Honam Mathematical J. **43** (2021), No. 1, pp. 78–87
https://doi.org/10.5831/HMJ.2021.43.1.78

A NEW CONSTRUCTION OF BIENERGY AND BIANGLE IN LORENTZ 5-SPACE

Talat Korpinar and Yasin Unluturk*

Abstract. In this study, we firstly compute the energies and the angles of Frenet vector fields in Lorentz 5-space $\mathbb{L}^5$. Then we obtain the bienergies and biangels of Frenet vector fields in $\mathbb{L}^5$ by using the values of energies and angles. Finally, we present the relations among energies, angles, bienergies, and biangles with graphics.

1. Introduction

Lorentz space-time helps to fill the gap between one of the major mathematical and physical topic known as general relativity and differential geometry by introducingLorentzian geometry. Expressing relativity theory in terms of this state of art geometry enables us penetrating into not only some of the well-known and newly developed physical concepts such as gravitational dilation of time, length contraction, cosmology, black holes, string theory etc., but also some traditional geometrical topics [15].

On the other hand on account of Lorentz space, we may employ methods of hyperbolic geometry to generate the calculations easier. In particular, full possibilities of field equations of Einstein by negative cosmological constant may be resolved in Anti de Sitter space. It is both the hyperquadric of semi Euclidean space with index 2 and a maximally symmetric. Therefore it is significant to research in Anti de Sitter space since it has a important position on general relativity which try to cope with the theory of energy and subject agreement [2]. Some studies of curves have been seen in higher dimensional Lorentzian spaces such as Lorentz 5-space, Lorentz 6-space, and Lorentzian $n-$space [5, 6, 16].

The study of computing an energy of given vector field depending on the structure of the geometrical spaces has become more popular research area recently [12, 13, 14, 17]. For instance, the energy of distrubutions and corrected energy of distrubutions have been investigated on Riemannian manifolds [3, 4].

Received September 29, 2020. Revised December 4, 2020. Accepted December 18, 2020.
2010 Mathematics Subject Classification. 53C41, 53A10.
Key words and phrases. Lorentz $5-$space, Energy, Angle, Bienergy, Biangle, Vector Fields.
*Corresponding author. Yasin Unluturk

Energy of Frenet vector fields for a given nonlightlike curves has been computed by Altin [1]. Körpınar [8], discussed timelike biharmonic particle's energy in Heisenberg spacetime. Some characterizations of energy have been studied by some authors [9, 10, 11].

The organization of the manuscript is prepared as follows: in Section 2, we state fundamental definitions for Frenet vectors in Lorentz 5-space firstly. In section 3, we recall interpretation of geometrical meaning of the energy for vector fields. In Section 4, in the light of given information we compute the energy and angle of Frenet vectors defined on the Lorentz 5-space and show their close relations with the bienergies and biangles. In Section 5, we present the relations among energies, angles, bienergies, and biangles with graphics.

2. Metric and Frenet frame field in $\mathbb{L}^5$

Let $\mathbf{y} = (y_1, y_2, y_3, y_4, y_5)$, $\mathbf{z} = (z_1, z_2, z_3, z_4, z_5)$ be two vectors in a five dimensional real vector space

$$\mathbb{R}^5 = \left\{ \mathbf{y} = (y_1, y_2, y_3, y_4, y_5) : y_i \in \mathbb{R}, \left(i = \overline{1,5} \right) \right\},$$

then, pseudo scalar multiplication of $\mathbf{y}$ and $\mathbf{z}$ is defined as

$$\langle \mathbf{y}, \mathbf{z} \rangle = -y_1 z_1 + \textstyle\sum_{i=2}^{5} y_i z_i.$$

In this case it is said that real vector space $\mathbb{R}^5$ with the above metric defines a new geometrical structure called as Lorentzian $5-$space with index 1 and it is denoted by $\mathbb{L}^5$ [16].

For any vector $\mathbf{y}$ in $\mathbb{L}^5$ has different characteristics with respect to value of given metric. That is,

$$\begin{cases} \mathbf{y} \text{ is spacelike if } \langle \mathbf{y}, \mathbf{y} \rangle \text{ is positive or } \mathbf{y} = 0, \\ \mathbf{y} \text{ is timelike } \langle \mathbf{y}, \mathbf{y} \rangle \text{ is negative}, \\ \mathbf{y} \text{ is lightlike } \langle \mathbf{y}, \mathbf{y} \rangle \text{ is zero}. \end{cases}$$

A new type of characterization in spacelike and timelike vectors can also be described in case of necessary conditions hold. These definitions can be extended to curves in $\mathbb{L}^5$ naturally [7, 16].

