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CERTAIN RESULTS ON INVARIANT SUBMANIFOLDS OF PARA-KENMOTSU MANIFOLDS

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Abstract. The purpose of this paper is to study invariant pseudoparallel, Ricci generalized pseudoparallel and 2-Ricci generalized pseudoparallel submanifold of a para-Kenmotsu manifold and I obtained some equivalent conditions of invariant submanifolds of para-Kenmotsu manifolds under some conditions which the submanifolds are totally geodesic. Finally, a non-trivial example of invariant submanifold of paracontact metric manifold is constructed in order to illustrate our results.

1. Introduction

The geometry of almost paracontact manifolds is a natural counterpart of the almost para-Hermitian geometry. The study of almost paracontact metric manifolds started in [6]. A systematic study of almost paracontact metric manifolds was considered by Zamkovoy[7]. Almost paracontact metric manifolds have been extensively studied under several points of view in[6, 7, 8, 9, 10, 11, 13].

Many geometers studied paracontact metric manifolds and researched some important properties of these manifolds. The geometry of paracontact metric manifolds can be related to the theory of Legendre foliations. In [8], authors introduced the class of paracontact metric manifolds for which the characteristic vector field ξ belongs to the (κ,μ)-nullity condition for some real constants κ and μ. Such manifolds are also known as (κ,μ)-paracontact metric manifolds.

The study of submanifolds of a paracontact metric manifold is a topic of interest in differential geometry. According to the behaviour of the tangent bundle of a submanifold with respect to action of the paracontact structure φ of the ambient manifold, there are two well known classes of submanifolds such as invariant and anti-invariant.

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Also, invariant submanifolds are used to discuss properties of non-linear antronomous systems. Also totally geodesic submanifolds play an important role in the relativity theory even though they are simplest submanifolds.

Pseudoparallel submanifolds have been studied intensively by many geometers[1, 2, 4, 5].

Motivated by the above studies, in this paper, we are deal with an invariant submanifold of a para-Kenmotsu manifold which have not been attempted so far. Also, we give some characterizations of an invariant submanifold to be totally geodesic.

2. Preliminaries

A $(2n+1)$-dimensional smooth manifold $\widetilde{M}^{2n+1}$ has an almost paracontact structure (φ, ξ, η, g) if it admits a tensor field φ of type $(1,1)$, a vector field ξ, a 1-form η and a semi-Riemannian metric tensor g satisfying the following conditions;

$$\varphi^2 X = X - \eta(X)\xi, \quad \eta(\xi) = 1, \quad \varphi\xi = \eta \circ \varphi = 0 \tag{1}$$

$$g(\varphi X, \varphi Y) = -g(X,Y) + \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi) \tag{2}$$

and

$$d\eta(X,Y) = g(X, \varphi Y), \tag{3}$$

for all vector fields X, Y on $\widetilde{M}^{2n+1}$.

An almost paracontact metric manifold $\widetilde{M}^{2n+1}(\varphi, \xi, \eta, g)$ is said to be para-Kenmotsu manifold if the Levi-Civita connection $\widetilde{\nabla}$ of g satisfies

$$(\widetilde{\nabla}_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X, \tag{4}$$

for all $X, Y \in \Gamma(T\widetilde{M})$, where $\Gamma(T\widetilde{M})$ denote the set of all differentiable vector fields on $\widetilde{M}^{2n+1}$[16].
From (1) and (4), we have

$$\widetilde{\nabla}_X \xi = \varphi^2 X = X - \eta(X)\xi. \tag{5}$$

In a para-Kenmotsu $\widetilde{M}^{2n+1}(\varphi, \xi, \eta, g)$, we have the following formulas.

$$\widetilde{R}(X,Y)\xi = \eta(X)Y - \eta(Y)X \tag{6}$$

$$\widetilde{R}(\xi, X)Y = \eta(Y)X - g(X,Y)\xi \tag{7}$$

$$S(\xi, X) = -2n\eta(X), \tag{8}$$

for any vector fields $X, Y \in \Gamma(\widetilde{M})$, where $\widetilde{R}$ and S denote the Riemannian cuvature tensor and Ricci tensor of $\widetilde{M}^{2n+1}$, respectively.

Now, let M be an immersed submanifold of a paracontact metric manifold $\widetilde{M}^{2n+1}$. By $\Gamma(TM)$ and $\Gamma(T^{\perp}M)$, we denote the tangent and normal subspaces of M in $\widetilde{M}$. Then the Gauss and Weingarten formulae are, respectively, given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y), \tag{9}$$

and

$$\widetilde{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \tag{10}$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, where ∇ and $\nabla^{\perp}$ are the connections on M and $\Gamma(T^{\perp}M)$ and σ and A are called the second fundamental form and shape operator of M, respectively. They are related by

$$g(A_V X, Y) = g(\sigma(X,Y), V). \tag{11}$$

The covariant derivative of σ is defined by

$$(\widetilde{\nabla}_X \sigma)(Y,Z) = \nabla^{\perp}_X \sigma(Y,Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z), \tag{12}$$

for all $X, Y, Z \in \Gamma(TM)$. If $\widetilde{\nabla}\sigma = 0$, then submanifold M is said to be its second fundamental form is parallel.

