symmetry identiities of the degenerate Carlitz-type (p, q)-Euler numbers and polynomials, degenerate q-poly-Bernoulli numbers and polynomials, (p, q)-Hurwitz zeta function, degenerate Carlitz-type q-Euler numbers and polynomials, (h, q)-Euler numbers and polynomials (see [4, 5, 10, 12, 13, 14, 15, 16, 17]). In this paper we define a new form of degenerate Carlitz-type twisted q-Euler numbers and polynomials and study some theories of the degenerate Carlitz-type twisted q-Euler numbers and polynomials. Throughout this paper, we always make use of the following notations: $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}_0 = \mathbb{N} \cup \{0\}$ denotes the set of nonnegative integers, $\mathbb{Z}_0^- = \{0, -1, -2, -3, \ldots\}$ denotes the set of nonpositive integers, and $\mathbb{C}$ denotes the set of complex numbers.

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We recall that the degenerate Euler numbers $\mathcal{E}_n(\mu)$ and Euler polynomials $\mathcal{E}_n(z,\mu)$, which are defined by generating functions like (1), and (2)(see [2, 3, 4])

$$\frac{2}{(1+\mu t)^{\frac{1}{\mu}}+1}=\sum_{n=0}^{\infty}\mathcal{E}_n(\mu)\frac{t^n}{n!}, \tag{1}$$

and

$$\frac{2}{(1+\mu t)^{\frac{1}{\mu}}+1}(1+\mu t)^{\frac{z}{\mu}}=\sum_{n=0}^{\infty}\mathcal{E}_n(z,\mu)\frac{t^n}{n!}, \tag{2}$$

respectively.

We remind that well-known Stirling numbers of the first kind $S_1(n,j)$ and the second kind $S_2(n,j)$ are defined by this(see [2, 3, 20])

$$(z)_n=\sum_{j=0}^{n}S_1(n,j)z^j \text{ and } z^n=\sum_{j=0}^{n}S_2(n,j)(z)_j,$$

respectively. Here $(z)_j=z(z-1)\cdots(z-j+1)$. The generalized falling factorial $(z|\mu)_m$ with increment μ is defined by

$$(z|\mu)_m=\prod_{j=0}^{m-1}(z-\mu j)$$

for positive integer n, with $(z|\mu)_0=1$; as we know,

$$(z|\mu)_m=\sum_{j=0}^{m}S_1(m,j)\mu^{m-j}z^j.$$

$(z|\mu)_m=\mu^m(\mu^{-1}z|1)_m$ for $\mu\neq 0$. Clearly $(z|0)_m=z^m$. The binomial theorem is this for a variable z,

$$(1+\mu t)^{z/\mu}=\sum_{n=0}^{\infty}(z|\mu)_n\frac{t^n}{n!}.$$

For $z\in\mathbb{C}$, the q-number is defined by

$$[z]_q=\frac{1-q^z}{1-q},(q\neq 1).$$

By using q-number, we define define a new form of degenerate Carlitz-type twisted q-Euler numbers and polynomials, which generalized the previously known numbers and polynomials, including the degenerate Euler numbers and polynomials, degenerate Carlitz-type twisted q-Euler numbers and polynomials(see [2, 3, 8, 13]. Here we first recall the Carlitz's type twisted q-Euler numbers and polynomials(see [17]). Let ζ be rth root of 1 and $\zeta\neq 1$(see [11, 16]).

Definition 1.1. The Carlitz's type twisted q-Euler polynomials $E_{n,q,\zeta}(z)$ are defined by means of the generating function

$$\sum_{n=0}^{\infty}E_{n,q,\zeta}(z)\frac{t^n}{n!}=[2]_q\sum_{m=0}^{\infty}(-1)^m q^m\zeta^m e^{[m+z]_q t}.$$

and their values at $z = 0$ are called the Carlitz's type q-Euler numbers and denoted $E_{n,q,\zeta}$.