Definition 2.1. *Let α be a unit speed non-null curve in $\mathbb{L}^5$. The curve is said to be the Frenet curve of osculating order 5 if its 5th order derivatives are linearly independent. For each Frenet curve of order 5, one can associate an orthonormal 5-frame $\{\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3, \mathbf{V}_4, \mathbf{V}_5\}$ along α which is called the Frenet frame and* the functions $k_i : I \to \mathbb{R}$; $1 \leq i \leq 4$ *are called the Frenet curvatures, such that the Frenet formulas are defined in the usual way*

$$\begin{aligned} \nabla_{\mathbf{V}_1} \mathbf{V}_1 &= \sigma_2 k_1 \mathbf{V}_2, \\ \nabla_{\mathbf{V}_1} \mathbf{V}_2 &= -\sigma_1 k_1 \mathbf{V}_1 + \sigma_3 k_2 \mathbf{V}_3, \\ \nabla_{\mathbf{V}_1} \mathbf{V}_3 &= -\sigma_2 k_2 \mathbf{V}_2 + \sigma_4 k_3 \mathbf{V}_4, \\ \nabla_{\mathbf{V}_1} \mathbf{V}_4 &= -\sigma_3 k_3 \mathbf{V}_3 + \sigma_5 k_4 \mathbf{V}_5, \\ \nabla_{\mathbf{V}_1} \mathbf{V}_5 &= -\sigma_4 k_4 \mathbf{V}_4, \end{aligned}$$

where $\sigma_i = \langle \mathbf{V}_i, \mathbf{V}_i \rangle = \pm 1$, [16].

Also, the i th curvature function m_i, $1 \leq i \leq 5$ is defined ([5]) by

$$m_i = \left\{ \begin{array}{ll} 0, & i = 1 \\ \dfrac{\sigma_1 \sigma_2}{k_1}, & i = 2 \\ [\dfrac{d}{dt}(m_{i-1}) + \sigma_{i-2} m_{i-2} k_{i-2}], & 2 < i \leq 5 \end{array} \right\}.$$

Definition 2.2. *A non-null curve α is called a $W-$curve (or helix) of rank 5, if α is a Frenet curve of osculating order 5 and the Frenet curvatures k_i, $1 \leq i \leq 5$ are non-zero constants* [6].

Definition 2.3. *Let α be a non-null curve of osculating order 5. The harmonic functions*

$$\mathcal{H}_j : I \to \mathbb{R}, 1 \leq i \leq 4,$$

defined by

$$\begin{aligned} &\mathcal{H}_0 = 0, \mathcal{H}_1 = \frac{k_1}{k_2}, \\ &\mathcal{H}_j = \{\nabla_{\mathbf{V}_1}(\mathcal{H}_{j-1}) + \sigma_{j-2}\mathcal{H}_{j-2}k_j\}\frac{\sigma_j}{k_{j+1}}, 2 \leq j \leq 4, \end{aligned}$$

are called the harmonic curvatures of α where k_i, for $1 \leq i \leq 5$, are Frenet curvatures of α which are not necessarily constant [6].

Proposition 2.3. *Let α be a curve in $\mathbb{L}^5$ of osculating order 5. Then, the Frenet formulas in terms of harmonic curvatures are given as follows*

$$\begin{aligned} \nabla_{\mathbf{V}_1}\mathbf{V}_1 &= \sigma_2 k_2 \mathcal{H}_1 \mathbf{V}_2, \\ \nabla_{\mathbf{V}_1}\mathbf{V}_2 &= -\sigma_1 k_2 \mathcal{H}_1 \mathbf{V}_1 + \sigma_3 \frac{k_1}{\mathcal{H}_1}\mathbf{V}_3, \\ \nabla_{\mathbf{V}_1}\mathbf{V}_3 &= -\sigma_2 \frac{k_1}{\mathcal{H}_1}\mathbf{V}_2 + \sigma_4 \sigma_2 \frac{\mathcal{H}_1'}{\mathcal{H}_2}\mathbf{V}_4, \\ \nabla_{\mathbf{V}_1}\mathbf{V}_4 &= -\sigma_3 \sigma_2 \frac{\mathcal{H}_1'}{\mathcal{H}_2}\mathbf{V}_3 + \sigma_5 \sigma_3 (\frac{\mathcal{H}_2 \mathcal{H}_2' + \sigma_1 \sigma_2 \mathcal{H}_1 \mathcal{H}_1'}{\mathcal{H}_2 \mathcal{H}_3})\mathbf{V}_5, \\ \nabla_{\mathbf{V}_1}\mathbf{V}_5 &= -\sigma_4 \sigma_3 (\frac{\mathcal{H}_2 \mathcal{H}_2' + \sigma_1 \sigma_2 \mathcal{H}_1 \mathcal{H}_1'}{\mathcal{H}_2 \mathcal{H}_3})\mathbf{V}_4, \end{aligned} \tag{1}$$

where $\mathcal{H}_i$, for $1 \leq i \leq 4$, are harmonic curvatures of α [7, 16].

3. Energy and angle of frame fields of a curve in $\mathbb{L}^5$

Definition 3.1. *For two Riemannian manifolds (M, ρ) and $(N, \mathcal{H})$, the energy of a differentiable map $f : (M, \rho) \to (N, \mathcal{H})$ can be defined as*

$$\varepsilon nergy\,(f) = \frac{1}{2}\int_M \sum_{a=1}^{n} \mathcal{H}\left(df\left(e_a\right), df\left(e_a\right)\right) v, \tag{2}$$

where $\{e_a\}$ is a local basis of the tangent space and v is the canonical volume form in M [3, 4].

