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CONNECTEDNESS IN IDEAL PROXIMITY SPACES

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Abstract. Two new concepts, namely, δ^*-connectedness and δ^*-component are introduced by using ideal in proximity spaces. A relation of δ^*-connectedness with different types of connectedness that are considered in literature before is studied. It is shown that δ^*-connectedness is a contractive property.

1. Introduction

Kuratowski [6] and Vaidyanathaswamy [11] introduced the concept of ideal topological spaces. Subsequently, Ekici *et al.* [2, 3] defined the notion of connectedness in ideal topological spaces. Recently, various types of connectedness in ideal topological spaces are further investigated by Modak *et al.* [7]. Also, Hosny *et al.* [4] studied the notion of generalized proximity using ideal and proximity.

The aim of this paper is to introduce the notion of δ^*-connectedness by using ideal in proximity spaces that is analogous to the notion of $*_s$-connectedness in ideal topological spaces [2]. Also, we study δ^*-component and the relation of it with δ-component and $*$-component. In Section 2, we recall some basic definitions and results which will be used in further sections. We define δ^*-connectedness and examine the relationship between δ^*-connectedness and different types of connectedness that are already in literature in Section 3. In the last section, we discuss the characterizations of δ^*-connectedness and, examples are given for those characterizations that do not hold under this connectedness.

Throughout this paper, by a proximity space (X, δ) (or X) [10], we mean a nonempty set X with an Efremovič proximity δ. Also, an ideal proximity space $(X, \delta, \mathcal{I})$ will denote a proximity space (X, δ) with an ideal $\mathcal{I}$ in X. Further, a δ-closed (or δ^*-closed) set in an ideal proximity space is a closed set with respect to the topology $\mathcal{T}_\delta$ generated by δ (or closed set with respect to $\mathcal{T}_{\delta^*}$).

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2. Preliminaries

Definition 2.1. [5] *Let $\mathcal{I}$ be an ideal in X and Y be a subset of X. Then the collection $\mathcal{I}_Y = \{A \cap Y : A \in \mathcal{I}\}$ is an ideal in Y.*

Definition 2.2. [4] *Let $(X, \delta, \mathcal{I})$ be an ideal proximity space. Then a subset A is said to be locally in $\mathcal{I}$ at $x \in X$ if there is a δ-neighbourhood U of x such that $(U \cap A) \in \mathcal{I}$. Thus, the local function A^* of A with respect to δ and $\mathcal{I}$ is defined as:*

$$A^* = \bigcup\{x \in X : (U \cap A) \notin \mathcal{I} \text{ for every } \delta\text{-neighbourhood } U \text{ of } x\}.$$

Theorem 2.3. [4] *Let $(X, \delta, \mathcal{I})$ be an ideal proximity space and A, B be the subsets of X. Then*

- (i) *The operator $\mathcal{C}$ defined by $\mathcal{C}(A) = A \cup A^*$ is Kuratowski closure operator.*
- (ii) *The relation δ^* defined by $(A, B) \in \delta^*$ if and only if $\mathcal{C}(A) \cap \mathcal{C}(B) \neq \phi$ is a basic proximity on X. Moreover, δ^* is finer than δ.*
- (iii) *$\delta^*_A < (\delta_A)^*$, the equality holds if A is δ-closed.*
- (iv) *$\mathcal{T}_{\delta^*} \subset (\mathcal{T}_\delta)^*$.*
- (v) *$Cl_{\delta^*}(A) \subset Cl_\delta(A)$ and every δ-closed is δ^*-closed.*

Theorem 2.4. [4] *Let (X, δ) be a proximity space and $\mathcal{I}$, $\mathcal{J}$ be the ideals in X. For $A \subset X$, the following statements hold:*

- (i) *If $\mathcal{I} \subset \mathcal{J}$, then $A^*(\mathcal{J}) \subset A^*(\mathcal{I})$.*
- (ii) *$A^*(\mathcal{I} \cap \mathcal{J}) = A^*(\mathcal{I}) \cup A^*(\mathcal{J})$.*

Definition 2.5. [9, 10] *Let (X, δ) and $(Y, \delta^{'})$ be two proximity spaces, a function $f : (X, \delta) \longrightarrow (Y, \delta^{'})$ is δ-continuous if $(f(A), f(B)) \in \delta^{'}$ whenever $(A, B) \in \delta$ for all $A, B \subset X$.*