By R, we denote the Riemannian curvature tensor of M, we have the following Gauss equation;

$$\begin{aligned} \widetilde{R}(X,Y)Z &= R(X,Y)Z + A_{\sigma(X,Z)}Y - A_{\sigma(Y,Z)}X + (\widetilde{\nabla}_X \sigma)(Y,Z) \\ &- (\widetilde{\nabla}_Y \sigma)(X,Z), \end{aligned} \tag{13}$$

for all $X, Y, Z \in \Gamma(T\widetilde{M})$, where if $(\widetilde{\nabla}_X \sigma)(Y,Z) - (\widetilde{\nabla}_Y \sigma)(X,Z) = 0$, then submanifold is called curvature-invariant submanifold.

For a $(0,k)$-type tensor field T, $k \geq 1$ and a $(0,2)$-type tensor field A on a Riemannian manifold (M,g), $Q(A,T)$-tensor field is defined by

$$\begin{aligned} Q(A,T)(X_1, X_2, ..., X_k; X, Y) &= -T((X \Lambda_A Y)X_1, X_2, ..., X_k)... \\ &- T(X_1, X_2, ...X_{k-1}, (X \Lambda_A Y)X_k),, \end{aligned} \tag{14}$$

for all $X_1, X_2, ..., X_k, X, Y \in \Gamma(TM)$, where

$$(X \wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y. \tag{15}$$

Definition 2.1. *A submanifold of a Riemannian manifold (M,g) is said to be pseudoparallel, 2-pseudoparallel, Ricci-generalized pseudoparallel and 2-Ricci-generalized pseudoparallel if*

$$\begin{aligned} \widetilde{R} \cdot \sigma \ \ &\textit{and} \ \ Q(g,\sigma) \\ \widetilde{R} \cdot \widetilde{\nabla}\sigma \ \ &\textit{and} \ \ Q(g,\widetilde{\nabla}\sigma) \\ \widetilde{R} \cdot \sigma \ \ &\textit{and} \ \ Q(S,\sigma) \\ \widetilde{R} \cdot \widetilde{\nabla}\sigma \ \ &\textit{and} \ \ Q(S,\widetilde{\nabla}\sigma) \end{aligned}$$

are linearly dependent, respectively.

Equivalently, these can be formulated by the following equations;

$$\widetilde{R}\cdot\sigma = L_1Q(g,\sigma), \tag{16}$$

$$\widetilde{R}\cdot\widetilde{\nabla}\sigma = L_2Q(g,\widetilde{\nabla}\sigma), \tag{17}$$

$$\widetilde{R}\cdot\sigma = L_3Q(S,\sigma), \tag{18}$$

$$\widetilde{R}\cdot\widetilde{\nabla}\sigma = L_4Q(S,\widetilde{\nabla}\sigma), \tag{19}$$

where functions L_1, L_2, L_3 and L_4 are, respectively, defined on $M_1 = \{x \in M : \sigma(x) \neq g(x)\}$, $M_2 = \{x \in M : \widetilde{\nabla}\sigma(x) \neq g(x)\}$, $M_3 = \{x \in M : S(x) \neq \sigma(x)\}$ and $M_4 = \{x \in M : S(x) \neq \widetilde{\nabla}\sigma(x)\}$.

Particularly, if $L_1 = 0$, then submanifold is said to be semiparallel, if $L_2 = 0$, submanifold is said to be 2-semiparallel.

3. Certain Results on Invarinat Submanifolds of Para Kenmotsu Manifolds

Now, we will investigate the above cases for the invariant submanifold M of a para-Kenmotsu manifold $\widetilde{M}^{2n+1}(\varphi,\xi,\eta,g)$.

Now, let M be an immersed submanifold of a para-Kenmotsu manifold manifold $\widetilde{M}^{2n+1}(\varphi,\xi,\eta,g)$. If $\varphi(T_xM) \subseteq T_xM$, for each point at $x \in M$, then M is said to be invariant submanifold. We note that all of the properties of an invariant submanifold inherit the ambient manifold.

In the rest of this paper, we will assume that M is invariant submanifold of a para Kenmotsu manifold $\widetilde{M}^{2n+1}(\varphi,\xi,\eta,g)$. Thus by using (4), (9), (10) and (11) we have

$$\sigma(X,\xi) = 0, \quad \sigma(\varphi X, Y) = \sigma(X,\varphi Y) = \varphi\sigma(X,Y), \tag{20}$$

and

$$\nabla_X\xi = X - \eta(X)\xi, \tag{21}$$

for all $X, Y \in \Gamma(TM)$.