Proposition 3.2. *Let $Q : T\left(T^1M\right) \to T^1M$ be the connection map. Then following two conditions hold:*

i) *$\omega \circ Q = \omega \circ d\omega$ and $\omega \circ Q = \omega \circ \tilde{\omega}$, where $\tilde{\omega} : T\left(T^1M\right) \to T^1M$ is the tangent bundle projection;*

ii) *for $\varrho \in T_xM$ and a section $\xi : M \to T^1M$; we have the following equality*

$$Q\left(d\xi\left(\varrho\right)\right) = \nabla_{\varrho}\xi, \tag{3}$$

where ∇ is the Levi-Civita covariant derivative [3, 4].

Definition 3.3. *For $\varsigma_1, \varsigma_2 \in T_{\xi}\left(T^1M\right)$, we define*

$$\rho_S\left(\varsigma_1, \varsigma_2\right) = \rho\left(d\omega\left(\varsigma_1\right), d\omega\left(\varsigma_2\right)\right) + \rho\left(Q\left(\varsigma_1\right), Q\left(\varsigma_2\right)\right). \tag{4}$$

This yields a Riemannian metric on TM: As we know, the metric ρ_S is said to be the Sasaki metric that also makes the projection $\omega : T^1M \to M$ a Riemannian submersion [3, 4].

Definition 3.4. ([1].) *Angle between arbitrary Frenet vectors is given by the following formula*

$$\mathcal{A}(\mathbf{V}_i) = \textstyle\int_{\vartheta}^{s} \left\|\nabla_{\mathbf{V}_1}\mathbf{V}_i\right\| du.$$

4. Bienergy and biangle of the frame fields of a curve in $\mathbb{L}^5$

Definition 4.1. *Let $\mathbf{X}$ be a vector field in $\mathbb{L}^5$, then the bienergy of the vector field $\mathbf{X}$ is defined as the formula*

$$\varepsilon nergy_2\left(\mathbf{X}\right) = \textstyle\int_0^s \rho_S\left(\nabla^2_{\mathbf{V}_1}\mathbf{X}, \nabla^2_{\mathbf{V}_1}\mathbf{X}\right) ds. \tag{5}$$

Definition 4.2. *Biangle between arbitrary Frenet vectors is given by*

$$\mathcal{A}_2(\mathbf{V}_i) = \textstyle\int_{\vartheta}^{s} \left\|\nabla^2_{\mathbf{V}_1}\mathbf{V}_i\right\| du. \tag{6}$$

Theorem 4.3. *Let $\{\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3, \mathbf{V}_4, \mathbf{V}_5\}$ be the Frenet frame fields of a non-null curve α* in *$\mathbb{L}^5$. Then, the energy of each field of this frame is, respectively, stated by using Sasaki metric as follows*

$$\begin{aligned}
\varepsilon nergy\left(\mathbf{V}_1\right) &= \frac{1}{2}(s\sigma_1 + \textstyle\int_0^s \sigma_2 k_2^2 \mathcal{H}_1^2 ds),\\
\varepsilon nergy\left(\mathbf{V}_2\right) &= \frac{1}{2}(s\sigma_1 + \textstyle\int_0^s (\sigma_1 k_2^2 \mathcal{H}_1^2 + \sigma_3 \dfrac{k_1^2}{\mathcal{H}_1^2}) ds),\\
\varepsilon nergy\left(\mathbf{V}_3\right) &= \frac{1}{2}(s\sigma_1 + \textstyle\int_0^s (\sigma_2 \dfrac{k_1^2}{\mathcal{H}_1^2} + \sigma_4 \dfrac{(\mathcal{H}_1')^2}{\mathcal{H}_2^2}) ds),\\
\varepsilon nergy\left(\mathbf{V}_4\right) &= \frac{1}{2}(s\sigma_1 + \textstyle\int_0^s (\sigma_3 \dfrac{(\mathcal{H}_1')^2}{\mathcal{H}_2^2} + \sigma_5 (\dfrac{\mathcal{H}_2\mathcal{H}_2' + \sigma_1\sigma_2\mathcal{H}_1\mathcal{H}_1'}{\mathcal{H}_2\mathcal{H}_3})^2) ds),\\
\varepsilon nergy\left(\mathbf{V}_5\right) &= \frac{1}{2}(s\sigma_1 + \textstyle\int_0^s (\sigma_4 (\dfrac{\mathcal{H}_2\mathcal{H}_2' + \sigma_1\sigma_2\mathcal{H}_1\mathcal{H}_1'}{\mathcal{H}_2\mathcal{H}_3})^2) ds).
\end{aligned}$$

