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A NOTE ON SEMI-SLANT LIGHTLIKE SUBMANIFOLDS OF AN INDEFINITE KENMOTSU MANIFOLD

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Abstract. In this paper, we study the geometry of semi-slant lightlike submanifolds of an indefinite Kenmotsu manifold. The integrability conditions of distributions $D_1 \oplus \{V\}, D_2 \oplus \{V\}$ and $RadTM$ on semi-slant lightlike submanifolds of an indefinite Kenmotsu manifold are defined. Furthermore, we derive necessary and sufficient conditions for the above distributions to have totally geodesic foliations.

1. Introduction

In the theory of submanifolds of semi-Riemannian manifolds, it is interesting to study the geometry of lightlike submanifolds since the intersection of the normal vector bundle, and the tangent bundle is non-trivial. For example, Duggal and Bejancu [1] first studied the geometry of lightlike submanifolds of indefinite Kähler manifolds, and Duggal and Sahin [2] introduced a general notion of lightlike submanifolds of indefinite Sasakian manifolds. In [14], Yano introduced the notion of a f-structure on a differential manifold M, i.e., a tensor field f of type $(1, 1)$ and rank $2n$ satisfying $f^3 + f = 0$ as a generalization of both almost contact (for $s = 1$) and almost complex structures (for $s = 0$). Nakagawa [10, 11] introduced the notion of globally framed f-manifolds, later developed and studied by Goldberg [4], Goldberg, and Yano [5, 6]. In 1972, Kenmotsu [9] studied a class of contact Riemannian manifolds satisfying some special conditions, which are known as Kenmotsu manifolds. A Kenmotsu manifold equipped with the non-degenerate indefinite metric is called Indefinite Kenmotsu manifold. On the other hand, Shukla and Yadav [13] introduced the geometry of semi-slant submanifolds of indefinite Sasakian manifolds. Recently, Gupta and Sharfuddin [8, 7] studied the geometry of slant lightlike submanifolds, invariant submanifolds, contact CR-lightlike submanifolds, and

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contact SCR-lightlike submanifolds of indefinite Kenmotsu manifolds. In [12], Sachdeva et al. studied totally contact umbilical slant lightlike submanifolds of indefinite Kenmotsu manifolds. It should be noted that the integrability and totally geodesic foliation did not consider in previous literature. Therefore, in the present paper, we will fill up this gape.

The paper is organized as follows: In section 2, it includes basic information on the lightlike geometry as needed in this paper. In section 3, we introduce the concept of semi-slant lightlike submanifolds. We obtain integrability conditions of distributions $D_1 \oplus \{V\}, D_2 \oplus \{V\}$ and $RadTM$. In section 4, we obtain necessary and sufficient conditions for the distributions to have totally geodesic foliation that involves the definition of semi-slant lightlike submanifolds

2. Preliminaries

An odd-dimensional semi-Riemannian manifold $\overline{M}$ is called an indefinite almost contact metric manifold if there is an indefinite almost contact structure $(\phi, V, \eta, \overline{g})$ consisting of a $(1,1)$-tensor field ϕ, a structure vector field V, a 1-form η and $\overline{g}$ is the semi-Riemannian metric on $\overline{M}$ satisfying

$$\phi^2 X = -X + \eta(X)V,\ \eta(V) = 1,\ \eta \circ \phi = 0,\ \phi V = 0,\ \eta(V) = 1, \tag{1}$$

$$\overline{g}(X, V) = \eta(V),\ \overline{g}(\phi X, \phi Y) = \overline{g}(X, Y) - \eta(X)\eta(Y) \tag{2}$$

for $X, Y \in T\overline{M}$. An indefinite almost contact metric manifold $\overline{M}$ is called an indefinite Kenmotsu manifold if [9],

$$(\overline{\nabla}_X \phi)Y = -\overline{g}(\phi X, Y)V + \eta(Y)\phi X,\ \overline{\nabla}_X V = -X + \eta(X)V \tag{3}$$

for $X, Y \in T\overline{M}$, where $\overline{\nabla}$ denotes the Levi-Civita connection on $\overline{M}$. A submanifold M^m immersed in a semi-Riemannian manifold $(\overline{M}^{m+n}, \overline{g})$ is called a lightlike submanifold [1] if it admits a degenerate metric g induced from $\overline{g}$ on M. If g is degenerate on the tangent bundle TM of M, then M is called a lightlike submanifold. For a degenerate metric g on M, $TM^{\perp}$ is a degenerate n-dimensional subspace of $T_x\overline{M}$. Thus both T_xM and $T_xM^{\perp}$ are degenerate orthogonal subspaces but not complementary to each other. Therefore there exists a subspace $Rad(TM) = T_xM \cap T_xM^{\perp}$, known as Radical subspace. If the mapping $Rad(TM) : M \longrightarrow TM$, such that $x \in M \mapsto Rad(T_xM)$, defines a smooth distribution of rank $r > 0$ on M, then M is said to be an r-lightlike submanifold and the distribution $Rad(TM)$ is said to be radical distribution on M. The non-degenerate complementary subbundles $S(TM)$ and $S(TM^{\perp})$ of $Rad(TM)$ are known as screen distribution in TM and screen transversal distribution in $TM^{\perp}$ respectively, i.e.,

$$TM = Rad(TM) \perp S(TM)\ \&\ TM^{\perp} = Rad(TM) \perp S(TM^{\perp}). \tag{4}$$