In the following section, we define a new form of degenerate Carlitz-type twisted q-Euler numbers $\mathcal{E}_{n,q,\zeta}(\mu)$ and polynomials $\mathcal{E}_{n,q,\zeta}(z,\mu)$. After that we will investigate some their properties and identities. In Sect. 2, a new form of degenerate Carlitz-type twisted q-Euler numbers $\mathcal{E}_{n,q,\zeta}(\mu)$ and polynomials $\mathcal{E}_{n,q,\zeta}(z,\mu)$ are defined. We derive some of their relevant properties and symmetric identities. In Sect. 3, first, we derive the symmetric properties for degenerate Carlitz-type twisted q-Euler numbers $\mathcal{E}_{n,q,\zeta}(\mu)$ and polynomials $\mathcal{E}_{n,q,\zeta}(z,\mu)$.

2. Degenerate Carlitz-type twisted q-Euler numbers and polynomials

In this section, we construct a new form of degenerate Carlitz-type twisted q-Euler numbers $\mathcal{E}_{n,q,\zeta}(\mu)$ and polynomials $\mathcal{E}_{n,q,\zeta}(z,\mu)$ and provide some of their relevant identities and properties. Firstly, we construct the degenerate Carlitz-type twisted q-Euler numbers and polynomials as follows:

Definition 2.1. For $0 < q < 1$, the degenerate Carlitz-type twisted q-Euler numbers $\mathcal{E}_{n,q,\zeta}(\mu)$ and polynomials $\mathcal{E}_{n,q,\zeta}(z,\mu)$ are defined by means of the generating functions

$$\sum_{n=0}^{\infty}\mathcal{E}_{n,q,\zeta}(\mu)\frac{t^n}{n!} = [2]_q\sum_{m=0}^{\infty}(-1)^m q^m\zeta^m(1+\mu t)^{\frac{[m]_q}{\mu}}, \tag{1}$$

and

$$\sum_{n=0}^{\infty}\mathcal{E}_{n,q,\zeta}(z,\mu)\frac{t^n}{n!} = [2]_q\sum_{m=0}^{\infty}(-1)^m q^m\zeta^m(1+\mu t)^{\frac{[m+z]_q}{\mu}}, \tag{2}$$

respectively.

The degenerate Carlitz-type twisted q-Euler numbers $\mathcal{E}_{n,q,\zeta}(\mu)$ can be determined explicitly. A few of them are

$$\begin{aligned}
\mathcal{E}_{0,q,\zeta}(\mu) &= \frac{[2]_q}{1+\zeta q},\\
\mathcal{E}_{1,q,\zeta}(\mu) &= \frac{[2]_q}{(1-q)(1+\zeta q)} - \frac{[2]_q}{(1-q)(1+\zeta q^2)},\\
\mathcal{E}_{2,q,\zeta}(\mu) &= -\frac{[2]_q\mu}{(1-q)(1+\zeta q)} + \frac{[2]_q}{(1-q)^2(1+\zeta q)} + \frac{[2]_q\mu}{(1-q)(1+\zeta q^2)}\\
&\quad - \frac{2[2]_q}{(1-q)^2(1+\zeta q^2)} + \frac{[2]_q}{(1-q)^2(1+\zeta q^3)}.
\end{aligned}$$

Putting $\zeta = 1$, we have

$$\lim_{q\to 1}\mathcal{E}_{n,q,\zeta}(z,\mu) = \mathcal{E}_n(z,\mu), \quad \lim_{q\to 1}\mathcal{E}_{n,q,\zeta}(\mu) = \mathcal{E}_n(\mu).$$

Since

$$(1+\mu t)^{\frac{[z+w]_q}{\mu}} = e^{\frac{[z+w]_q}{\mu}\log(1+\mu t)} = \sum_{n=0}^{\infty}\left(\frac{[z+w]_q}{\mu}\right)^n \frac{(\log(1+\mu t))^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} S_1(n,m)\mu^{n-m}[z+w]_q^m\right)\frac{t^n}{n!}, \tag{3}$$

we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\mathcal{E}_{n,q,\zeta}(z,\mu)\frac{t^n}{n!}\\
&= [2]_q\sum_{m=0}^{\infty}(-1)^m q^m\zeta^m(1+\mu t)^{\frac{[m+z]_q}{\mu}}\\
&= [2]_q\sum_{m=0}^{\infty}(-1)^m q^m\zeta^m\sum_{n=0}^{\infty}\sum_{l=0}^{n}S_1(n,l)\mu^{n-l}\frac{\sum_{j=0}^{l}\binom{l}{j}(-1)^j q^{(z+m)j}}{(1-q)^l}\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty}\left([2]_q\sum_{l=0}^{n}\sum_{j=0}^{l}\frac{S_1(n,l)\mu^{n-l}\binom{l}{j}(-1)^j q^{zj}}{(1-q)^l}\frac{1}{1+\zeta q^{j+1}}\right)\frac{t^n}{n!}.
\end{aligned} \tag{4}$$

Comparing coefficients $t^n/n!$ in the above equation, we get the following theorem.