Proof. We prove only the energy of the tangent vector $\mathbf{V}_1$. The energies of other vector fields can be computed if the same method is followed. From (2) and (3) we know

$$\varepsilon nergy\,(\mathbf{V}_1) = \frac{1}{2}\textstyle\int_0^s \rho_S\,(d\mathbf{V}_1(\mathbf{V_1}), d\mathbf{V}_1(\mathbf{V_1}))\,ds. \tag{7}$$

Using Eq. (4) we have the Sasaki metric

$$\begin{aligned}\rho_S\,(d\mathbf{V}_1(\mathbf{V_1}), d\mathbf{V}_1(\mathbf{V_1})) = \rho(d\omega(\mathbf{V}_1(\mathbf{V_1})), d\omega(\mathbf{V}_1(\mathbf{V_1})))\\ +\rho(Q(\mathbf{V}_1(\mathbf{V_1})), Q(\mathbf{V}_1(\mathbf{V_1}))).\end{aligned} \tag{8}$$

Since $\mathbf{V}_1$ is a section, we get

$$d(\omega) \circ d(\mathbf{V}_1) = d(\omega \circ \mathbf{V}_1) = d(id_C) = id_{TC},$$

then, the equation (8) turns into

$$\begin{aligned}\rho_S\,(d\mathbf{V}_1(\mathbf{V_1}), d\mathbf{V}_1(\mathbf{V_1})) &= 1+\rho(Q(\mathbf{V}_1(\mathbf{V_1})), Q(\mathbf{V}_1(\mathbf{V_1})))\\ &= 1+\rho(\nabla_{\mathbf{V}_1}\mathbf{V}_1, \nabla_{\mathbf{V}_1}\mathbf{V}_1).\end{aligned} \tag{9}$$

Using (1) in (9) and then substituting (9) into (7), the energy of the tangent vector field is obtained. The energies of other frame fields can be computed with similar methodology.

Theorem 4.4. *Let* $\{\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3, \mathbf{V}_4, \mathbf{V}_5\}$ *be the Frenet frame fields of a non-null curve* α in $\mathbb{L}^5$. *Then, the angles of Frenet vectors are, respectively, given* as

$$\begin{aligned}
\mathcal{A}(\mathbf{V}_1) &= \textstyle\int_0^s \left|\sigma_2 k_2 \mathcal{H}_1\right| ds,\\
\mathcal{A}(\mathbf{V}_2) &= \textstyle\int_0^s (\left|\sigma_1 k_2^2 \mathcal{H}_1^2 + \sigma_3 \frac{k_1^2}{\mathcal{H}_1^2}\right|^{\frac{1}{2}} ds,\\
\mathcal{A}(\mathbf{V}_3) &= \textstyle\int_0^s \left|(\sigma_2 \frac{k_1^2}{\mathcal{H}_1^2} + \sigma_4 \frac{(\mathcal{H}_1')^2}{\mathcal{H}_2^2})\right|^{\frac{1}{2}} ds,\\
\mathcal{A}(\mathbf{V}_4) &= \textstyle\int_0^s \left|\sigma_3 \frac{(\mathcal{H}_1')^2}{\mathcal{H}_2^2} + \sigma_5 (\frac{\mathcal{H}_2\mathcal{H}_2' + \sigma_1\sigma_2\mathcal{H}_1\mathcal{H}_1'}{\mathcal{H}_2\mathcal{H}_3})^2\right|^{\frac{1}{2}} ds,\\
\mathcal{A}(\mathbf{V}_5) &= \textstyle\int_0^s \left|\sigma_4 (\frac{\mathcal{H}_2\mathcal{H}_2' + \sigma_1\sigma_2\mathcal{H}_1\mathcal{H}_1'}{\mathcal{H}_2\mathcal{H}_3})^2\right|^2 ds.
\end{aligned} \tag{10}$$

Proof. By Def. 3.4, the results in (10) are straightforwardly obtained in $\mathbb{L}^5$.

Theorem 4.5. *Let* $\{\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3, \mathbf{V}_4, \mathbf{V}_5\}$ *be the Frenet frame fields of a non-null curve* α in $\mathbb{L}^5$. *Then, the bienergy of each field of this frame is, respectively, stated by using Sasaki metric as follows*

$$\varepsilon nergy_2\,(\mathbf{V}_1) = \frac{1}{2}(s\sigma_1 + \textstyle\int_0^s (\sigma_1 k_2^4 \mathcal{H}_1^4 + \sigma_2((k_2\mathcal{H}_1)')^2 + \sigma_3(k_2k_1)^2)ds),$$

$$\begin{aligned}\varepsilon nergy_2\,(\mathbf{V}_2) &= \frac{1}{2}(s\sigma_1 + \textstyle\int_0^s (\sigma_1((k_2\mathcal{H}_1)')^2 + \sigma_2(\sigma_1\sigma_2 k_2^2\mathcal{H}_1^2 + \sigma_2\sigma_3\frac{k_1^2}{\mathcal{H}_1^2})^2\\ &+\sigma_3((\frac{k_1}{\mathcal{H}_1})')^2 + \sigma_4(\frac{k_1}{\mathcal{H}_1}\frac{\mathcal{H}_1'}{\mathcal{H}_2})^2)ds),\end{aligned}$$