Theorem 2.6. [8] *Let (X, δ) be a proximity space. Then the following statements are equivalent:*

- (i) *X is δ-connected.*
- (ii) *$(A, X\backslash A) \in \delta$ for each nonempty subset A with $A \neq X$.*
- (iii) *Every δ-continuous function from X to a discrete space is constant.*
- (iv) *If $X = A \cup B$ and $(A, B) \notin \delta$, then either $A = \phi$ or $B = \phi$.*

Definition 2.7. [1] *Let (X, δ) be a proximity space. Then δ-component of x in X is the union of all δ-connected subsets of X containing x.*

Definition 2.8. [1] *Let $\mathcal{C}$ be a cover of proximity space X. Then $\mathcal{C}$ is called proximity cover of X, if for any two near sets P, Q there is some $U \in \mathcal{C}$ such that $P \cap U \neq \phi$ and $Q \cap U \neq \phi$.*

Definition 2.9. [2] *Let $(X, \mathcal{T}, \mathcal{I})$ be an ideal space. A subset M is said to be $*_s$-connected if it cannot be written as $M = P \cup Q$ with $Cl^*(P) \cap Q = \phi$ and $P \cap Cl(Q) = \phi$.*

Definition 2.10. [2] *Let $(X, \mathcal{T}, \mathcal{I})$ be an ideal space and $x \in X$. Then $*$-component of x in X is the union of all $*_s$-connected subsets containing x.*

3. δ^*-Connectedness

In this section, the notion of δ^*-connectedness is defined and the relationship between different connectednesses is studied.

Definition 3.1. *Let $(X, \delta, \mathcal{I})$ be an ideal proximity space. Then a pair P, Q of nonempty subsets of X is said to be δ^*-separation for X if $X = P \cup Q$ with $(Cl_\delta(P), Q) \notin \delta^*$.*

Definition 3.2. *Let $(X, \delta, \mathcal{I})$ be an ideal proximity space. Then X is called δ^*-connected if it has no δ^*-separation. Otherwise, X is said to be δ^*-disconnected.*

Theorem 3.3. *For an ideal proximity space $(X, \delta, \mathcal{I})$, the following statements are equivalent:*

(i) *X is δ^*-connected.*
(ii) *$(Cl_\delta(P), X \backslash P) \in \delta^*$ for every nonempty subset $P \subsetneq X$.*
(iii) *If $X = P \cup Q$ with $(Cl_\delta(P), Q) \notin \delta^*$, then either $P = \phi$ or $Q = \phi$.*

Proof. (i) $\Rightarrow$ (ii). Let $(Cl_\delta(P), X \backslash P) \notin \delta^*$ for some nonempty subset P of X, then the pair P and $X \backslash P$ forms δ^*-separation for X.

(ii) $\Rightarrow$ (iii). Suppose there are two nonempty subsets P and Q such that $X = P \cup Q$ with $(Cl_\delta(P), Q) \notin \delta^*$. Thus, $(Cl_\delta(P), X \backslash P) = (Cl_\delta(P), Q) \notin \delta^*$, a contradiction.

(iii) $\Rightarrow$ (i). If X is not δ^*-connected, then there exists a pair P, Q of nonempty subsets such that $X = P \cup Q$ with $(Cl_\delta(P), Q) \notin \delta^*$. □

Definition 3.4. *Let $(X, \delta, \mathcal{I})$ be an ideal proximity space. Then a subset Y of X is said to be:*

(i) *δ^*-connected if $Y = P \cup Q$ with $(Cl_\delta(P), Q) \notin \delta^*$, then either $P = \phi$ or $Q = \phi$.*
(ii) *δ^*_Y-connected if $Y = P \cup Q$ with $(Cl_{\delta_Y}(P), Q) \notin \delta^*_Y$, then either $P = \phi$ or $Q = \phi$.*
(iii) *$(\delta_Y)^*$-connected if $Y = P \cup Q$ with $(Cl_{\delta_Y}(P), Q) \notin (\delta_Y)^*$, then either $P = \phi$ or $Q = \phi$.*

We observe that every $(\delta_Y)^*$-connected proximity subspace is δ^*_Y-connected as $(\delta_Y)^* > \delta^*_Y$. However, both connectedness are same if Y is δ-closed. Also, every δ^*_Y-connected subspace is δ^*-connected.