Theorem 2.2. *For $n\in\mathbb{Z}_0$, we have*

$$\begin{aligned}
\mathcal{E}_{n,q,\zeta}(z,\mu) &= [2]_q\sum_{l=0}^{n}\sum_{j=0}^{l}\frac{S_1(n,l)\mu^{n-l}\binom{l}{j}(-1)^j q^{zj}}{(1-q)^l}\frac{1}{1+\zeta q^{j+1}}\\
&= [2]_q\sum_{m=0}^{\infty}\sum_{l=0}^{n}S_1(n,l)\mu^{n-l}(-1)^m q^m\zeta^m[z+m]_q^l.
\end{aligned}$$

By replacing t by $\dfrac{e^{\mu t}-1}{\mu}$ in (2), we have

$$\begin{aligned}
E_{n,q,\zeta}(z) &= \sum_{n=0}^{\infty}\mathcal{E}_{n,q,\zeta}(z,\mu)\left(\frac{e^{\mu t}-1}{\mu}\right)^n\frac{1}{n!}\\
&= \sum_{n=0}^{\infty}\mathcal{E}_{n,q,\zeta}(z,\mu)\mu^{-n}\sum_{m=n}^{\infty}S_2(m,n)\mu^m\frac{t^m}{m!}\\
&= \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m}\mathcal{E}_{n,q,\zeta}(z,\mu)\mu^{m-n}S_2(m,n)\right)\frac{t^m}{m!}.
\end{aligned} \tag{5}$$

Thus, by (5), we have the following theorem.

Theorem 2.3. *For $n \in \mathbb{Z}_0$, we have*

$$E_{n,q,\zeta}(z) = \sum_{n=0}^{m} \mathcal{E}_{n,q,\zeta}(z,\mu)\mu^{m-n} S_2(m,n).$$

By replacing t by $\log(1+\mu t)^{1/\mu}$ in Definition 1.1 and Definition 2.1, we have

$$\begin{aligned}\sum_{n=0}^{\infty} E_{n,q,\zeta}(z)\left(\log(1+\mu t)^{1/\mu}\right)^n \frac{1}{n!} &= [2]_q \sum_{m=0}^{\infty}(-1)^m q^m \zeta^m (1+\mu t)^{\frac{[m+z]_q}{\mu}} \\ &= \sum_{m=0}^{\infty} \mathcal{E}_{m,q,\zeta}(z,\mu)\frac{t^m}{m!},\end{aligned} \tag{6}$$

and

$$\begin{aligned}&\sum_{n=0}^{\infty} E_{n,q,\zeta}(z)\left(\log(1+\mu t)^{1/\mu}\right)^n \frac{1}{n!} \\ &= \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m} E_{n,q,\zeta}(z)\mu^{m-n} S_1(m,n)\right)\frac{t^m}{m!}.\end{aligned} \tag{7}$$

Therefore, by (6) and (7), we have the following theorem.

Theorem 2.4. *For $m \in \mathbb{Z}_0$, we have*

$$\mathcal{E}_{m,q,\zeta}(z,\mu) = \sum_{k=0}^{m} E_{k,q,\zeta}(z)\mu^{m-k} S_1(m,k)$$

We introduce the q-analogue of the generalized falling factorial $(z|\mu)_m$ with increment μ. The q-generalized falling factorial $([z]_q|\mu)_m$ with increment μ is defined by

$$([z]_q|\mu)_m = \prod_{j=0}^{m-1}([z]_q - \mu j)$$

for positive integer m, with the convention $([z]_q|\mu)_0 = 1$.