Let $ltr(TM)$(lightlike transversal bundle) and $tr(TM)$(transversal bundle) be complementary but not orthogonal vector bundles to $Rad(TM)$ in $S(TM^{\perp})^{\perp}$ and TM in $T\bar{M}|_M$ respectively. Then, the transversal vector bundle $tr(TM)$ is given by[3]

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}). \tag{5}$$

From (4) and (5), we get

$$T\bar{M}|_M = TM \oplus tr(TM) = (Rad(TM) \oplus ltr(TM)) \perp S(TM) \perp S(TM^{\perp}). \tag{6}$$

Theorem 2.1. [1] *Let $(M, g, S(TM), S(TM^{\perp}))$ be an r-lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then there exists a complementary vector bundle $ltr(TM)$ of $Rad(TM)$ in $S(TM^{\perp})^{\perp}$ and a basis of $\Gamma(ltr(TM)|u)$ consisting of a smooth section $\{N_i\}$ of $S(TM^{\perp})^{\perp}|_u$, where u is a coordinate neighbourhood of M such that*

$$\bar{g}_{ij}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}_{ij}(N_i, N_j) = 0 \tag{7}$$

for any $i, j \in \{1, 2, \ldots, r\}$.

A submanifold $(M, g, S(TM), S(TM^{\perp}))$ of $\overline{M}$ is said to be
(i) r-lightlike if $r < min\{m, n\}$;
(ii) coisotropic if $r = n < m, S(TM^{\perp}) = 0$;
(iii) isotropic if $r = m = n, S(TM) = 0$;
(iv) totally lightlike if $r = m = n, S(TM) = 0 = S(TM^{\perp})$.

Let $\overline{\nabla}, \nabla$ and ∇^t denote the linear connections on $\overline{M}, M$ and vector bundle $tr(TM)$, respectively. Then the Gauss and Weingarten formulae are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \forall\ X, Y \in \Gamma(TM), \tag{8}$$

$$\overline{\nabla}_X U = -A_U X + \nabla^t_X U, \forall\ U \in \Gamma(tr(TM)), \tag{9}$$

where $\{\nabla_X Y, A_U X\}$ and $\{h(X, Y), \nabla^t_X U\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$ respectively, the linear connections ∇ and ∇^t are on M and on the vector bundle $tr(TM)$ respectively, the second fundamental form h is a symmetric $F(M)$-bilinear form on $\Gamma(TM)$ with values in $\Gamma(tr(TM))$ and the shape operator A_V is a linear endomorphism of $\Gamma(TM)$.
From (8) and (9) , for any $X, Y \in \Gamma(tr(TM))$, $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$, we have

$$\overline{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \tag{10}$$

$$\overline{\nabla}_X N = -A_N X + \nabla^l_X(N) + D^s(X, N), \tag{11}$$

$$\overline{\nabla}_X W = -A_W X + \nabla^s_X(W) + D^l(X, W), \tag{12}$$

where $D^l(X, W), D^s(X, N)$ are the projections of ∇^t on $\Gamma(ltr(TM))$ and $\Gamma(S(TM^{\perp}))$ respectively, ∇^l, ∇^s are linear connections on $\Gamma(ltr(TM))$ and $\Gamma(S(TM^{\perp}))$, respectively and A_N, A_W are shape operators on M with respect to N and W,

respectively.
Using (8) and (10)-(12) , we obtain

$$\overline{g}(h^s(X,Y),W) + \overline{g}(Y, D^1(X,W)) = g(A_W X, Y), \tag{13}$$

$$\overline{g}(D^s(X,N),W) = g(N, A_W X). \tag{14}$$

for $X, Y \in \Gamma(TM)$, $\xi \in \Gamma(Rad(TM))$, $W \in \Gamma(S(TM^{\perp}))$ and $N \in \Gamma(ltr(TM))$.

If the induced connection ∇ and transversal connection ∇^t_X are not metric connections, then for $X, Y, Z \in \Gamma(TM)$ and $U, U' \in \Gamma(tr(TM))$, following formulae represent induced connection and transversal connection respectively

$$(\nabla_X g)(Y,Z) = \overline{g}(h^l(X,Y),Z) + \overline{g}(h^l(X,Z),Y), \tag{15}$$

$$(\nabla^t_X \overline{g})(U,U') = -\{\overline{g}(A_U X, U') + \overline{g}(A_{U'} X, U)\}. \tag{16}$$

Let $\overline{P}$ denote the projection of TM on $S(TM)$ and let ∇^*, ∇^{*t} denote the linear connections on $S(TM)$ and $Rad(TM)$, respectively. Then from the decomposition of tangent bundle of lightlike submanifolds, we have

$$\nabla_X \overline{P}Y = \nabla^*_X \overline{P}Y + h^*(X, \overline{P}Y), \tag{17}$$

$$\nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X(\xi) \tag{18}$$

for $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(RadTM)$, where h^*, A^* are the second fundamental form and shape operator of distributions $S(TM)$ and $Rad(TM)$, respectively. From (14) and (15), we get

$$\overline{g}(h^l(X, \overline{P}Y), \xi) = g(A^*_\xi X, \overline{P}Y), \tag{19}$$

$$\overline{g}(h^*(X, \overline{P}Y), N) = g(A_N X, \overline{P}Y), \tag{20}$$

$$\overline{g}(h^l(X,\xi),\xi) = 0, \;\; A^*_\xi \xi = 0. \tag{21}$$

3. Semi-Slant Lightlike Submanifolds

In this section, before introducing the semi-slant lightlike submanifolds of an indefinite Kenmotsu manifold, we state the following Lemmas for later use:

Lemma 3.1. [8] *Let M be a q-lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$ of index $2q$ with structure vector field V tangent to M. Suppose that $\phi RadTM$ is a distribution on M such that $RadTM \cap \phi RadTM = \{0\}$. Then $\phi ltrTM$ is a subbundle of the screen distribution $S(TM)$ and $\phi RadTM \cap \phi ltrTM = \{0\}$.*

Lemma 3.2. [8] *Let M be a q-lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$ of index $2q$ with structure vector field V tangent to M. Suppose that $\phi RadTM$ is a distribution on M such that $RadTM \cap \phi RadTM = \{0\}$. Then any complementary distribution to $\phi RadTM \oplus \phi ltr(TM)$ in $S(TM)$ is Riemannian.*

Definition 3.3. *Let M be a q-lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$ of index $2q$ such that $2q < dim(M)$ with structure vector field V tangent to M. Then we say that M is a semi-slant lightlike submanifold of $\overline{M}$ if the following conditions are satisfied:*

(i) *$\phi RadTM$ is distribution on M such that $RadTM \cap \phi RadTM = \{0\}$,*

(ii) *there exist non-degenerate orthogonal distributions D_1 and D_2 on M such that*

$$S(TM) = (\phi RadTM \oplus \phi ltr(TM)) \oplus_{orth} D_1 \oplus_{orth} D_2 \oplus_{orth} \{V\},$$

(iii) *the distribution D_1 is an invariant distribution, i.e. $\phi D_1 = D_1$,*

(iv) *the distribution $\overline{D}_2 = D_2 \perp \{V\}$ is slant with angle $\theta(\neq 0)$, i.e. for each $x \in M$ and each non-zero vector $X \in (\overline{D}_2)_x$, if X and V are linearly independent, then the angle θ between ϕX and the vector subspace $(\overline{D}_2)_x$ is a non-zero constant, which is independent of choice of $x \in M$ and $X \in (\overline{D}_2)_x$.*
This constant angle θ is called the slant angle of distribution $\overline{D}_2$. A semi-slant lightlike submanifold is said to be proper if $D_1 \neq \{0\}, \overline{D}_2 \neq \{0\}$ and $\theta \neq \{0\}$.

Example 3.4. *Let $(\bar{M} = R_2^{11}, \bar{g})$ be a semi-Euclidean space of signature (-,+,+,+,+,+,+,+,+,+,+) with respect to the canonical basis*

$$\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial x_7, \partial x_8, \partial x_9, \partial x_{10}, \partial Z\}.$$

Consider a submanifold M of R_2^{11}, defined by $x_1 = x_8 = u_1,\ x_2 = u_2,\ x_3 = sinu_3,\ x_4 = cosu_3,\ x_5 = u_5,\ x_6 = -u_3 sinu_6,\ x_7 = -u_3 cosu_6,\ x_9 = u_7,\ x_{10} = u_8,\ \partial Z = V$. The local frame of TM is given by

$$\begin{aligned}
Z_1 &= e^{-z}(\partial x_1 + \partial x_8)\\
Z_2 &= e^{-z}\partial x_2\\
Z_3 &= e^{-z}(cosu_3\partial x_3 - sinu_3\partial x_4 - sinu_6\partial x_6 - cosu_6\partial x_7)\\
Z_4 &= e^{-z}(-u_3 cosu_6\partial x_6 + u_3 sinu_6\partial x_7)\\
Z_5 &= e^{-z}\partial x_9\\
Z_6 &= e^{-z}\partial x_{10}\\
Z_7 &= e^{-z}\partial x_5\\
Z_8 &= V = \partial Z.
\end{aligned}$$

Hence, $RadTM = span\{Z_1\}$ *and* $\phi RadTM = span\{Z_2 + Z_7\}$.
Next, we have $\bar{D}_2 = D_2 \perp \{V\} = \{Z_3, Z_4\} \perp V$.
Then M *is slant lightlike with slant angle* $\pi/4$. *By direct calculations, we get* $S(TM^{\perp})$=*span*

$$\begin{cases} W_1 = e^{-z}(cosu_3\partial x_3 - sinu_3\partial x_4 - sinu_6\partial x_6 - cosu_6\partial x_7) \\ W_2 = e^{-z}(-u_3cosu_6\partial x_6 + u_3sinu_6\partial x_7) \end{cases}$$

and $ltr(TM)$ *is spanned by* $N = e^{-z}/2(-\partial x_1+\partial x_9)$ *such that* $\phi N = -Z_2+Z_7 \in S(TM)$.
Now, $\phi Z_5 = -Z_6$, *which implies that* $D_1 = \{Z_5, Z_6\}$ *is invariant with respect to* ϕ.
Hence, M *is semi-slant lightlike submanifold of* R^{11}_2.

