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COMPARISON OF TWO DESINGULARIZATIONS OF THE MODULI SPACE OF ELLIPTIC STABLE MAPS

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Abstract. We study the geometry of the moduli space of elliptic stable maps to projective space. The main component of the moduli space of elliptic stable maps is singular. There are two different ways to desingularize this space. One is Vakil-Zinger's desingularization and the other is via the moduli space of logarithmic stable maps. Our main result is a proof of the direct geometric relationship between these two spaces. For degree less than or equal to 3, we prove that the moduli space of logarithmic stable maps can be obtained by blowing up Vakil-Zinger's desingularization.

1. Introduction

1.1. Overviews

The moduli space of stable maps $\overline{\mathbf{M}}_{g,k}(X,d)$ is the moduli space which parameterizes maps from k-marked nodal curve of arithmetic genus g to a projective variety X satisfying a certain stability condition. We refer the reader to [1] for an introduction to the moduli space of stable maps.

In our paper we focus on the geometry of the moduli space

$$\overline{\mathbf{M}}_{1,0}(\mathbb{P}^n,d)$$

of elliptic stable maps to $\mathbb{P}^n$. $\overline{\mathbf{M}}_{1,0}(\mathbb{P}^n,d)$ has several irreducible components ([8]). We define the main component $\overline{\mathbf{M}}_{1,0}(\mathbb{P}^n,d)_0$ by the component parameterizing elliptic stable maps whose domain curve have non-contracted elliptic subcurve. The space $\overline{\mathbf{M}}_{1,0}(\mathbb{P}^n,d)_0$ is singular. See [7].

1.2. Desingularizations

Several birational model of $\overline{\mathbf{M}}_{1,0}(\mathbb{P}^n,d)_0$ have been introduced by many authors. In [8], Vakil and Zinger found a canonical desingularization

$$\widetilde{\mathbf{M}}_{1,0}(\mathbb{P}^n,d)_0$$

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of $\overline{\mathbf{M}}_{1,0}(\mathbb{P}^n, d)_0$ by explicit blowing-ups.

In [4], Kim introduced another desingularization of $\overline{\mathbf{M}}_{1,0}(\mathbb{P}^n, d)_0$ via the moduli space of logarithmic stable maps,

$$\overline{\mathbf{M}}_{1,0}^{log,ch}(\mathbb{P}^n, d)\,.$$

In [5], Marian, Oprea and Pandharipande constructed the moduli space of stable quotients

$$\mathbf{Q}_{g,n}(\mathbb{P}^n, d)\,,$$

as the moduli space of quotients of the rank n trivial sheaf on nodal curves. For genus one, one can prove $\mathbf{Q}_{1,0}(\mathbb{P}^n, d)$ is smooth by studying the deformation theory of the space.

In [9], Viscardi constructed the moduli space of (m)-stable maps

$$\overline{\mathbf{M}}_{1,k}^{(m)}(\mathbb{P}^n, d)\,,$$

via (m)-stable curves introduced by Smyth as a generalization of usual stable curves ([6]). He also proved $\overline{\mathbf{M}}_{1,k}^{(m)}(\mathbb{P}^n, d)$ is smooth if $d+k \leq m \leq 5$.

1.3. Geometric relationship

In general, it is not clear how the birational models in Section 1.2 are related to each others. In this paper, we study the geometric relationship between Vakil-Zinger's desingularization and the moduli space of logarithmic stable maps. More explicitly, we show that

$$\overline{\mathbf{M}}_{1,0}^{log,ch}(\mathbb{P}^n, 3)$$

can be obtained by blowing up

$$\widetilde{\mathbf{M}}_{1,0}(\mathbf{P}^n, 3)_0$$

along the locus $\sum_2$, Γ_2, $\sum_1$, Γ_1 which is the sub-locus of $\overline{\mathbf{M}}_{1,0}(\mathbb{P}^n, d)_0$ defined as follows:

- $\sum_1$ is the closure of the locus of $\overline{\mathbf{M}}_{1,0}(\mathbb{P}^n, d)_0$ parameterizing stable maps whose domain curves consist of the elliptic component of the degree 0 and the rational component of the degree 3 and the morphism restricted to the rational component has ramification order 3 at the nodal point.
- $\sum_2$ is the closure of the locus of $\overline{\mathbf{M}}_{1,0}(\mathbb{P}^n, d)_0$ parameterizing stable maps whose domain curves consist of the elliptic component of the degree 0, and two rational components with the degree 1 and 2. Each rational component meets the elliptic component at one point and the morphism restricted to degree 2 rational component has ramification order 2 at the nodal point.
- Γ_1 is the closure of the locus of $\overline{\mathbf{M}}_{1,0}(\mathbb{P}^n, d)_0$ parameterizing stable maps whose domain curves consist of the elliptic component of the degree 0 and the rational component of the degree three. There exists

a smooth point q on the rational component such that p and q have the same image, where p is the node point.