By (1) and (2), we get

$$\begin{aligned}&-[2]_q(-1)^n q^n \zeta^n \sum_{l=0}^{\infty}(-1)^l q^l \zeta^l (1+\mu t)^{\frac{[l+n]_q}{\mu}} \\ &+ [2]_q \sum_{l=0}^{\infty}(-1)^l q^l \zeta^l (1+\mu t)^{\frac{[l+n]_q}{\mu}} \\ &= [2]_q \sum_{l=0}^{n-1}(-1)^l q^l \zeta^l (1+\mu t)^{\frac{[l]_q}{\mu}}.\end{aligned} \tag{8}$$

Hence, by (8), we also have

$$
\begin{aligned}
&(-1)^{n+1}q^n\zeta^n\sum_{m=0}^{\infty}\mathcal{E}_{m,q,\zeta}(n,\mu)\frac{t^m}{m!}+\sum_{m=0}^{\infty}\mathcal{E}_{m,q,\zeta}(\mu)\frac{t^m}{m!}\\
&=\sum_{m=0}^{\infty}\left([2]_q\sum_{l=0}^{n-1}(-1)^l q^l\zeta^l([l]_q|\mu)_m\right)\frac{t^m}{m!}.
\end{aligned}
\tag{9}
$$

Comparing coefficients $t^m/m!$ on both sides of (9), we get the following theorem.

Theorem 2.5. *For $n \in \mathbb{Z}_0$, we have*

$$
\sum_{l=0}^{n-1}(-1)^l q^l\zeta^l([l]_q|\mu)_m=\frac{(-1)^{n+1}q^n\zeta^n\mathcal{E}_{m,q,\zeta}(n,\mu)+\mathcal{E}_{m,q,\zeta}(\mu)}{[2]_q}.
$$

We observe that

$$
\begin{aligned}
(1+\mu t)^{\frac{[z+y]_q}{\mu}}&=(1+\mu t)^{\frac{[z]_q}{\mu}}(1+\mu t)^{\frac{q^z[y]_q}{\mu}}\\
&=\sum_{m=0}^{\infty}([z]_q|\mu)_m\frac{t^m}{m!}e^{\log(1+\mu t)\frac{q^z[y]_q}{\mu}}\\
&=\sum_{m=0}^{\infty}([z]_q|\mu)_m\frac{t^m}{m!}\sum_{l=0}^{\infty}\left(\frac{q^z[y]_q}{\mu}\right)^l\frac{\log(1+\mu t)^l}{l!}\\
&=\sum_{m=0}^{\infty}([z]_q|\mu)_m\frac{t^m}{m!}\sum_{l=0}^{\infty}\left(\frac{q^z[y]_q}{\mu}\right)^l\sum_{k=l}^{\infty}S_1(k,l)\mu^k\frac{t^k}{k!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}([z]_q|\mu)_{n-k}\mu^{k-l}q^{zl}[y]_q^l S_1(k,l)\right)\frac{t^n}{n!}.
\end{aligned}
\tag{10}
$$

By (2), we get

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\mathcal{E}_{n,q,\zeta}(z,\mu)\frac{t^n}{n!}\\
&=[2]_q\sum_{m=0}^{\infty}(-1)^m q^m\zeta^m(1+\mu t)^{\frac{[m+z]_q}{\mu}}\\
&=[2]_q\sum_{m=0}^{\infty}(-1)^m q^m\zeta^m\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}([z]_q|\mu)_{n-k}\mu^{k-l}q^{zl}[m]_q^l S_1(k,l)\right)\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}([z]_q|\mu)_{n-k}\mu^{k-l}q^{zl}S_1(k,l)E_{l,q,\zeta}\right)\frac{t^n}{n!}.
\end{aligned}
$$

Comparing coefficients $t^n/n!$ in the above equation, we obtain the result as follows:

Theorem 2.6. *For $n \in \mathbb{Z}_0$, we have*

$$\mathcal{E}_{n,q,\zeta}(z,\mu)=\sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}([z]_q|\mu)_{n-k}\mu^{k-l}q^{zl}S_1(k,l)E_{l,q,\zeta}.$$