From above definition, we have the following decomposition:

$$TM = RadTM \oplus_{orth} (\phi RadTM \oplus \phi ltr(TM)) \oplus_{orth} D_1 \oplus_{orth} D_2 \oplus_{orth} \{V\}. \tag{22}$$

For any vector field X tangent to M, we put

$$\phi X = fX + FX \tag{23}$$

where fX and FX are tangential and transversal part of ϕX respectively. we denote the projections on $RadTM, \phi RadTM, \phi ltr(TM), D_1$ and $D_2 \perp \{V\}$ in TM by P_1, P_2, P_3, Q_1 and $\overline{Q}_2$ respectively. Then, for any $X \in \Gamma(TM)$, we get

$$X = P_1X + P_2X + P_3X + Q_1X + \overline{Q}_2X, \tag{24}$$

where $\overline{Q}_2X = Q_2X + \eta(X)V$. Now applying ϕ to (24), we get

$$\phi X = \phi P_1X + \phi P_2X + FP_3X + fQ_1X + fQ_2X + FQ_2X, \tag{25}$$

where $\phi P_1X \in \Gamma(\phi RadTM)$, $\phi P_2X \in \Gamma(RadTM)$, $FP_3X \in \Gamma(ltrTM)$, $fQ_1X \in \Gamma(D_1)$, $fQ_2X \in \Gamma(D_2)$, $FQ_2X \in \Gamma(S(TM^{\perp}))$. Using (3), (25) and (9) -(11) and identifying the components on $RadTM, \phi RadTM, \phi ltr(TM), D_1,\ D_2, ltr(TM), (S(TM^{\perp}))$ and $\{V\}$, we obtain

$$\begin{aligned} &P_1(\nabla_X \phi P_1Y) + P_1(\nabla_X \phi P_2Y) + P_1(\nabla_X fQ_1Y) + P_1(\nabla_X fQ_2Y) \\ &= P_1(A_{FP_3Y}X) + P_1(A_{FQ_2Y}X) + \phi P_2\nabla_X Y + \eta(Y)\phi P_2X, \end{aligned} \tag{26}$$

$$\begin{aligned} &P_2(\nabla_X \phi P_1Y) + P_2(\nabla_X \phi P_2Y) + P_2(\nabla_X fQ_1Y) + P_2(\nabla_X fQ_2Y) \\ &= P_2(A_{FP_3Y}X) + P_2(A_{FQ_2Y}X) + \phi P_1\nabla_X Y + \eta(Y)\phi P_1X, \end{aligned} \tag{27}$$

$$\begin{aligned} &P_3(\nabla_X \phi P_1Y) + P_3(\nabla_X \phi P_2Y) + P_3(\nabla_X fQ_1Y) + P_3(\nabla_X fQ_2Y) \\ &= P_3(A_{FP_3Y}X) + P_3(A_{FQ_2Y}X) + Bh^l(X,Y), \end{aligned} \tag{28}$$

$$\begin{aligned} &Q_1(\nabla_X \phi P_1Y) + Q_1(\nabla_X \phi P_2Y) + Q_1(\nabla_X fQ_1Y) + Q_1(\nabla_X fQ_2Y) \\ &= Q_1(A_{FP_3Y}X) + Q_1(A_{FQ_2Y}X) + fQ_1\nabla_X Y + \eta(Y)fQ_1X, \end{aligned} \tag{29}$$

(30)
$$Q_2(\nabla_X \phi P_1 Y) + Q_2(\nabla_X \phi P_2 Y) + Q_2(\nabla_X fQ_1 Y) + Q_2(\nabla_X fQ_2 Y) = Q_2(A_{FP_3Y}X) + Q_2(A_{FQ_2Y}X) + fQ_2\nabla_X Y + Bh^s(X,Y) + \eta(Y)fQ_2X,$$

(31)
$$h^l(X, \phi P_1 Y) + h^l(X, \phi P_2 Y) + h^l(X, fQ_1 Y) + h^l(X, fQ_2 Y) = FP_3\nabla_X Y - \nabla^l_X(X, FP_3Y) - D^l(X, FQ_2Y) + \eta(Y)FP_3X,$$

(32)
$$h^s(X, \phi P_1 Y) + h^s(X, \phi P_2 Y) + h^s(X, fQ_1 Y) + h^s(X, fQ_2 Y) = FQ_2\nabla_X Y - \nabla^s_X(X, FQ_2Y) - D^s(X, FP_3Y) + Ch^s(X,Y),$$

(33)
$$\eta(\nabla_X \phi P_1 Y) + \eta(\nabla_X \phi P_2 Y) + \eta(\nabla_X fQ_1 Y) + \eta(\nabla_X fQ_2 Y) = \eta(A_{FP_3Y}X) + \eta(A_{FQ_2Y}X) - \overline{g}(\phi X, Y)V.$$

Theorem 3.5. *Let M be a semi-slant lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$ with structure vector field V tangent to M. Then, for any $X, Y \in \Gamma(RadTM)$, $RadTM$ is integrable if and only if*

(i) $P_1(\nabla_X \phi P_1 Y) = P_1(\nabla_Y \phi P_1 X)$, $Q_1(\nabla_X \phi P_1 Y) = Q_1(\nabla_Y \phi P_1 X)$ *and* $Q_2(\nabla_X \phi P_1 Y) = Q_2(\nabla_Y \phi P_1 X)$,

(ii) $h^l(Y, \phi P_1 X) = h^l(X, \phi P_1 Y)$ *and* $h^s(Y, \phi P_1 X) = h^s(X, \phi P_1 Y)$.