$$\varepsilon nergy_2\,(\mathbf{V}_3) = \frac{1}{2}(s\sigma_1 + \int_0^s(\sigma_1(\tfrac{k_1}{\mathcal{H}_1}k_2\mathcal{H}_1)^2 + \sigma_2((\frac{k_1}{\mathcal{H}_1})')^2$$
$$+\sigma_3((\sigma_3\sigma_2\sigma_4\sigma_2 + \sigma_3\sigma_2)(\frac{\mathcal{H}_1'}{\mathcal{H}_2})^2)^2 + \sigma_4((\frac{\mathcal{H}_1'}{\mathcal{H}_2})')^2$$
$$+\sigma_5(\frac{\mathcal{H}_1'}{\mathcal{H}_2}(\frac{\mathcal{H}_2\mathcal{H}_2' + \sigma_1\sigma_2\mathcal{H}_1\mathcal{H}_1'}{\mathcal{H}_2\mathcal{H}_3}))^2)ds),$$

$$\varepsilon nergy_2\,(\mathbf{V}_4) = \frac{1}{2}(s\sigma_1 + \int_0^s(\sigma_2(\frac{\mathcal{H}_1'}{\mathcal{H}_2}\frac{k_1}{\mathcal{H}_1})^2 + \sigma_3((\frac{\mathcal{H}_1'}{\mathcal{H}_2})')^2$$
$$+\sigma_4(\sigma_5\sigma_4(\frac{\mathcal{H}_2\mathcal{H}_2' + \sigma_1\sigma_2\mathcal{H}_1\mathcal{H}_1'}{\mathcal{H}_2\mathcal{H}_3})^2 + \sigma_4\sigma_3(\frac{\mathcal{H}_1'}{\mathcal{H}_2})^2)^2$$
$$+\sigma_5((\frac{\mathcal{H}_2\mathcal{H}_2' + \sigma_1\sigma_2\mathcal{H}_1\mathcal{H}_1'}{\mathcal{H}_2\mathcal{H}_3})')^2)ds),$$

$$\varepsilon nergy_2\,(\mathbf{V}_5) = \frac{1}{2}(s\sigma_1 + \int_0^s(\sigma_3((\frac{\mathcal{H}_2\mathcal{H}_2' + \sigma_1\sigma_2\mathcal{H}_1\mathcal{H}_1'}{\mathcal{H}_2\mathcal{H}_3})\frac{\mathcal{H}_1'}{\mathcal{H}_2})^2$$
$$+\sigma_4((\frac{\mathcal{H}_2\mathcal{H}_2' + \sigma_1\sigma_2\mathcal{H}_1\mathcal{H}_1'}{\mathcal{H}_2\mathcal{H}_3})')^2$$
$$+\sigma_5(\frac{\mathcal{H}_2\mathcal{H}_2' + \sigma_1\sigma_2\mathcal{H}_1\mathcal{H}_1'}{\mathcal{H}_2\mathcal{H}_3})^4)ds).$$

Proof. First we calculate second order covariant derivative of each of Frenet field by using the equation (1) as follows:

$$\begin{aligned}
\nabla_{\mathbf{V}_1}\nabla_{\mathbf{V}_1}\mathbf{V}_1 &= -\sigma_1\sigma_2k_2^2\mathcal{H}_1^2\mathbf{V}_1 + \sigma_2(k_2\mathcal{H}_1)'\mathbf{V}_2 + \sigma_3\sigma_2k_2k_1\mathbf{V}_3,\\
\nabla_{\mathbf{V}_1}\nabla_{\mathbf{V}_1}\mathbf{V}_2 &= -\sigma_1(k_2\mathcal{H}_1)'\mathbf{V}_1 - (\sigma_1\sigma_2k_2^2\mathcal{H}_1^2 + \sigma_2\sigma_3\frac{k_1^2}{\mathcal{H}_1^2})\mathbf{V}_2\\
&\quad +\sigma_3(\frac{k_1}{\mathcal{H}_1})'\mathbf{V}_3 + \sigma_4\sigma_2\sigma_3\frac{k_1}{\mathcal{H}_1}\frac{\mathcal{H}_1'}{\mathcal{H}_2}\mathbf{V}_4,\\
\nabla_{\mathbf{V}_1}\nabla_{\mathbf{V}_1}\mathbf{V}_3 &= \sigma_1\sigma_2\frac{k_1}{\mathcal{H}_1}k_2\mathcal{H}_1\mathbf{V}_1 - \sigma_2(\frac{k_1}{\mathcal{H}_1})'\mathbf{V}_2 - (\sigma_3\sigma_2\sigma_4\sigma_2 + \sigma_3\sigma_2)(\frac{\mathcal{H}_1'}{\mathcal{H}_2})^2\mathbf{V}_3\\
\text{(11)}\qquad &\quad +\sigma_4\sigma_2(\frac{\mathcal{H}_1'}{\mathcal{H}_2})'\mathbf{V}_4 + \sigma_5\sigma_3\sigma_4\sigma_2\frac{\mathcal{H}_1'}{\mathcal{H}_2}(\frac{\mathcal{H}_2\mathcal{H}_2' + \sigma_1\sigma_2\mathcal{H}_1\mathcal{H}_1'}{\mathcal{H}_2\mathcal{H}_3})\mathbf{V}_5,\\
\nabla_{\mathbf{V}_1}\nabla_{\mathbf{V}_1}\mathbf{V}_4 &= \sigma_3\frac{\mathcal{H}_1'}{\mathcal{H}_2}\frac{k_1}{\mathcal{H}_1}\mathbf{V}_2 - \sigma_3\sigma_2(\frac{\mathcal{H}_1'}{\mathcal{H}_2})'\mathbf{V}_3 - (\sigma_5\sigma_4(\frac{\mathcal{H}_2\mathcal{H}_2' + \sigma_1\sigma_2\mathcal{H}_1\mathcal{H}_1'}{\mathcal{H}_2\mathcal{H}_3})^2\\
&\quad +\sigma_4\sigma_3(\frac{\mathcal{H}_1'}{\mathcal{H}_2})^2)\mathbf{V}_4 + \sigma_5\sigma_3(\frac{\mathcal{H}_2\mathcal{H}_2' + \sigma_1\sigma_2\mathcal{H}_1\mathcal{H}_1'}{\mathcal{H}_2\mathcal{H}_3})'\mathbf{V}_5,\\
\nabla_{\mathbf{V}_1}\nabla_{\mathbf{V}_1}\mathbf{V}_5 &= \sigma_2\sigma_4(\frac{\mathcal{H}_2\mathcal{H}_2' + \sigma_1\sigma_2\mathcal{H}_1\mathcal{H}_1'}{\mathcal{H}_2\mathcal{H}_3})\frac{\mathcal{H}_1'}{\mathcal{H}_2}\mathbf{V}_3 - \sigma_4\sigma_3(\frac{\mathcal{H}_2\mathcal{H}_2' + \sigma_1\sigma_2\mathcal{H}_1\mathcal{H}_1'}{\mathcal{H}_2\mathcal{H}_3})'\mathbf{V}_4\\
&\quad -\sigma_5\sigma_4(\frac{\mathcal{H}_2\mathcal{H}_2' + \sigma_1\sigma_2\mathcal{H}_1\mathcal{H}_1'}{\mathcal{H}_2\mathcal{H}_3})^2\mathbf{V}_5.
\end{aligned}$$

As similar to the proof of Theorem 4.3., we find the Sasaki metric for the tangent vector field $\mathbf{V}_1$

$$\text{(12)}\qquad \rho_S\,(d\mathbf{V}_1d\mathbf{V}_1(\mathbf{V_1}), d\mathbf{V}_1d\mathbf{V}_1(\mathbf{V_1})) = 1+\rho(\nabla^2_{\mathbf{V}_1}\mathbf{V}_1, \nabla^2_{\mathbf{V}_1}\mathbf{V}_1),$$

Using (11) in (12) and then substituting (12) into (5), the bienergy of the tangent vector field $\mathbf{V}_1$ is obtained. The bienergies of other frame fields can be computed with similar methodology.

Corollary 4.6. *Let* $\{\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3, \mathbf{V}_4, \mathbf{V}_5\}$ *be the Frenet frame fields of a non-null curve* α *in* $\mathbb{L}^5$. *Then, the conditions for the bienergy of each field of this frame is to be fixed are given as follows:*

(13)

$$\sigma_1 k_2^4 \mathcal{H}_1^4 + \sigma_2((k_2\mathcal{H}_1)')^2 + \sigma_3(k_2k_1)^2 = 0,$$

$$\sigma_1((k_2\mathcal{H}_1)')^2 + \sigma_2(\sigma_1\sigma_2 k_2^2\mathcal{H}_1^2 + \sigma_2\sigma_3\frac{k_1^2}{\mathcal{H}_1^2})^2 + \sigma_3((\frac{k_1}{\mathcal{H}_1})')^2 + \sigma_4(\frac{k_1}{\mathcal{H}_1}\frac{\mathcal{H}_1'}{\mathcal{H}_2})^2 = 0,$$

$$\sigma_1(\frac{k_1}{\mathcal{H}_1}k_2\mathcal{H}_1)^2 + \sigma_2((\frac{k_1}{\mathcal{H}_1})')^2 + \sigma_3((\sigma_3\sigma_2\sigma_4\sigma_2 + \sigma_3\sigma_2)(\frac{\mathcal{H}_1'}{\mathcal{H}_2})^2)^2 + \sigma_4((\frac{\mathcal{H}_1'}{\mathcal{H}_2})')^2$$
$$+\sigma_5(\frac{\mathcal{H}_1'}{\mathcal{H}_2}(\frac{\mathcal{H}_2\mathcal{H}_2' + \sigma_1\sigma_2\mathcal{H}_1\mathcal{H}_1'}{\mathcal{H}_2\mathcal{H}_3}))^2 = 0,$$