Lemma 3.1. *Let M be an invariant submanifold of a para Kenmotsu manifold $\widetilde{M}^{2n+1}(\varphi,\xi,\eta,g)$. The second fundamental form σ of M is parallel if and only if M is totally geodesic.*

Proof. Let us assume that σ is parallel. Then (12) yields to

$$(\widetilde{\nabla}_X\sigma)(Y,Z) = \nabla^{\perp}_X\sigma(Y,Z) - \sigma(\nabla_XY,Z) - \sigma(Y,\nabla_XZ) = 0,$$

for all $X, Y, Z \in \Gamma(TM)$. Here, taking $Z = \xi$, by virtue of (4), (20) and (21), we can verify

$$-\sigma(\nabla_XY,\xi) + \sigma(Y,\nabla_X\xi) = \sigma(Y, X - \eta(X)\xi) = \sigma(Y,X) = 0$$

This proves our assertion. The converse is obvious. □

Lemma 3.1 is important for later theorems and corollaries.

Theorem 3.2. *Let M be an invariant pseudoparallel submanifold of a para-Kenmotsu manifold $\widetilde{M}^{2n+1}(\varphi,\xi,\eta,g)$. Then M is either totally geodesic or $L_1=-1$.*

Proof. Let M be pseudoparallel, then from (16) we have

$$(\widetilde{R}(X,Y)\cdot\sigma)(U,V)=L_1Q(g,\sigma)(U,V;X,Y),$$

for all $X,Y,U,V\in\Gamma(TM)$. Taking into account of (13) and (20), this leads to

$$\begin{aligned} R^{\perp}(X,Y)\sigma(U,V) &- \sigma(R(X,Y)U,V)-\sigma(U,R(X,Y)V)\\ &= -L_1\{\sigma((X\wedge_g Y)U,V)+\sigma(U,(X\wedge_g Y)V)\}\\ &= -L_1\{\sigma(g(Y,U)X-g(X,U)Y,V)\\ &+ \sigma(U,g(Y,V)X-g(X,V)Y)\} \end{aligned} \tag{22}$$

for all $X,Y,U,V\in\Gamma(TM)$. Taking $V=\xi$ in (22) and taking into account of (6), (7) and (20), we obtain

$$\begin{aligned} \sigma(R(X,Y)\xi,U) &= L_1\{\eta(Y)\sigma(X,U)-\eta(X)\sigma(U,Y)\}\\ \sigma(\eta(X)Y-\eta(Y)X,U) &= L_1\{\eta(Y)\sigma(X,U)-\eta(X)\sigma(U,Y)\} \end{aligned}$$

This completes the proof. □

From the Theorem 3.2, we have the following corollary.

Corollary 3.3. *Let M be an invariant pseudoparallel submanifold of a para-Kenmotsu manifold $\widetilde{M}^{2n+1}(\varphi,\xi,\eta,g)$. Then M is semiparallel if and only if M is totally geodesic.*

Theorem 3.4. *Let M be an invariant 2-pseudoparallel submanifold of a para Kenmotsu manifold $\widetilde{M}^{2n+1}(\varphi,\xi,\eta,g)$. Then M is either totally geodesic or $L_2=-1$.*

Proof. Let M be 2-pseudoparallel of a para Kenmotsu manifold $\widetilde{M}^{2n+1}(\varphi,\xi,\eta,g)$. Then from (17), we have

$$(\widetilde{R}(X,Y)\cdot\widetilde{\nabla}\sigma)(U,V,Z) \;=\; L_2Q(g,\widetilde{\nabla}\sigma)(U,V,Z;X,Y),$$

for all $X,Y,U,V,Z\in\Gamma(TM)$. Also, making use use of (15), we have

$$\begin{aligned} & R^{\perp}(X,Y)(\widetilde{\nabla}_U\sigma)(V,Z)-(\widetilde{\nabla}_{R(X,Y)U}\sigma)(V,Z)-(\widetilde{\nabla}_U\sigma)(R(X,Y)V,Z)\\ - & (\widetilde{\nabla}_U\sigma)(V,R(X,Y)Z)=-L_2\{(\widetilde{\nabla}_{(X\wedge_g Y)U}\sigma)(V,Z)+(\widetilde{\nabla}_U\sigma)((X\wedge_g Y)V,Z)\\ + & (\widetilde{\nabla}_U\sigma)(V,(X\wedge_g Y)Z)\}, \end{aligned}$$

that is,

$$\begin{aligned} & R^{\perp}(X,Y)(\widetilde{\nabla}_U\sigma)(V,Z)-(\widetilde{\nabla}_{R(X,Y)U}\sigma)(V,Z)-(\widetilde{\nabla}_U\sigma)(R(X,Y)V,Z) \\ - & (\widetilde{\nabla}_U\sigma)(V,R(X,Y)Z)=-L_2\{g(Y,U)(\widetilde{\nabla}_X\sigma)(V,Z)-g(X,U)(\widetilde{\nabla}_Y\sigma)(V,Z) \\ + & (\widetilde{\nabla}_U\sigma)(g(Y,V)X-g(X,V)Y,Z)+(\widetilde{\nabla}_U\sigma)(V,g(Y,Z)X-g(X,Z)Y)\}. \end{aligned}$$