Proof. Let M be a semi-slant lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$. Let $X, Y \in \Gamma(RadTM)$. From (26), we have

$$P_1(\nabla_X \phi P_1 Y) = \phi P_2 \nabla_X Y. \tag{34}$$

Interchanging X and Y in (34) and subtracting resulting equation from (34) we obtain

$$P_1(\nabla_X \phi P_1 Y) - P_1(\nabla_Y \phi P_1 X) = \phi P_2[X,Y]. \tag{35}$$

From (29), we have

$$Q_1(\nabla_X \phi P_1 Y) = \phi Q_1 \nabla_X Y. \tag{36}$$

Interchanging X and Y in (36) and subtracting resulting equation from (36), we get

$$Q_1(\nabla_X \phi P_1 Y) - Q_1(\nabla_Y \phi P_1 X) = \phi Q_1[X,Y]. \tag{37}$$

From (30), we obtain

$$Q_2(\nabla_X \phi P_1 Y) = fQ_2 \nabla_X Y + Bh^s(X,Y). \tag{38}$$

Interchanging X and Y in (38) and subtracting resulting equation from (38), we get

$$Q_2(\nabla_X \phi P_1 Y) - Q_2(\nabla_Y \phi P_1 X) = fQ_2[X,Y]. \tag{39}$$

In view of (31), we obtain

$$h^l(X, \phi P_1 Y) = FP_3 \nabla_X Y. \tag{40}$$

Interchanging X and Y in (40) and subtracting resulting equation from (40), we have

$$h^l(X, \phi P_1 Y) - h^l(Y, \phi P_1 X) = FP_3[X, Y]. \tag{41}$$

Similarly, from (32), we get

$$h^s(X, \phi P_1 Y) = Ch^s(X, Y) + FQ_2 \nabla_X Y, \tag{42}$$

which gives

$$h^s(X, \phi P_1 Y) - h^s(Y, \phi P_1 X) = FQ_2[X, Y]. \tag{43}$$

From equations (35), (37), (39), (41) and (43), we conclude that $Rad(TM)$ is integrable. □

Theorem 3.6. *Let M be a semi-slant lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$ with structure vector field V tangent to M. Then, for any $X, Y \in \Gamma(D_1 \oplus \{V\})$, $D_1 \oplus \{V\}$ is integrable if and only if*

(i) $P_1(\nabla_X fQ_1 Y) = P_1(\nabla_Y fQ_1 X)$, $P_2(\nabla_X fQ_1 Y) = P_2(\nabla_Y fQ_1 X)$ *and* $Q_2(\nabla_X fQ_1 Y) = Q_2(\nabla_Y fQ_1 X)$,

(ii) $h^l(Y, fQ_1 X) = h^l(X, fQ_1 Y)$ *and* $h^s(Y, fQ_1 X) = h^s(X, fQ_1 Y)$.

Proof. Let M be a semi-slant lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$. Let $X, Y \in \Gamma(D_1 \oplus \{V\})$. From (26), we have

$$P_1(\nabla_X fQ_1 Y) = \phi P_2 \nabla_X Y. \tag{44}$$

Interchanging X and Y in (44) and subtracting resulting equation from (44), we get

$$P_1(\nabla_X fQ_1 Y) - P_1(\nabla_Y fQ_1 X) = \phi P_2[X, Y]. \tag{45}$$

From (27), we obtain

$$P_2(\nabla_X fQ_1 Y) = \phi P_1 \nabla_X Y. \tag{46}$$

Interchanging X and Y in (46) and subtracting resulting equation from (46), we get

$$P_2(\nabla_X fQ_1 Y) - P_2(\nabla_Y fQ_1 X) = \phi P_1[X, Y]. \tag{47}$$

From (30), we have

$$Q_2(\nabla_X fQ_1 Y) = fQ_2 \nabla_X Y + Bh^s(X, Y). \tag{48}$$

Interchanging X and Y in (48) and subtracting resulting equation from (48), we get

$$Q_2(\nabla_X fQ_1 Y) - Q_2(\nabla_Y fQ_1 X) = fQ_2[X, Y]. \tag{49}$$

In view of (31), we get

$$h^l(X, fQ_1 Y) = FP_3 \nabla_X Y. \tag{50}$$

Interchanging X and Y in (50) and subtracting resulting equation from (50), we obtain

$$h^l(X, fQ_1Y) - h^l(Y, fQ_1X) = FP_3[X, Y]. \tag{51}$$

Similarly, from (32), we get

$$h^s(X, fQ_1Y) = Ch^s(X, Y) + FQ_2\nabla_X Y, \tag{52}$$

which gives

$$h^s(X, fQ_1Y) - h^s(Y, fQ_1X) = FQ_2[X, Y]. \tag{53}$$

From equations (45), (47), (49), (51) and (53), we find that $D_1 \oplus \{V\}$ is integrable. □

Theorem 3.7. *Let M be a semi-slant lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$ with structure vector field V tangent to M. Then, for all $X, Y \in \Gamma(D_2 \oplus \{V\})$, $D_2 \oplus \{V\}$ is integrable if and only if*

(i) $P_1(\nabla_X fQ_2Y - \nabla_Y fQ_2X) = P_1(A_{FQ_2Y}X - A_{FQ_2X}Y)$,
(ii) $P_2(\nabla_X fQ_2Y - \nabla_Y fQ_2X) = P_2(A_{FQ_2Y}X - A_{FQ_2X}Y)$,
(iii) $Q_1(\nabla_X fQ_2Y - \nabla_Y fQ_2X) = Q_1(A_{FQ_2Y}X - A_{FQ_2X}Y)$,
(iv) $h^l(X, fQ_2Y) - h^l(Y, fQ_2X) = D^l(Y, FQ_2X) - D^l(X, FQ_2Y)$.