$$\sigma_2(\frac{\mathcal{H}_1'}{\mathcal{H}_2}\frac{k_1}{\mathcal{H}_1})^2 + \sigma_3((\frac{\mathcal{H}_1'}{\mathcal{H}_2})')^2 + \sigma_4(\sigma_5\sigma_4(\frac{\mathcal{H}_2\mathcal{H}_2' + \sigma_1\sigma_2\mathcal{H}_1\mathcal{H}_1'}{\mathcal{H}_2\mathcal{H}_3})^2 + \sigma_4\sigma_3(\frac{\mathcal{H}_1'}{\mathcal{H}_2})^2)^2$$
$$+\sigma_5((\frac{\mathcal{H}_2\mathcal{H}_2' + \sigma_1\sigma_2\mathcal{H}_1\mathcal{H}_1'}{\mathcal{H}_2\mathcal{H}_3})')^2 = 0,$$

$$\sigma_3((\frac{\mathcal{H}_2\mathcal{H}_2' + \sigma_1\sigma_2\mathcal{H}_1\mathcal{H}_1'}{\mathcal{H}_2\mathcal{H}_3})\frac{\mathcal{H}_1'}{\mathcal{H}_2})^2 + \sigma_4((\frac{\mathcal{H}_2\mathcal{H}_2' + \sigma_1\sigma_2\mathcal{H}_1\mathcal{H}_1'}{\mathcal{H}_2\mathcal{H}_3})')^2$$
$$+\sigma_5(\frac{\mathcal{H}_2\mathcal{H}_2' + \sigma_1\sigma_2\mathcal{H}_1\mathcal{H}_1'}{\mathcal{H}_2\mathcal{H}_3})^4 = 0.$$

Proof. By taking the norm of each of frame vector fields in (11) and equaling zero, we obtain the conditions for these fields to be fixed.

Theorem 4.7. *Let* $\{\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3, \mathbf{V}_4, \mathbf{V}_5\}$ *be the Frenet frame fields of a non-null curve* α. *Then, the biangles of Frenet vectors are given by*

$$\mathcal{A}_2(\mathbf{V}_1) = \int_0^s (\sigma_1 k_2^4\mathcal{H}_1^4 + \sigma_2((k_2\mathcal{H}_1)')^2 + \sigma_3(k_2k_1)^2)^{\frac{1}{2}} du,$$

$$\mathcal{A}_2(\mathbf{V}_2) = \int_0^s ((\sigma_1((k_2\mathcal{H}_1)')^2 + \sigma_2(\sigma_1\sigma_2k_2^2\mathcal{H}_1^2 + \sigma_2\sigma_3\frac{k_1^2}{\mathcal{H}_1^2})^2 + \sigma_3((\frac{k_1}{\mathcal{H}_1})')^2$$
$$+\sigma_4(\frac{k_1}{\mathcal{H}_1}\frac{\mathcal{H}_1'}{\mathcal{H}_2})^2)^{\frac{1}{2}} du,$$

(14)

$$\mathcal{A}_2(\mathbf{V}_3) = \int_0^s (\sigma_1(\frac{k_1}{\mathcal{H}_1}k_2\mathcal{H}_1)^2 + \sigma_2((\frac{k_1}{\mathcal{H}_1})')^2 + \sigma_3((\sigma_3\sigma_2\sigma_4\sigma_2 + \sigma_3\sigma_2)(\frac{\mathcal{H}_1'}{\mathcal{H}_2})^2)^2$$
$$+\sigma_4((\frac{\mathcal{H}_1'}{\mathcal{H}_2})')^2 + \sigma_5(\frac{\mathcal{H}_1'}{\mathcal{H}_2}(\frac{\mathcal{H}_2\mathcal{H}_2' + \sigma_1\sigma_2\mathcal{H}_1\mathcal{H}_1'}{\mathcal{H}_2\mathcal{H}_3}))^2)^{\frac{1}{2}} du,$$