In the last equality, taking $X=Z=\xi$ and the necessary arrangements are made, we obtain

$$\begin{aligned} R^{\perp}(\xi,Y)(\widetilde{\nabla}_U\sigma)(V,\xi) & - (\widetilde{\nabla}_{R(\xi,Y)U}\sigma)(V,\xi)-(\widetilde{\nabla}_U\sigma)(R(\xi,Y)V,\xi) \\ & - (\widetilde{\nabla}_U\sigma)(V,R(\xi,Y)\xi)=-L_2\{g(Y,U)(\widetilde{\nabla}_\xi\sigma)(V,\xi) \\ & - \eta(U)(\widetilde{\nabla}_Y\sigma)(V,\xi)+(\widetilde{\nabla}_U\sigma)(g(Y,V)\xi-\eta(V)Y,\xi) \\ & + (\widetilde{\nabla}_U\sigma)(V,\eta(Y)\xi-Y)\}. \end{aligned} \tag{23}$$

Now, let us calculate each of these expressions. Making use of (4), (12) and (20), we obtain

$$\begin{aligned} R^{\perp}(\xi,Y)(\widetilde{\nabla}_U\sigma)(V,\xi) & = R^{\perp}(\xi,Y)\{\nabla^{\perp}_U\sigma(V,\xi)-\sigma(\nabla_U V,\xi)-\sigma(V,\nabla_U\xi)\} \\ & = R^{\perp}(\xi,Y)\{-\sigma(V,\nabla_U\xi)\} \\ & = -R^{\perp}(\xi,Y)\sigma(V,U-\eta(U)\xi) \\ & = -R^{\perp}(\xi,Y)\sigma(V,U). \end{aligned} \tag{24}$$

Moreover, taking into account of (4) and (20), we have

$$\begin{aligned} (\widetilde{\nabla}_{R(\xi,Y)U}\sigma)(V,\xi) & = \nabla^{\perp}_{R(\xi,Y)U}\sigma(V,\xi)-\sigma(\nabla_{R(\xi,Y)U}V,\xi) \\ & - \sigma(\nabla_{R(\xi,Y)U}\xi,V) \\ & = -\sigma(R(\xi,Y)U-\eta(R(\xi,Y)U)\xi,V) \\ & = -\sigma(R(\xi,Y)U,V)=-\sigma(\eta(U)Y-g(U,Y)\xi,V). \\ & = -\eta(U)\sigma(Y,V). \end{aligned} \tag{25}$$

$$\begin{aligned} (\widetilde{\nabla}_U\sigma)(R(\xi,Y)V,\xi) & = \nabla^{\perp}_U\sigma(R(\xi,Y)V,\xi)-\sigma(\nabla_U R(\xi,Y)V,\xi) \\ & - \sigma(R(\xi,Y)V,\nabla_U\xi) \\ & = -\sigma(\eta(V)Y-g(Y,V)\xi,U-\eta(U)\xi) \\ & = -\eta(V)\sigma(Y,U). \end{aligned} \tag{26}$$

$$\begin{aligned} (\widetilde{\nabla}_U\sigma)(V,R(\xi,Y)\xi) & = (\widetilde{\nabla}_U\sigma)(V,Y-\eta(Y)\xi) \\ & = (\widetilde{\nabla}_U\sigma)(V,Y)-(\widetilde{\nabla}_U\sigma)(V,\eta(Y)\xi) \\ & = (\widetilde{\nabla}_U\sigma)(V,Y)-\nabla^{\perp}_U\sigma(V,\eta(Y)\xi) \\ & + \sigma(\nabla_U V,\eta(Y)\xi)+\sigma(V,\nabla_U\eta(Y)\xi) \\ & = (\widetilde{\nabla}_U\sigma)(V,Y)+\sigma(V,U\eta(Y)\xi+\eta(Y)\nabla_U\xi) \\ & = (\widetilde{\nabla}_U\sigma)(V,Y)+\eta(Y)\sigma(V,U). \end{aligned} \tag{27}$$