3. Symmetric properties about degenerate Carlitz-type twisted q-Euler numbers and polynomials

In this section, we are going to have the main results of degenerate Carlitz-type twisted q-Euler numbers $\mathcal{E}_{n,q,\zeta}(\mu)$ and polynomials $\mathcal{E}_{n,q,\zeta}(z,\mu)$. We also establish some interesting symmetric identities for degenerate Carlitz-type twisted q-Euler numbers $\mathcal{E}_{n,q,\zeta}(\mu)$ and polynomials $\mathcal{E}_{n,q,\zeta}(z,\mu)$. Let a and b be odd positive integers. Observe that $[zy]_q=[z]_{q^y}[y]_q$ for any $z,y\in\mathbb{C}$.
By substitute $az+\frac{ai}{b}$ for z in Definition 2.1, replace q by q^b, replace ζ by ζ^b, and replace μ by $\dfrac{\mu}{[b]_q}$, respectively, we derive

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left([2]_{q^a}[b]_q^n\sum_{i=0}^{b-1}(-1)^iq^{ai}\zeta^{ai}\mathcal{E}_{n,q^b,\zeta^b}\left(az+\frac{ai}{b},\frac{\mu}{[b]_q}\right)\right)\frac{t^n}{n!}\\
&=[2]_{q^a}\sum_{i=0}^{b-1}(-1)^iq^{ai}\zeta^{ai}\sum_{n=0}^{\infty}\mathcal{E}_{n,q^b,\zeta^b}\left(az+\frac{ai}{b},\frac{\mu}{[b]_q}\right)\frac{([b]_qt)^n}{n!}\\
&=[2]_{q^a}\sum_{i=0}^{b-1}(-1)^iq^{ai}\zeta^{ai}[2]_{q^b}\sum_{n=0}^{\infty}(-1)^nq^{bn}\zeta^{bn}\left(1+\frac{\mu}{[b]_q}[b]_qt\right)^{\frac{[az+\frac{ai}{b}+n]_{q^b}}{\frac{\mu}{[b]_q}}}\\
&=[2]_{q^a}\sum_{i=0}^{b-1}(-1)^iq^{ai}\zeta^{ai}[2]_{q^b}\sum_{n=0}^{\infty}(-1)^nq^{bn}\zeta^{bn}\left(1+\mu t\right)^{\frac{[abz+ai+nb]_q}{\mu}}.
\end{aligned}$$

Since for any non-negative integer n and odd positive integer a, there exist unique non-negative integer r such that $n=ar+j$ with $0\le j\le a-1$. Hence, this can be written as

$$\begin{aligned}
&[2]_{q^a}[2]_{q^b}\sum_{i=0}^{b-1}(-1)^iq^{ai}\zeta^{ai}\sum_{n=0}^{\infty}(-1)^nq^{bn}\zeta^{bn}\left(1+\mu t\right)^{\frac{[abz+ai+nb]_q}{\mu}}.\\
&=[2]_{q^a}[2]_{q^b}\sum_{i=0}^{b-1}(-1)^iq^{ai}\zeta^{ai}\sum_{\substack{ar+j=0\\0\le j\le a-1}}^{\infty}(-1)^{ar+j}q^{b(ar+j)}\zeta^{b(ar+j)}\\
&\qquad\times(1+\mu t)^{\frac{[abz+ai+(ar+j)b]_q}{\mu}}.
\end{aligned}$$

$$= [2]_{q^a}[2]_{q^b}\sum_{i=0}^{b-1}(-1)^i q^{ai}\zeta^{ai}\sum_{j=0}^{a-1}\sum_{r=0}^{\infty}(-1)^{ar}(-1)^j q^{bar}q^{bj}\zeta^{bar}\zeta^{bj}$$

$$\times (1+\mu t)^{\frac{[abz+ai+abr+bj]_q}{\mu}}$$

$$= [2]_{q^a}[2]_{q^b}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}\sum_{r=0}^{\infty}(-1)^i(-1)^r(-1)^j q^{ai}q^{bar}q^{bj}\zeta^{ai}\zeta^{bar}\zeta^{bj}$$

$$\times (1+\mu t)^{\frac{[abz+ai+abr+bj]_q}{\mu}}.$$

It follows from the above equation that

$$\sum_{n=0}^{\infty}\left([2]_{q^b}[b]_q^n\sum_{i=0}^{b-1}(-1)^i q^{ai}\zeta^{ai}\mathcal{E}_{n,q^b,\zeta^b}\left(az+\frac{ai}{b},\frac{\mu}{[b]_q}\right)\right)\frac{t^n}{n!}$$