Proof. Let M be a semi-slant lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$. Let $X, Y \in \Gamma(D_1 \oplus \{V\})$.
From (26), we have

$$P_1(\nabla_X fQ_2Y) - P_1(A_{FQ_2Y}X) = \phi P_2\nabla_X Y. \tag{54}$$

Interchanging X and Y in (54) and subtracting resulting equation from (54), we obtain

$$P_1(\nabla_X fQ_2Y - \nabla_Y fQ_2X) - P_1(A_{FQ_2Y}X - (A_{FQ_2X}Y)) = \phi P_2[X, Y]. \tag{55}$$

From (27), we get

$$P_2(\nabla_X fQ_2Y) - P_2(A_{FQ_2Y}X) = \phi P_1\nabla_X Y. \tag{56}$$

Interchanging X and Y in (56) and subtracting resulting equation from (56), we have

$$P_2(\nabla_X fQ_2Y - \nabla_Y fQ_2X) - P_2(A_{FQ_2Y}X - (A_{FQ_2X}Y)) = \phi P_1[X, Y]. \tag{57}$$

In view of (29), we obtain

$$Q_1(\nabla_X fQ_2Y) - Q_1(A_{FQ_2Y}X) = fQ_1\nabla_X Y. \tag{58}$$

Interchanging X and Y in (58) and subtracting resulting equation from (58), we have

(59) $Q_1(\nabla_X fQ_2Y - (\nabla_Y fQ_2X) - Q_1(A_{FQ_2Y}X - (A_{FQ_2X}Y)) = fQ_1[X,Y].$

Similarly, from (31), we get

(60) $$h^l(X, fQ_2Y) + D^l(X, FQ_2Y) = FP_3\nabla_X Y,$$

which gives

(61)
$$h^l(X, fQ_2Y) - h^l(Y, fQ_2X) + D^l(X, FQ_2Y) - D^l(Y, FQ_2X) = FP_3[X,Y].$$

From equations (55), (57), (59) and (61), we find that $D_2 \oplus \{V\}$ is integrable. □

4. Foliations Determined by Distributions

In this section, we obtain necessary and sufficient conditions for foliations determined by distributions on a semi-slant lightlike submanifold of an indefinite Kenmotsu manifold to be totally geodesic.

Definition 4.1. [13] *A semi-slant lightlike submanifold M of an indefinite Kenmotsu manifold $\overline{M}$ is said to be mixed geodesic if its second fundamental form h satisfies $h(X,Y) = 0$, $\forall X \in \Gamma(D_1)$ and $Y \in \Gamma(D_2)$. Thus M is a mixed geodesic semi-slant lightlike submanifold if $h^l(X,Y) = 0$, $h^s(X,Y) = 0$, $\forall X \in \Gamma(D_1)$ and $Y \in \Gamma(D_2)$.*

Theorem 4.2. *Let M be a semi-slant lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$ with structure vector field V tangent to M. Then, for any $X, Y \in \Gamma(RadTM)$ and $Z \in \Gamma(S(TM))$, $RadTM$ defines a totally geodesic foliation if and only if*

$$\begin{aligned}\overline{g}(\nabla_X \phi P_2 Z + &\nabla_X fQ_1 Z + \nabla_X fQ_2 Z - \eta(Z)\phi P_1 X, \phi Y)\\ &= \overline{g}(A_{FP_3Z}X + A_{FQ_2Z}X, \phi Y).\end{aligned}$$

Proof. Let M be a semi-slant lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$ with structure vector field V tangent to M. To prove that $RadTM$ defines totally geodesic foliation it is sufficient to show that $\nabla_X Y \in \Gamma(RadTM), \forall X, Y \in \Gamma(RadTM)$. Since $\overline{\nabla}$ is a metric connection, using equation (10), we get

(62) $$g(\nabla_X Y, Z) = \overline{g}(\overline{\nabla}_X Y, Z),$$

which implies

$$-\overline{g}(\nabla_X Y, Z) = \overline{g}(Y, \overline{\nabla}_X Z).$$

Using (2) in $\overline{g}(\nabla_X Y, Z)$, we obtain

$$g(\overline{\nabla}_X Z, Y) = g(\phi\overline{\nabla}_X Z, \phi Y) + \eta(\overline{\nabla}_X Z)\eta(Y),$$

Since $\eta(Y) = g(Y, V) = 0$, above equation reduces to

$$g(\nabla_X Y, Z) = -\overline{g}(\phi\overline{\nabla}_X Z, \phi Y). \tag{63}$$

From (3), we have

$$\overline{\nabla}_X \phi Z - \phi\overline{\nabla}_X Z = -\overline{g}(\phi X, Z)V + \eta(Z)\phi X,$$

which implies

$$\overline{\nabla}_X \phi Z + \overline{g}(\phi X, Z)V - \eta(Z)\phi X = \phi\overline{\nabla}_X Z, \tag{64}$$

using (64) in (63) , we get

$$\begin{aligned} -\overline{g}(\phi\overline{\nabla}_X Z, \phi Y) = &- \overline{g}(\overline{\nabla}_X \phi Z \\ &+ \overline{g}(\phi X, Z)V - \eta(Z)\phi X, \phi Y). \end{aligned}$$

Using (25), we obtain

$$\begin{aligned} -\overline{g}(\phi\overline{\nabla}_X Z, \phi Y) = &- \overline{g}(\overline{\nabla}_X(\phi P_2 Z + FP_3 Z + fQ_1 Z + fQ_2 Z + FQ_2 Z) \\ &+ \overline{g}(\phi X, Z)V - \eta(Z)\phi X, \phi Y), \end{aligned}$$

which reduces to

$$\begin{aligned} \overline{g}(\nabla_X Y, Z) = &- \overline{g}(-A_{FP_3 Z}X - A_{FQ_2 Z}X + \nabla_X \phi P_2 Z \\ &+ \nabla_X fQ_1 Z + \nabla_X fQ_2 Z - \eta(Z)\phi P_1 X, \phi Y). \end{aligned}$$

This proves the theorem. □

Theorem 4.3. *Let M be a semi-slant lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$ with structure vector field V tangent to M. Then, for any $X, Y \in \Gamma(D_1 \oplus \{V\})$, $Z \in \Gamma(D_2), W \in \Gamma(\phi ltr(TM))$ and $N \in \Gamma(ltr(TM))$, $D_1 \oplus \{V\}$ defines a totally geodesic foliation if and only if*

(i) $\overline{g}(A_{FQ_2 Z}X, \phi Y) = \overline{g}(\nabla_X fQ_2 Z, \phi Y)$,
(ii) $A_{FP_3 Y}X$ *and* $\nabla_X \phi N$ *have no component in* $D_1 \oplus \{V\}$.