A pair of nonempty subsets P, Q is said to be $*_s$-separation [2] for a subset Y of ideal space $(X, \mathcal{T}, \mathcal{I})$ if $Y = P \cup Q$ with $Cl^*(P) \cap Q = P \cap Cl(Q) = \phi$.

Proposition 3.5. *Every $*_s$-connected subset of ideal proximity space is δ^*-connected.*

Proof. Suppose the pair P, Q be a δ^*-separation for subspace Y of ideal proximity space X. Then, $Cl_\delta(P) \cap Q = \phi$ and $P \cap Cl_{\delta^*}(Q) = \phi$. Since

$\mathcal{T}_{\delta^*} \subset (\mathcal{T}_\delta)^*$, therefore, $Cl_{\mathcal{T}_\delta}(P) \cap Q = \phi = P \cap Cl_{(\mathcal{T}_\delta)^*}(Q)$. Thus, the pair P, Q forms a $*_s$-separation for Y. $\square$

However, the converse of Proposition 3.5 may not be true.

Example 3.6. *Let $\mathbb{Q}$ be the space of rational numbers with usual proximity and $\mathcal{I}_f$ be an ideal consisting of all the finite subsets of $\mathbb{Q}$. Then $\mathbb{Q}$ is δ^*-connected but not $*_s$-connected.*

Every δ^*-connected space is δ-connected. Let $\mathcal{I}$ be an ideal consisting of empty set, then δ^*-connectedness and δ-connectedness coincides. Thus, δ^*-connectedness naturally generalizes the δ-connectedness.

Following example shows that there may exist an ideal other than the empty set for which δ-connectedness and δ^*-connectedness are same.

Example 3.7. *Let X be any compact T_1-proximity space and $\mathcal{I}_f$ be the ideal consisting of all finite subsets of X. Then δ-connectedness and δ^*-connectedness are same. A similar result holds for X if the ideal $\mathcal{I}_{cd}$ consists all closed discrete subsets of X.*

An example of δ-connected proximity space which is not δ^*-connected.

Example 3.8. (i). *Let $X = [0, 1] \cup (\mathbb{Q} \backslash (\mathbb{Q} \cap [0, 1]))$ with usual subspace proximity induced from $\mathbb{R}$. Let I_c be an ideal consisting of all countable subsets of X. Then X is δ-connected but not δ^*-connected. To verify the latter, take $A = [0, 1]$ then $Cl_\delta(A) = A$. Therefore, $(Cl_\delta(A), X \backslash A) \in \delta^*$ if and only if $\mathcal{C}(Cl_\delta(A)) \cap \mathcal{C}(X \backslash A) \neq \phi$, that is, $\mathcal{C}(A) \cap \mathcal{C}(X \backslash A) \neq \phi$. Since $\mathcal{C}(A) = A$ and $\mathcal{C}(X \backslash A) = X \backslash A$, therefore $(Cl_\delta(A), X \backslash A) \notin \delta^*$.*

(ii). *Let $\mathbb{R}$ be the Real line with usual proximity δ and $\mathcal{I}$ be the ideal consisting of all subsets of $\mathbb{R}$. Then the proximity δ^* (generated by $\mathcal{I}$) is discrete proximity. Therefore, $\mathbb{R}$ is δ-connected but not δ^*-connected.*

Following diagram shows the relationship between connectednesses in an ideal proximity space.

$$\begin{array}{ccc} *_s\text{-connected} & \longrightarrow & \text{connected} \\ \downarrow & & \downarrow \\ \delta^*\text{-connected} & \longrightarrow & \delta\text{-connected} \end{array}$$

Proposition 3.9. *Let $\mathcal{I}$ and $\mathcal{J}$ be two ideals in proximity space X such that $\mathcal{I} \subset \mathcal{J}$. Then X is $\delta^*(\mathcal{I})$-connected if it is $\delta^*(\mathcal{J})$-connected.*

Proof. Suppose X is $\delta^*(\mathcal{J})$-connected. Then, $(Cl_\delta(P), X \backslash P) \in \delta^*(\mathcal{J})$ for all nonempty proper subset P of X. Since $\mathcal{I} \subset \mathcal{J}$, so by Theorem 2.4, $P^*(\mathcal{J}) \subset P^*(\mathcal{I})$. Therefore, $\mathcal{C}_{(\delta,\mathcal{J})}(P) \subset \mathcal{C}_{(\delta,\mathcal{I})}(P)$. Thus, $(Cl_\delta(P), X \backslash P) \in \delta^*(\mathcal{I})$ for all nonempty proper subset P of X. Therefore, X is $\delta^*(\mathcal{I})$-connected. $\square$

Proposition 3.9 shows that δ^*-connectedness is a contractive property.