$$\mathcal{A}_2(\mathbf{V}_4) = \int_0^s (\sigma_2(\frac{\mathcal{H}_1'}{\mathcal{H}_2}\frac{k_1}{\mathcal{H}_1})^2 + \sigma_3((\frac{\mathcal{H}_1'}{\mathcal{H}_2})')^2 + \sigma_4(\sigma_5\sigma_4(\frac{\mathcal{H}_2\mathcal{H}_2' + \sigma_1\sigma_2\mathcal{H}_1\mathcal{H}_1'}{\mathcal{H}_2\mathcal{H}_3})^2$$
$$+\sigma_4\sigma_3(\frac{\mathcal{H}_1'}{\mathcal{H}_2})^2)^2 + \sigma_5((\frac{\mathcal{H}_2\mathcal{H}_2' + \sigma_1\sigma_2\mathcal{H}_1\mathcal{H}_1'}{\mathcal{H}_2\mathcal{H}_3})')^2)^{\frac{1}{2}} du,$$
$$\mathcal{A}_2(\mathbf{V}_5) = \int_0^s (\sigma_3((\frac{\mathcal{H}_2\mathcal{H}_2' + \sigma_1\sigma_2\mathcal{H}_1\mathcal{H}_1'}{\mathcal{H}_2\mathcal{H}_3})\frac{\mathcal{H}_1'}{\mathcal{H}_2})^2 + \sigma_4((\frac{\mathcal{H}_2\mathcal{H}_2' + \sigma_1\sigma_2\mathcal{H}_1\mathcal{H}_1'}{\mathcal{H}_2\mathcal{H}_3})')^2$$
$$+\sigma_5(\frac{\mathcal{H}_2\mathcal{H}_2' + \sigma_1\sigma_2\mathcal{H}_1\mathcal{H}_1'}{\mathcal{H}_2\mathcal{H}_3})^4)^{\frac{1}{2}} du.$$

Proof. Rewriting the second order covariant derivatives obtained in (11) into the formula (6), we have the results in (14).

5. Application

In this section we transmit our geometric understanding of the energy for different type of particle into graphs for different cases. By doing this practice we have a chance to observe differentiation of the energy of the particle with respect to time and different curves.

Energy and angle of Frenet vectors $\{\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3, \mathbf{V}_4, \mathbf{V}_5\}$ are drawn for our particles in Figures 1,2, respectively,

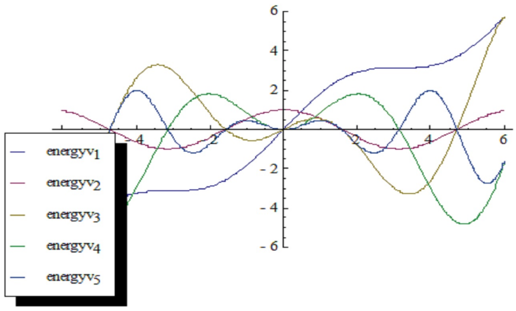

Figure 1

Bienergy and biangle of Frenet vectors $\{\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3, \mathbf{V}_4, \mathbf{V}_5\}$ are drawn for our particles in Figures 3,4, respectively,

6. Conclusion

In this research, we obtained the energies and the angles of Frenet vector fields in Lorentz 5-space $\mathbb{L}^5$. Then we found the bienergies and biangels of Frenet vector fields in $\mathbb{L}^5$ by using the values of energies and angles. Finally, we presented the relations among energies, angles, bienergies, and biangles with graphics.

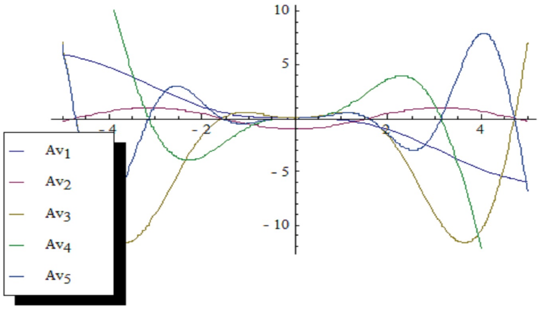

Figure 2

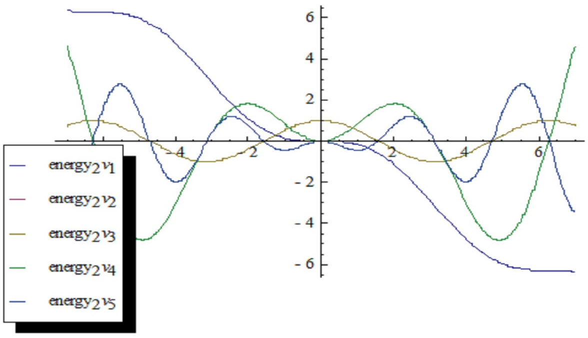

Figure 3

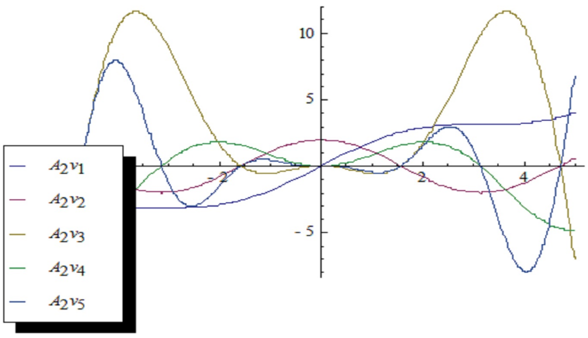

Figure 4

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Talat Korpinar
Department of Mathematics, Mus Alparslan University,
Mus 49250, Turkey.
E-mail: talatkorpinar@gmail.com

Yasin Unluturk
Department of Mathematics, Kirklareli University,
Mus 39100, Turkey.
E-mail: yasinunluturk@yahoo.com