- Γ_2 is the closure of the locus of $\overline{\mathbf{M}}_{1,0}(\mathbb{P}^n, d)_0$ parameterizing stable maps whose domain curves consist of the elliptic component of the degree 0 and two rational components with the degree 1 and 2. Each rational component meets the elliptic component at one point and there exists a smooth point q on degree 2 rational component such that p, q have same image, where p is nodal point on degree 2 rational component.

Now we are ready to state our main theorem.

Theorem 1. *$\overline{\mathbf{M}}_{1,0}^{log,ch}(\mathbb{P}^n, 3)$ is obtained by blowing-up $\widetilde{\mathbf{M}}_{1,0}(\mathbb{P}^n, 3)_0$ along the locus $\sum_2$, Γ_2, $\sum_1$, Γ_1.*

1.4. Plan of the paper

After brief introductions to Vakil-Zinger desingularization and logarithmic stable map in Section 2, we study explicit examples of degenerations where a nontrivial elliptic logarithmic stable map occurs in Section 3. In Section 4, we calculate the fiber of the natural morphism from the moduli space of admissible stable maps to the moduli space of stable maps. In Section 5, we prove two desingularizations are equal if the degree of map is 2. In Section 6, after complete descriptions of local charts of the main component of the moduli space of stable maps of degree 3, we give the proof of Theorem 1.

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2. Desingularizations

Here we review the definitions of Vakil-Zinger's desingularization and the moduli space of logarithmic stable maps. Throughout the paper we work over the base field k which is an algebraic closed field with characteristic zero.

2.1. Dual graph of domain curves

For every nodal curve, we can associate a graph called the dual graph. Each irreducible component of the nodal curve corresponds to a vertex of the graph and each nodal point of nodal curve corresponds to an edge of the graph.

While the dual graph is well-defined for arbitrary genus curves, only connected curves of arithmetic genus 1 are used in the paper. Every connected curve of arithmetic genus 1 has the unique minimal subcurve of arithmetic genus 1.

Definition 2. Let C be a connected curve of arithmetic genus 1. Let C' be the minimal subcurve of arithmetic genus 1 of C. We call C' the *essential part* of C.

For a connected curve C of arithmetic genus 1 whose essential part is irreducible, we express its dual graph as follows. We explain this via the examples whose generalization is obvious.

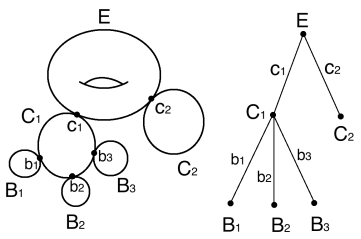
Suppose C has 6 irreducible components

$$E\,, C_1\,, C_2\,, B_1\,, B_2\, B_3\,.$$

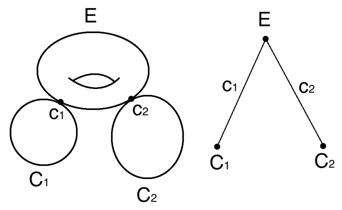
Here E is the smooth curve of arithmetic genus 1. Two smooth rational components C_1, C_2 are connected to E and three smooth rational components B_1, B_2, B_3 are connected to C_1. Then we represent the dual graph of C by

$$E[C_1[B_1, B_2, B_3], C_2]\,.$$

Here we say curve C is of the type $E[C_1[B_1, B_2, B_3], C_2]$. Denote the intersection point of E and C_1 by c_1 and denote the intersection point of C_1 and B_1 as b_1 and so on.



The real picture and the dual graph of the curve of the type E[C1[B1,B2,B3],C2]



The real picture and the dual graph of the curve of the type E[C1,C2]

Furthermore, if a curve C is the domain curve of the moduli space of elliptic stable maps, we record information of the degree in the parenthesis. For example, if the dual graph of the curve C is represented as $E[B_1[C_1, C_2]]$ and the degrees of the maps restricted to the components E, B_1, C_1, C_2 are 0, 0, 1, 2, respectively, we say that the domain curve C of the stable maps is of the type $E(0)[B_1(0)[C_1(1), C_2(2)]]$.