4. Characterizations of δ^*-Connectedness

Lemma 4.1. *Let $(Y, \delta_Y, \mathcal{I}_Y)$ be a δ^*-connected subspace of $(X, \delta, \mathcal{I})$. If P and Q are subsets of X such that $Y \subset P \cup Q$ with $(Cl_\delta(P), Q) \notin \delta^*$, then either $Y \subset P$ or $Y \subset Q$.*

Proof. $Y = (P \cap Y) \cup (Q \cap Y)$ with $(Cl_\delta(P \cap Y), (Q \cap Y)) \notin \delta^*$ as $Cl_\delta(P \cap Y) \subset Cl_\delta(P)$. Therefore, either $P \cap Y = \phi$ or $Q \cap Y = \phi$. □

Theorem 4.2. *Let $\{(Y_i, \delta_{Y_i}, \mathcal{I}_{Y_i}) : i \in J\}$ be a collection of δ^*-connected subspaces of $(X, \delta, \mathcal{I})$. Suppose there is some i_0 such that $(Y_{i_0}, Y_i) \in \delta^*$ for every $i \in J$. Then $Y = \bigcup_{i \in J} Y_i$ is δ^*-connected.*

Proof. Suppose Y is not δ^*-connected. Then there exists a pair P, Q of nonempty subsets such that $Y = P \cup Q$ with $(Cl_\delta(P), Q) \notin \delta^*$. By Lemma 4.1, either $Y_{i_0} \subset P$ or $Y_{i_0} \subset Q$. If $Y_{i_0} \subset P$, then $Y_i \subset P$ for all $i \in J$ because if $Y_i \subset Q$ for some $i \in J$, then $(Y_{i_0}, Y_i) \notin \delta^*$, a contradiction. Similarly, if $Y_{i_0} \subset Q$, then $Y_i \subset Q$ for all $i \in J$. □

Corollary 4.3. *Let $\{(Y_i, \delta_{Y_i}, \mathcal{I}_{Y_i}) : i \in J\}$ be a collection of δ^*-connected subspaces of $(X, \delta, \mathcal{I})$. If $Y_i \cap Y_j \neq \phi$ for all $i, j \in J$, then $Y = \bigcup_{i \in J} Y_i$ is δ^*-connected.*

Lemma 4.4. *Let $(Y, \delta_Y, \mathcal{I}_Y)$ be a δ^*-connected subspace of $(X, \delta, \mathcal{I})$. Then every subspace W such that $Y \subset W \subset Cl_{\delta^*}(Y)$ is δ^*-connected.*

Proof. Consider a collection $\{Y \cup \{p\} : p \in W\}$ of δ^*-connected subspaces of X. By Corollary 4.3, W is δ^*-connected. □

Lemma 4.5. *Let $(X, \delta, \mathcal{I})$ be an ideal proximity space. Suppose for every pair of points $x, y \in X$, there is δ^*-connected subspace which joins them. Then X is δ^*-connected.*

Proof. Fix some $x_0 \in X$. Assume that Y_x be the δ^*-connected subspace joining x to x_0. Then by Corollary 4.3, X being union of $\{Y_x : x \in X\}$ is δ^*-connected. □

Corollary 4.6. *Let $\mathcal{I}$ be an ideal in proximity space X and Y be another proximity space such that $f : (X, \delta_X) \longrightarrow (Y, \delta_Y)$ is δ-continuous surjective map. Then Y is δ-connected if X is δ^*-connected.*

Proof. Since every δ^*-connected proximity space is δ-connected and δ-connectedness is preserved under δ-continuous map, therefore Y is δ-connected. □