$$
\begin{aligned}
(\widetilde{\nabla}_{(\xi\wedge_g Y)U}\sigma)(V,\xi) &= \nabla^{\perp}_{(\xi\wedge_g Y)U}\sigma(V,\xi)-\sigma(\nabla_{(\xi\wedge_g Y)U}V,\xi)\\
&- \sigma(V,\nabla_{(\xi\wedge_g Y)U}\xi)=-\sigma(V,\nabla_{g(Y,U)\xi-\eta(U)Y}\xi)\\
&= -\sigma(V,g(Y,U)\xi-\eta(U)Y-\eta(g(Y,U)\xi-\eta(U)Y)\xi)\\
&= \eta(U)\sigma(V,Y). \qquad (28)
\end{aligned}
$$

$$
\begin{aligned}
(\widetilde{\nabla}_U\sigma)((\xi\wedge_g Y)V,\xi) &= \nabla^{\perp}_U\sigma((\xi\wedge_g Y)V,\xi)-\sigma(\nabla_U(\xi\wedge_g Y)V,\xi)\\
&- \sigma((\xi\wedge_g Y)V,\nabla_U\xi)\\
&= -\sigma(g(Y,V)\xi-\eta(V)Y,U-\eta(U)\xi)\\
&= \eta(V)\sigma(Y,U). \qquad (29)
\end{aligned}
$$

$$
\begin{aligned}
(\widetilde{\nabla}_U\sigma)(V,(\xi\wedge_g Y)\xi) &= (\widetilde{\nabla}_U\sigma)(V,\eta(Y)\xi-Y)\\
&= (\widetilde{\nabla}_U\sigma)(V,\eta(Y)\xi)-(\widetilde{\nabla}_U\sigma)(V,Y)\\
&= \nabla^{\perp}_U\sigma(V,\eta(Y)\xi)-\sigma(\nabla_UV,\eta(Y)\xi)\\
&- \sigma(V,\nabla_U\eta(Y)\xi)-(\widetilde{\nabla}_U\sigma)(V,Y)\\
&= -\sigma(V,U\eta(Y)\xi+\eta(Y)\nabla_U\xi)-(\widetilde{\nabla}_U\sigma)(V,Y)\\
&= -\eta(Y)\sigma(V,U-\eta(U)\xi)-(\widetilde{\nabla}_U\sigma)(V,Y)\\
&= -\eta(Y)\sigma(V,U)-(\widetilde{\nabla}_U\sigma)(V,Y). \qquad (30)
\end{aligned}
$$

Consequently, if we put (24), (25), (26), (27), (28), (29) and (30) in (23), we reach at

$$
\begin{aligned}
&- R^{\perp}(\xi,Y)\sigma(V,U)+\eta(U)\sigma(Y,V)+\eta(V)\sigma(Y,U)-(\widetilde{\nabla}_U\sigma)(V,Y)\\
&- \eta(Y)\sigma(U,V)=-L_2\{\eta(U)\sigma(V,Y)+\eta(V)\sigma(Y,U)-\eta(Y)\sigma(V,U)\\
&- (\widetilde{\nabla}_U\sigma)(V,Y)\} \qquad (31)
\end{aligned}
$$

If ξ is taken of V at (31), considering (20) and (5), we get

$$
\sigma(Y,U)-(\widetilde{\nabla}_U\sigma)(Y,\xi)=-L_2\{\sigma(U,Y)-(\widetilde{\nabla}_U\sigma)(Y,\xi)\}, \qquad (32)
$$

where

$$
\begin{aligned}
(\widetilde{\nabla}_U\sigma)(\xi,Y) &= \nabla^{\perp}_U\sigma(Y,\xi)-\sigma(\nabla_UY,\xi)-\sigma(Y,\nabla_U\xi)\\
&= -\sigma(Y,U). \qquad (33)
\end{aligned}
$$

From (32) and (33), we conclude that

$$
L_2\{\sigma(U,Y)\}=-\sigma(U,Y)
$$

which is proves our assertions. □

From Theorem 3.4, we have the following corollary.

Corollary 3.5. *Let M be an invariant pseudoparallel submanifold of a para Kenmotsu manifold $\widetilde{M}^{2n+1}(\varphi,\xi,\eta,g)$. Then M is 2-semiparallel if and only if M is totally geodesic.*

Theorem 3.6. *Let M be an invariant Ricci-generalized pseudoparallel submanifold of a para Kenmotsu manifold $\widetilde{M}^{2n+1}(\varphi,\xi,\eta,g)$. Then M is either totally geodesic or the function $L_3 = \frac{1}{2n}$.*

Proof. If M is Ricci-generalized pseudoparallel of para Kenmotsu manifold $\widetilde{M}(\varphi,\xi,\eta,g)$, then from (14) and (18), we have

$$\begin{aligned}(\widetilde{R}(X,Y)\cdot\sigma)(U,V) &= L_3 Q(S,\sigma)(U,V;X,Y)\\ &= -L_3\left\{\sigma((X\wedge_S Y)U,V)+\sigma(U,(X\wedge_S Y)V)\right\},\end{aligned}$$

for all $X,Y,U,V\in\Gamma(TM)$. This means that

$$\begin{aligned}R^{\perp}(X,Y)\sigma(U,V) &- \sigma(R(X,Y)U,V)-\sigma(U,R(X,Y)V)\\ &= -L_3\{\sigma(S(Y,U)X-S(X,U)Y,V)\\ &+ \sigma(S(V,Y)X-S(X,V)Y,U)\}.\end{aligned}$$