Proof. Let M be a semi-slant lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$ with structure vector field V tangent to M.
To prove that $D_1 \oplus \{V\}$ defines a totally geodesic foliation, it is sufficient to show that $\nabla_X Y \in \Gamma(D_1 \oplus \{V\}), \forall X, Y \in \Gamma(D_1 \oplus \{V\})$.
Since $\overline{\nabla}$ is a metric connection, using equation (10), we get

$$g(\nabla_X Y, Z) = \overline{g}(\overline{\nabla}_X Y, Z), \tag{65}$$

which implies

$$-\overline{g}(\nabla_X Y, Z) = \overline{g}(Y, \overline{\nabla}_X Z).$$

Using (2), we obtain

$$\overline{g}(\overline{\nabla}_X Z, Y) = \overline{g}(\phi\overline{\nabla}_X Z, \phi Y) + \eta(\overline{\nabla}_X Z)\eta(Y). \tag{66}$$

Using (3) in (66), we get

$$\overline{g}(\overline{\nabla}_X Z, Y) = \overline{g}(\overline{\nabla}_X \phi Z, \phi Y) + \eta(\overline{\nabla}_X Z)\eta(Y),$$

$$= \overline{g}(\overline{\nabla}_X fQ_2 Z + \overline{\nabla}_X FQ_2 Z, \phi Y) + \eta(\overline{\nabla}_X Z)\eta(Y)$$

which reduces to

$$-g(\nabla_X Y, Z) = \overline{g}(\nabla_X fQ_2 Z - A_{FQ_2 Z} X, \phi Y) + \eta(\nabla_X Z)\eta(Y). \tag{67}$$

Now, for any $X, Y \in D_1 \oplus \{V\}$ and $N \in \Gamma(ltr(TM))$, we obtain

$$\overline{g}(\overline{\nabla}_X Y, N) = -g(\nabla_X N, Y).$$

From (2), we obtain

$$\overline{g}(\overline{\nabla}_X N, Y) = \overline{g}(\phi \overline{\nabla}_X N, \phi Y) + \eta(\overline{\nabla}_X N)\eta(Y). \tag{68}$$

Using (3) in (68), we get

$$-g(\overline{\nabla}_X Y, N) = \overline{g}(\nabla_X \phi N, \phi Y) + \eta(\overline{\nabla}_X N)\eta(Y). \tag{69}$$

Now, for any $X, Y \in D_1 \oplus \{V\}$ and $W \in \Gamma(\phi ltr(TM))$, we get

$$\overline{g}(\overline{\nabla}_X Y, W) = -g(\nabla_X W, Y).$$

From (2), we obtain

$$\overline{g}(\overline{\nabla}_X W, Y) = \overline{g}(\phi \overline{\nabla}_X W, \phi Y) + \eta(\overline{\nabla}_X W)\eta(Y). \tag{70}$$

Using (3) in (70), we have

$$g(\overline{\nabla}_X W, Y) = \overline{g}(\nabla_X \phi W, \phi Y) + \eta(\overline{\nabla}_X W)\eta(Y), \tag{71}$$

$$= \overline{g}(\nabla_X FP_3 Y, \phi Y) + \eta(-A_W X)\eta(Y)$$

which reduces to

$$-g(\nabla_X Y, W) = \overline{g}(-A_{FP_3 Y} X, \phi Y) - \eta(A_W X)\eta(Y). \tag{72}$$

Thus, we obtain the required results.

□

Theorem 4.4. *Let M be a semi-slant lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$ with structure vector field V tangent to M. Then, for any $X, Y \in \Gamma(D_2 \oplus \{V\})$, $Z \in \Gamma(D_1), W \in \Gamma(\phi ltr(TM))$ and $N \in \Gamma(ltr(TM))$, $D_2 \oplus \{V\}$ defines a totally geodesic foliation if and only if*

- (i) $\overline{g}(\nabla_X fQ_1 Z, fY) = -\overline{g}(h^s(X, fQ_1 Z), FY)$ *and* $\nabla_X Z$ *has no component in* $\{V\}$,
- (ii) $\overline{g}(\nabla_X \phi N, fY) = -\overline{g}(h^s(X, \phi N), FY)$ *and* $\nabla_X N$ *has no component in* $\{V\}$,
- (iii) $\overline{g}(A_{FP_3 W} X, fY) = \overline{g}(D^s(X, FP_3 W), FY)$ *and* $\nabla_X W$ *has no component in* $\{V\}$.