$$= [2]_{q^a}[2]_{q^b}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}\sum_{r=0}^{\infty}(-1)^i(-1)^r(-1)^j q^{ai}q^{bar}q^{bj}\zeta^{ai}\zeta^{bar}\zeta^{bj} \tag{11}$$

$$\times (1+\mu t)^{\frac{[abz+ai+abr+bj]_q}{\mu}}.$$

From a similar method, we can obtain that

$$\sum_{n=0}^{\infty}\left([2]_{q^b}[a]_q^n\sum_{i=0}^{a-1}(-1)^i q^{bi}\zeta^{bi}\mathcal{E}_{n,q^a,\zeta^a}\left(bz+\frac{bi}{a},\frac{\mu}{[a]_q}\right)\right)\frac{t^n}{n!}$$

$$= [2]_{q^a}[2]_{q^b}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}\sum_{r=0}^{\infty}(-1)^i(-1)^r(-1)^j q^{bi}q^{ar}q^{aj}\zeta^{bi}\zeta^{bar}\zeta^{aj} \tag{12}$$

$$\times (1+\mu t)^{\frac{[abz+bi+abr+aj]_q}{\mu}}.$$

Thus, we have the following theorem from (11) and (12).

Theorem 3.1. *Let a and b be odd positive integers. Then one has*

$$[2]_{q^a}[b]_q^n\sum_{i=0}^{b-1}(-1)^i q^{ai}\zeta^{ai}\mathcal{E}_{n,q^b,\zeta^b}\left(az+\frac{ai}{b},\frac{\mu}{[b]_q}\right)$$

$$= [2]_{q^b}[a]_q^n\sum_{j=0}^{a-1}(-1)^j q^{bj}\zeta^{bj}\mathcal{E}_{n,q^a,\zeta^a}\left(bz+\frac{bi}{a},\frac{\mu}{[a]_q}\right).$$

It follows that we show some special cases of Theorem 3.1. Setting $b=1$ in Theorem 3.1, we have the multiplication theorem for the degenerate Carlitz-type twisted q-Euler polynomials $\mathcal{E}_{n,q,\zeta}(z,\mu)$.

Corollary 3.2. *Let a be odd positive integer. Then one has*

$$\mathcal{E}_{n,q,\zeta}(z,\mu)=\frac{[2]_q}{[2]_{q^a}}[a]_q^n\sum_{j=0}^{a-1}(-1)^j q^j\zeta^j\mathcal{E}_{n,q^a,\zeta^a}\left(\frac{z+i}{a},\frac{\mu}{[a]_q}\right). \tag{13}$$

Let $x=0$ in Theorem 3.1, we have the following corollary.

Corollary 3.3. *Let a and b be odd positive integers. Then it has*

$$\begin{aligned}&[2]_{q^a}[b]_q^n\sum_{i=0}^{b-1}(-1)^i q^{ai}\zeta^{ai}\mathcal{E}_{n,q^b,\zeta^b}\left(\frac{ai}{b},\frac{\mu}{[b]_q}\right)\\&=[2]_{q^b}[a]_q^n\sum_{j=0}^{a-1}(-1)^j q^{bj}\zeta^{bj}\mathcal{E}_{n,q^a,\zeta^a}\left(\frac{bj}{a},\frac{\mu}{[a]_q}\right).\end{aligned}$$

By Theorem 2.4 and Corollary 3.3, we have the below theorem.

Theorem 3.4. *Let a and b be odd positive integers. Then*

$$\begin{aligned}&\sum_{l=0}^{n}S_1(n,l)\mu^{n-l}[b]_q^l[2]_{q^a}\sum_{i=0}^{b-1}(-1)^i q^{ai}\zeta^{ai}E_{l,q^b,\zeta^b}\left(\frac{a}{b}i\right)\\&=\sum_{l=0}^{n}S_1(n,l)\mu^{n-l}[a]_q^l[2]_{q^b}\sum_{j=0}^{a-1}(-1)^j q^{bj}\zeta^{bj}E_{l,q^a,\zeta^a}\left(\frac{b}{a}j\right).\end{aligned}$$