Here taking $X=V=\xi$ and by using (8) and (20), we reach at

$$\begin{aligned}R^{\perp}(\xi,Y)\sigma(U,\xi) &- \sigma(R(\xi,Y)U,\xi)-\sigma(U,R(\xi,Y)\xi)\\ &= -L_3\{\sigma(S(Y,U)\xi-S(\xi,U)Y,\xi)\\ &+ \sigma(S(\xi,Y)\xi-S(\xi,\xi)Y,U)\}. \quad (34)\end{aligned}$$

By using (8) and (20), (34) reduces

$$\begin{aligned}-\sigma(U,Y-\eta(Y)\xi) &= -L_3\{-S(\xi,\xi)\sigma(Y,U)\}\\ -\sigma(Y,U) &= -2nL_3\sigma(Y,U)\end{aligned}$$

This proves our assertion. □

Theorem 3.7. *Let M be an invariant 2-Ricci-generalized pseudoparallel submanifold of a para Kenmotsu manifold $\widetilde{M}^{2n+1}(\varphi,\xi,\eta,g)$. Then M is either totally geodesic or $L_4 = \frac{1}{2n}$.*

Proof. Let us assume that M is 2-Ricci-generalized pseudoparallel submanifold. Then from (19), we have

$$(\widetilde{R}(X,Y)\cdot\widetilde{\nabla}\sigma)(U,V,Z) = L_4 Q(S,\widetilde{\nabla}\sigma)(U,V,Z;X,Y),$$

for all $X,Y,U,V,Z\in\Gamma(TM)$. This implies that

$$\begin{aligned}R^{\perp}(X,Y)(\widetilde{\nabla}_U\sigma)(V,Z) &- (\widetilde{\nabla}_{R(X,Y)U}\sigma)(V,Z)-(\widetilde{\nabla}_U\sigma)(R(X,Y)V,Z)\\ &- (\widetilde{\nabla}_U\sigma)(V,R(X,Y)Z) = -L_4\{(\widetilde{\nabla}_{(X\wedge_S Y)U}\sigma)(V,Z)\\ &+ (\widetilde{\nabla}_U\sigma)((X\wedge_S Y)V,Z)+(\widetilde{\nabla}_U\sigma)(V,(X\wedge_S Y)Z)\}.\end{aligned}$$

Here taking $X=V=\xi$, we have

$$\begin{aligned}R^{\perp}(\xi,Y)(\widetilde{\nabla}_U\sigma)(\xi,Z) &- (\widetilde{\nabla}_{R(\xi,Y)U}\sigma)(\xi,Z)-(\widetilde{\nabla}_U\sigma)(R(\xi,Y)\xi,Z)\\ &- (\widetilde{\nabla}_U\sigma)(\xi,R(\xi,Y)Z) = -L_4\{(\widetilde{\nabla}_{(\xi\wedge_S Y)U}\sigma)(\xi,Z)\\ &+ (\widetilde{\nabla}_U\sigma)((\xi\wedge_S Y)\xi,Z)+(\widetilde{\nabla}_U\sigma)(\xi,(\xi\wedge_S Y)Z)\}. \quad (35)\end{aligned}$$

Now, let's calculate each of these expressions. Also taking into account of (4) and (20), we arrive at

$$\begin{aligned} R^{\perp}(\xi,Y)(\widetilde{\nabla}_U\sigma)(\xi,Z) &= R^{\perp}(\xi,Y)\{\nabla^{\perp}_U\sigma(\xi,Z)-\sigma(\nabla_U Z,\xi) \\ &- \sigma(Z,\nabla_U\xi)\} = R^{\perp}(\xi,Y)\{-\sigma(Z,U-\eta(U)\xi)\} \\ &= -R^{\perp}(\xi,Y)\sigma(Z,U). \end{aligned} \tag{36}$$