Definition 4.7. *A finite family $\{A_i : 1 \leq i \leq n\}$ of subsets of an ideal proximity space X is called δ^*-chain if $(A_i, A_{i+1}) \in \delta^*$ for each $i = 1, 2, 3, \cdots, n-1$. An arbitrary family $\mathcal{F}$ of subsets of X is said to be δ^*-chained if for every pair A, B of elements of$\mathcal{F}$, there is a δ^*-chain in $\mathcal{F}$ which joins A and B.*

Theorem 4.8. *Let $(X, \delta, \mathcal{I})$ be an ideal proximity space. If each member of the δ^*-chained family $\mathcal{F} = \{A_i : i \in J\}$ is δ^*-connected, then $A = \bigcup_{i \in J} A_i$ is also δ^*-connected.*

Proof. Using Theorem 4.2, the result is true for $J = \{1, 2\}$. By induction the result can be proved for any finite set $J = \{1, 2, \cdots, n\}$.

For an arbitrary ordered set J, Let $x, y \in A$. Then $x \in A_i$ and $y \in A_j$ for some $i, j \in J$. Therefore there is a δ^*-chain $\mathcal{C}$ in $\mathcal{F}$ joining A_i and A_j as $\mathcal{F}$ is δ^*-chained family. Since each member of $\mathcal{C}$ is δ^*-connected, therefore by induction hypothesis $\bigcup_{k \in \mathcal{C}} A_k$ is δ^*-connected. Hence, by Lemma 4.5, A is δ^*-connected. □

Theorem 4.9. *Let $(X, \delta, \mathcal{I})$ be a δ^*-connected ideal proximity space. Then every proximity cover of X is a δ^*-chained family.*

Proof. Let $\mathcal{F} = \{U_i : i \in J\}$ be a proximity cover of X. Suppose there exist U_i and U_j for some $i, j \in J$ such that U_i and U_j cannot be joined by any δ^*-chain.
Now put $P = \bigcup\{U_k \in \mathcal{F} \ : \ U_k$ can be joined with U_i by some δ^*-chain$\}$ and Q as the union of all other elements of $\mathcal{F}$. Then, $X = P \cup Q$. It is to show that $(Cl_\delta(P), Q) \notin \delta^*$. Let $(Cl_\delta(P), Q) \in \delta^*$ which implies $(P, Q) \in \delta$. By the definition of proximity cover there is some $U \in \mathcal{F}$ such that $U \cap P \neq \phi$ and $U \cap Q \neq \phi$. Therefore, there are $U_p \subset P$ and $U_q \subset Q$ such that $U \cap U_p \neq \phi$ and $U \cap U_q \neq \phi$. Hence, U_q can be joined with U_i by some δ^*-chain. Thus, U_j can be joined with U_i by some δ^*-chain, a contradiction. Hence, $(Cl_\delta(P), Q) \notin \delta^*$, which is a contradiction. □

Definition 4.10. *Let $(X, \delta, \mathcal{I})$ be an ideal proximity space and $x \in X$. Then the δ^*-component of x is the union of all δ^*-connected subsets of X which contain x and it is denoted by $C_{\delta^*}(x)$.*

By Corollary 4.3, for each x in ideal proximity space X, the δ^*-component $C_{\delta^*}(x)$ is δ^*-connected. Note that the δ^*-components of any two distinct points of X are either same or δ^*-far sets in X. The δ^*-components of an ideal proximity space not necessarily coincide with the $*$-components with respect to topology $\mathcal{T}_\delta$. From Example 3.6, $\mathbb{Q}$ is δ^*-connected, therefore the δ^*-component of any $x \in \mathbb{Q}$ is $\mathbb{Q}$ itself. But $*$-component of any $x \in \mathbb{Q}$ is $\{x\}$ itself because every $*$-component is contained in a component.

Also note that every δ^*-component is contained in some δ-component and every $*$-component is contained in some δ^*-component.

Corollary 4.11. *For an ideal proximity space $(X, \delta, \mathcal{I})$, Every δ^*-component of X is $*$-closed with respect to $\mathcal{T}_\delta$. (In fact, δ^*-closed)*

Proof. Let C be δ^*-component of X. Since $Cl^*(C) \subset Cl_{\delta^*}(C)$, therefore by Lemma 4.4, $Cl^*(C)$ is δ^*-connected. Thus, by maximality of δ^*-component $Cl^*(C) \subset C$, that is, C is $*$-closed in the topology $\mathcal{T}_\delta$. □

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