In particular, the case $a=3$ in Corollary 3.2 gives the triplication formula for degenerate Carlitz-type twisted q-Euler polynomials

$$\begin{aligned}&\mathcal{E}_{n,q^3,\zeta^3}\left(\frac{z}{3},\frac{\mu}{[3]_q}\right)+q^2\zeta^2\mathcal{E}_{n,q^3,\zeta^3}\left(\frac{z+2}{3},\frac{\mu}{[3]_q}\right)\\&=\frac{[2]_{q^3}}{[2]_q[3]_q^n}\mathcal{E}_{n,q,\zeta}(z,\mu)+q\zeta\mathcal{E}_{n,q^3,\zeta^3}\left(\frac{z+1}{3},\frac{\mu}{[3]_q}\right).\end{aligned} \tag{14}$$

Setting $p=1$ in (13) and (14) leads to the familiar multiplication theorem for the degenerate Carlitz-type twisted q-Euler polynomials

$$\mathcal{E}_{n,q,\zeta}(z,\mu)=\frac{[2]_q[a]_q^n}{[2]_{q^a}}\sum_{j=0}^{a-1}(-1)^j q^j\zeta^j\mathcal{E}_{n,q^a,\zeta^a}\left(\frac{z+i}{a},\frac{\mu}{[a]_q}\right), \tag{15}$$

and the triplication formula for degenerate Carlitz-type twisted q-Euler polynomials

$$\begin{aligned}&\mathcal{E}_{n,q^3,\zeta^3}\left(\frac{z}{3},\frac{\mu}{[3]_q}\right)+q^2\zeta^2\mathcal{E}_{n,q^3,\zeta^3}\left(\frac{z+2}{3},\frac{\mu}{[3]_q}\right)\\&=\frac{[2]_{q^3}}{[2]_q[3]_q^n}\mathcal{E}_{n,q,\zeta}(z,\mu)+q\zeta\mathcal{E}_{n,q^3,\zeta^3}\left(\frac{z+1}{3},\frac{\mu}{[3]_q}\right).\end{aligned} \tag{16}$$

Letting $q \to 1$ in (15) and (16) leads to the familiar multiplication theorem for the degenerate twisted Euler polynomials

$$\mathcal{E}_{n,\zeta}(z,\mu) = a^n \sum_{j=0}^{a-1} (-1)^j \zeta^j \mathcal{E}_{n,\zeta^a}\left(\frac{z+i}{a}, \frac{\mu}{a}\right), \tag{17}$$

and the triplication formula for degenerate twisted Euler polynomials

$$\begin{aligned} &\mathcal{E}_{n,\zeta^3}\left(\frac{z}{3}, \frac{\mu}{3}\right) + \zeta^2 \mathcal{E}_{n,\zeta^3}\left(\frac{z+2}{3}, \frac{\mu}{3}\right) \\ &= \frac{1}{3^n}\mathcal{E}_{n,\zeta}(z,\mu) + \zeta \mathcal{E}_{n,\zeta^3}\left(\frac{z+1}{3}, \frac{\mu}{3}\right). \end{aligned} \tag{18}$$

Letting $\zeta = 1$ in (17) and (18) leads to the familiar multiplication theorem for the degenerate Euler polynomials

$$\mathcal{E}_n(z,\mu) = a^n \sum_{j=0}^{a-1} (-1)^j \mathcal{E}_n\left(\frac{z+i}{a}, \frac{\mu}{a}\right), \tag{19}$$

and the triplication formula for degenerate Euler polynomials

$$\mathcal{E}_n\left(\frac{z}{3}, \frac{\mu}{3}\right) + \mathcal{E}_n\left(\frac{z+2}{3}, \frac{\mu}{3}\right) = \frac{1}{3^n}\mathcal{E}_n(z,\mu) + \mathcal{E}_n\left(\frac{z+1}{3}, \frac{\mu}{3}\right). \tag{20}$$

Letting $\mu \to 0$ in (19) and (20) leads to the familiar multiplication theorem for the Euler polynomials

$$E_n(z) = a^n \sum_{j=0}^{a-1} (-1)^j E_n\left(\frac{z+i}{a}\right).$$

and the triplication formula for Euler polynomials

$$E_n(z) = 3^n E_n\left(\frac{z}{3}\right) - 3^n E_n\left(\frac{z+1}{3}\right) + 3^n E_n\left(\frac{z+2}{3}\right).$$

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