On the other hand, by using (4) and (20), we have

$$\begin{aligned} (\widetilde{\nabla}_{R(\xi,Y)U}\sigma)(\xi,Z) &= \nabla^{\perp}_{R(\xi,Y)U}\sigma(\xi,Z)-\sigma(\nabla_{R(\xi,Y)U}\xi,Z) \\ &- \sigma(\xi,\nabla_{R(\xi,Y)U}Z) \\ &= -\sigma(R(\xi,Y)U-\eta(R(\xi,Y)U)\xi,Z) \\ &= -\sigma(\eta(U)Y-g(Y,U)\xi,Z) = -\eta(U)\sigma(Y,Z). \end{aligned} \tag{37}$$

$$\begin{aligned} (\widetilde{\nabla}_U\sigma)(R(\xi,Y)\xi,Z) &= (\widetilde{\nabla}_U\sigma)(Y-\eta(Y)\xi,Z) = (\widetilde{\nabla}_U\sigma)(Y,Z) \\ &- (\widetilde{\nabla}_U\sigma)(\eta(Y)\xi,Z) = (\widetilde{\nabla}_U\sigma)(Y,Z) \\ &- \nabla^{\perp}_U\sigma(\eta(Y)\xi,Z)+\sigma(\nabla_U\eta(Y)\xi,Z) \\ &+ \sigma(\eta(Y)\xi,\nabla_U Z) \\ &= (\widetilde{\nabla}_U\sigma)(Y,Z)+\sigma(U\eta(Y)\xi+\eta(Y)\nabla_U\xi,Z) \\ &= (\widetilde{\nabla}_U\sigma)(Y,Z)+\sigma(U\eta(Y)\xi+\eta(Y)(U-\eta(U)\xi),Z) \\ &= (\widetilde{\nabla}_U\sigma)(Y,Z)+\eta(Y)\sigma(U,Z). \end{aligned} \tag{38}$$

$$\begin{aligned} (\widetilde{\nabla}_U\sigma)(\xi,R(\xi,Y)Z) &= \nabla^{\perp}_U\sigma(\xi,R(\xi,Y)Z)-\sigma(\nabla_U\xi,R(\xi,Y)Z) \\ &- \sigma(\xi,\nabla_U R(\xi,Y)Z) = -\sigma(U-\eta(U)\xi,R(\xi,Y)Z) \\ &= -\sigma(U,\eta(Z)Y-g(Y,Z)\xi) = -\eta(Z)\sigma(U,Y). \end{aligned} \tag{39}$$

Now, let's calculate the left side of (35). Making use of (4), (6) and (20), we have

$$\begin{aligned} (\widetilde{\nabla}_{(\xi\wedge_S Y)U}\sigma)(\xi,Z) &= \nabla^{\perp}_{(\xi\wedge_S Y)U}\sigma(\xi,Z)-\sigma(\nabla_{(\xi\wedge_S Y)U}\xi,Z) \\ &- \sigma(\xi,\nabla_{(\xi\wedge_S Y)U}Z) \\ &= -\sigma(\nabla_{S(Y,U)\xi-S(\xi,U)Y}\xi,Z) \\ &= -S(Y,U)\sigma(\nabla_\xi\xi,Z)+S(\xi,U)\sigma(\nabla_Y\xi,U) \\ &= -2n\eta(U)\sigma(Y-\eta(Y)\xi,Z) = -2n\eta(U)\sigma(Y,Z). \end{aligned} \tag{40}$$

$$\begin{aligned}(\widetilde{\nabla}_U\sigma)((\xi\wedge_S Y)\xi,Z) &= (\widetilde{\nabla}_U\sigma)(S(Y,\xi)\xi - S(\xi,\xi)Y,Z)\\ &= (\widetilde{\nabla}_U\sigma)(2nY-2n\eta(Y)\xi,Z)\\ &= 2n\{(\widetilde{\nabla}_U\sigma)(Y,Z)-(\widetilde{\nabla}_U\sigma)(\eta(Y)\xi,Z)\}\\ &= 2n\{(\widetilde{\nabla}_U\sigma)(Y,Z)-\nabla^{\perp}_U\sigma(\eta(Y)\xi,Z)\\ &+ \sigma(\nabla_U\eta(Y)\xi,Z)+\sigma(\eta(Y)\xi,\nabla_U Z)\}\\ &= 2n\{(\widetilde{\nabla}_U\sigma)(Y,Z)+\sigma(U\eta(Y)\xi+\eta(Y)\nabla_U\xi,Z)\}\\ &= 2n\{(\widetilde{\nabla}_U\sigma)(Y,Z)+\eta(Y)\sigma(U,Z)\} \qquad (41)\end{aligned}$$

Finally,

$$\begin{aligned}(\widetilde{\nabla}_U\sigma)(\xi,(\xi\wedge_S Y)Z) &= (\widetilde{\nabla}_U\sigma)(\xi,S(Y,Z)\xi-S(\xi,Z)Y)\\ &= (\widetilde{\nabla}_U\sigma)(\xi,S(Y,Z)\xi)+2n(\widetilde{\nabla}_U\sigma)(\xi,\eta(Z)Y)\\ &= \nabla^{\perp}_U\sigma(\xi,S(Y,Z)\xi)-\sigma(\nabla_U\xi,S(Y,Z)\xi)\\ &- \sigma(\xi,\nabla_U S(Y,Z)\xi)+2n\{\nabla^{\perp}_U\sigma(\xi,\eta(Z)Y)\\ &- \sigma(\nabla_U\xi,\eta(Z)Y)-\sigma(\xi,\nabla_U\eta(Z)Y)\}\\ &= -2n\eta(Z)\sigma(U,Y). \qquad (42)\end{aligned}$$