Proof. Let M be a semi-slant lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$ with structure vector field V tangent to M. To prove that $D_2 \oplus \{V\}$ defines a totally geodesic foliation, it is sufficient to show that $\nabla_X Y \in \Gamma(D_2 \oplus \{V\}), \forall X, Y \in \Gamma(D_2 \oplus \{V\})$. Since $\overline{\nabla}$ is a metric connection, using equation (10), we get

$$g(\nabla_X Y, Z) = \overline{g}(\overline{\nabla}_X Y, Z) \tag{73}$$

which implies

$$-\overline{g}(\nabla_X Y, Z) = \overline{g}(Y, \overline{\nabla}_X Z).$$

Using (2), we obtain

$$\overline{g}(\overline{\nabla}_X Z, Y) = \overline{g}(\phi\overline{\nabla}_X Z, \phi Y) + \eta(\overline{\nabla}_X Z)\eta(Y) \tag{74}$$

using (3) in (74), we have

$$\begin{aligned}\overline{g}(\overline{\nabla}_X Z, Y) =& \overline{g}(\overline{\nabla}_X fQ_1 Z, \phi Y) + \eta(\overline{\nabla}_X Z)\eta(Y)\\ =& \overline{g}(\overline{\nabla}_X fQ_1 Z, fY + FY) + \eta(\overline{\nabla}_X Z)\eta(Y)\end{aligned}$$

which reduces to

$$-\overline{g}(\nabla_X Y, Z) = \overline{g}(\nabla_X fQ_1 Z, fY) + \overline{g}(h^s(X, fQ_1 Z), FY) + \eta(\overline{\nabla}_X Z)\eta(Y). \tag{75}$$

Now, for any $X, Y \in D_2 \oplus \{V\}$ and $N \in \Gamma(ltr(TM))$, we obtain

$$\overline{g}(\overline{\nabla}_X Y, N) = -g(\nabla_X N, Y).$$

Using (2), we obtain

$$\overline{g}(\overline{\nabla}_X N, Y) = \overline{g}(\phi\overline{\nabla}_X N, \phi Y) + \eta(\overline{\nabla}_X N)\eta(Y). \tag{76}$$

Inserting (3) in (76), we have

$$\begin{aligned}\overline{g}(\overline{\nabla}_X N, Y) =& \overline{g}(\nabla_X \phi N, fY + FY) + \eta(\overline{\nabla}_X N)\eta(Y)\\ =& \overline{g}(\nabla_X \phi N + h^s(X, \phi N), fY + FY) + \eta(\overline{\nabla}_X N)\eta(Y)\\ g(\overline{\nabla}_X Y, N) =& \overline{g}(\nabla_X \phi N, fY) + \overline{g}(h^s(X, \phi N), FY) + \eta(\overline{\nabla}_X N)\eta(Y). \qquad (77)\end{aligned}$$

From (2), we get

$$\overline{g}(\overline{\nabla}_X W, Y) = \overline{g}(\phi\overline{\nabla}_X W, \phi Y) + \eta(\overline{\nabla}_X W)\eta(Y). \tag{78}$$

Using (3) in (78), we have

$$\overline{g}(\overline{\nabla}_X W, Y) = \overline{g}(\overline{\nabla}_X FP_3 W, fY + FY) + \eta(\overline{\nabla}_X W)\eta(Y)$$

which implies

(79)

$$-g(\nabla_X Y, W) = \overline{g}(-A_{FP_3 W} X, fY) + \overline{g}(D^s(X, FP_3 W), FY) + \eta(\nabla_X W)\eta(Y).$$

This completes the proof. □

Theorem 4.5. *Let M be a mixed geodesic semi-slant lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$ with structure vector field V tangent to M. Then, for any $X, Y \in \Gamma(D_2 \oplus \{V\})$, $Z \in \Gamma(D_1), W \in \Gamma(\phi ltr(TM))$ and $N \in \Gamma(ltr(TM))$, $D_2 \oplus \{V\}$ defines a totally geodesic foliation if and only if*

(i) $\nabla_X fQ_1 Z$ *has no component in* $D_2 \oplus \{V\}$,

(ii) $\overline{g}(\nabla_X \phi N, fY) = -\overline{g}(h^s(X, \phi N), FY)$ *and* $\nabla_X N$ *has no component in* $\{V\}$,

(iii) $\overline{g}(A_{FP_3W}X, fY) = \overline{g}(D^s(X, FP_3W), FY)$ *and* $\nabla_X W$ *has no component in* $\{V\}$.

Proof. Let M be a mixed geodesic semi-slant lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$ with structure vector field V tangent to M. To prove that $D_2 \oplus \{V\}$ defines a totally geodesic foliation, it is sufficient to show that $\nabla_X Y \in \Gamma(D_2), \forall X, Y \in \Gamma(D_2) \oplus \{V\}$. Since M is a mixed geodesic semi-slant lightlike submanifold of an indefinite Kenmotsu manifold, $h(X, Y) = 0\ \forall X \in \Gamma(D_1), Y \in \Gamma(D_2)$, we get

$$h^l(X, Y) = 0, \ h^s(X, Y) = 0.$$

Putting $h^s(X, Y) = 0$, in (75), we get

$$-\overline{g}(\nabla_X Y, Z) = \overline{g}(\nabla_X fQ_1 Z, fY) + \eta(\overline{\nabla}_X Z)\eta(Y).$$

This gives $\nabla_X fQ_1 Z$ has no component in D_2 and $\nabla_X Z$ has no component in $\{V\}$.

(ii) $\overline{g}(\nabla_X \phi N, fY) = -\overline{g}(h^s(X, \phi N), FY)$ and $\nabla_X N$ has no component in $\{V\}$,

(iii) $\overline{g}(A_{FP_3W}X, fY) = \overline{g}(D^s(X, FP_3W), FY)$ and $\nabla_X W$ has no component in $\{V\}$,

are same as $(ii), (iii)$ part of Theorem 4.4. □

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