By substituting (36), (37), (38), (39), (40), (41) and (42) into (35) we reach at

$$\begin{aligned}&- R^{\perp}(\xi,Y)\sigma(U,Z)+\eta(U)\sigma(Y,Z)-(\widetilde{\nabla}_U\sigma)(Y,Z)-\eta(Y)\sigma(U,Z)\\ &+ \eta(Z)\sigma(U,Y)=-2nL_4\{-\eta(U)\sigma(Y,Z)+\eta(Y)\sigma(U,Z)\\ &+ (\widetilde{\nabla}_U\sigma)(Y,Z)-\eta(Z)\sigma(U,Y)\}. \qquad (43)\end{aligned}$$

Here if taking $Z=\xi$, then (43) reduce

$$-2nL_4\{(\widetilde{\nabla}_U\sigma)(Y,\xi)-\sigma(U,Y)\}=-(\widetilde{\nabla}_U\sigma)(Y,\xi)+\sigma(U,Y).$$

From (33), we conclude that

$$(2nL_4-1)\sigma(U,Y)=0,$$

which proves our assertion. □

Example 3.8. *Let us the 5-dimensional manifold*
$\widetilde{M}^5=\{(x_1,x_2,x_3,x_4,t):t\neq 0\}$,
where (x_i,t) *denote the coordinate of* $\mathbb{R}^5$. *Then the vector fields*

$$e_1=t\frac{\partial}{\partial x_1},e_2=t\frac{\partial}{\partial x_2},e_3=t\frac{\partial}{\partial x_3},e_4=t\frac{\partial}{\partial x_4},e_5=-t\frac{\partial}{\partial t}$$

are linearly independent at each point of $\widetilde{M}^5$. *By* g, *we denote the semi-Riemannian metric tensor such that*
$g(e_i,e_i)=-1$, *if* i *is even*
$g(e_i,e_i)=1$, *if* i *is odd*
$g(e_i,e_j)=0$, *if* $i\neq j$

Let η be the 1-form defined by $\eta(X) = g(X, e_5)$ for all $X \in \Gamma(T\widetilde{M})$. Now, we define the tensor field $(1,1)$-type φ such that

$$\varphi e_1 = e_2, \quad \varphi e_2 = e_1, \quad \varphi e_3 = e_4, \quad \varphi e_4 = e_3, \quad \varphi e_5 = 0.$$

Then we can easily to see that

$$\eta(e_5) = 1, \quad \varphi^2 X = X - \eta(X)\xi, \quad e_5 = \xi$$

and

$$g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y)$$

for all $X, Y \in \Gamma(\widetilde{M})$, that is, the equations(1), (2) and (3) are satisfied. Thus $\widetilde{M}(\varphi, \eta, \xi, g)$ defines an almost paracontact metric manifold. By $\widetilde{\nabla}$, we denote the Levi-Civita connection on $\widetilde{M}$. Then by direct calculations, we have

$$[e_i, e_5] = e_i, \quad \widetilde{\nabla}_{e_i} e_5 = e_i, \quad 1 \leq i \leq 4, \quad \widetilde{\nabla}_{e_i} e_j = 0, \quad otherwise$$

Thus one can easily verified

$$[\varphi, \varphi](e_i, e_j) - 2d\eta(e_i, e_j) = 0, \quad 1 \leq i, j \leq 5, \quad \widetilde{\nabla}_X e_5 = \varphi^2 X - \eta(X)\xi$$

This tell us that $\widetilde{M}(\varphi, \eta, \xi, g)$ is a para Kenmotsu manifold.

Now, let us a submanifolds M of $\widetilde{M}^5(\varphi, \eta, \xi, g)$ defined by immersion ψ as follows;

$$\psi(x_1, x_2, x_3, x_4, t) = (tx_1, tx_2, tx_3, tx_4, \frac{1}{2}t^2), \quad x_1 = x_3, \quad x_2 = x_4.$$

Then the tangent space of M is spanned by the vector fields

$$U = e_1 + e_3, \quad V = e_2 + e_4, \quad \xi = e_5 \quad and \quad \varphi U = V,$$

that is, M is a 3-dimensional invariant submanifold of a para Kenmotsu manifold $\widetilde{M}^5(\varphi, \eta, \xi, g)$. Furthermore, we can easily to see that

$$\nabla_U \xi = U, \quad \nabla_V \xi = V, \quad \nabla_U V = \nabla_V U = 0.$$

This tell us that M is pseudoparallel, Ricci generalized pseudoparallel submanifold because it is a totally geodesic submanifold of $\widetilde{M}^5(\varphi, \eta, \xi, g)$.

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