2.2. Expanded target

Let $\mathbb{P}^n$ be an n-dimensional projective space. Denote by $\mathbb{P}^n(1)$ the space

$$(Bl_{c(0)}\mathbb{P}^n)\bigcup \mathbb{P}^n$$

defined as follows.

Let $c(0)$ be a closed point in $\mathbb{P}^n$. $Bl_{c(0)}\mathbb{P}^n$ and $\mathbb{P}^n$ are glued along $D(1)$. Here we consider $D(1)$ as subscheme of $Bl_{c(0)}\mathbb{P}^n$ and $\mathbb{P}^n$ at the same time. In $Bl_{c(0)}\mathbb{P}^n$, $D(1)$ is the exceptional divisor. In $\mathbb{P}^n$, $D(1)$ is a hyperplane. We give the linear order to the set of irreducible components of $\mathbb{P}^n(1)$ such that component $\mathbb{P}^n$ is the largest. We denote the irreducible components of $\mathbb{P}^n(1)$ by $\mathbb{P}^n_1$, $\mathbb{P}^n_2$ according to this order. That is, $\mathbb{P}^n_2$ is the largest one.

Denote by $\mathbb{P}^n(2)$ the space

$$(Bl_{c(1)}\mathbb{P}^n(1))\bigcup\mathbb{P}^n$$

defined as follows. Let $c(1)$ be a closed point in $\mathbb{P}^n_2$ not contained in $D(1)$. $Bl_{c(1)}\mathbb{P}^n(1)$ and $\mathbb{P}^n$ are glued along $D(2)$. In $Bl_{c(1)}\mathbb{P}^n(1)$, $D(2)$ is the exceptional divisor associated to the blowing-up at the point $c(1)$. In $\mathbb{P}^n$, $D(2)$ is a hyperplane. We give the linear order to the set of irreducible components of $\mathbb{P}^n(2)$ such that component $\mathbb{P}^n$ is the largest. We denote the irreducible components of $\mathbb{P}^n(2)$ by $\mathbb{P}^n_1$, $\mathbb{P}^n_2$, $\mathbb{P}^n_3$ according to this order. That is, $\mathbb{P}^n_3$ is the largest one.

In this way, we define

$$\mathbb{P}^n(k)$$

and its components

$$\mathbb{P}^n_1, \mathbb{P}^n_2, \dots, \mathbb{P}^n_{k+1},$$

inductively for all positive integer k. Also the intersections between these components

$$D(1), D(2), \dots, D(k)$$

are defined in the obvious way.

2.3. Vakil-Zinger desingularization

In [7], Vakil and Zinger defined the m-tail locus of $\overline{\mathbf{M}}_{1,0}(\mathbb{P}^n, d)_0$ to be the locus parameterizing maps such that in the domain the contracted elliptic curve meets the rest of the curve at m points. Vakil and Zinger desingularized the space

$$\overline{\mathbf{M}}_{1,0}(\mathbb{P}^n, d)_0$$

as follows: First, blow up the 1-tail locus, then blow up the proper transform of the 2-tail locus, then blow up the proper transform of the 3-tail locus. This process stops at finite steps and the final space

$$\widetilde{\mathbf{M}}_{1,0}(\mathbb{P}^n, d)_0$$

is smooth Delign-Mumford stack.

Now we briefly review the local equations of the space

$$\overline{\mathbf{M}}_{1.0}(\mathbb{P}^n, d),$$

which were described in [2]. First the terminally weighted tree γ was defined. To each γ, the variety Z_γ called local model and the subvariety $Z^0_\gamma \subset Z_\gamma$ called the type γ loci in Z_γ were associated.

In [2] LI and Hu defined DM-stack S to have singularity type γ at a closed point $s \in S$ if there exists a scheme Y, a point $y \in Y$ and two smooth morphisms

$$q_1 : Y \to S, \quad q_2 : Y \to Z_\gamma$$

with $q_1(y) = s$ and $q_2(y) \in Z^0_\gamma$. To each element

$$[u] \in \overline{\mathbf{M}}_{1,0}(\mathbb{P}^n, d),$$

the terminally weighted rooted tree was associated. They defined the substack

$$\overline{\mathbf{M}}_{1,0}(\mathbb{P}^n, d)_\gamma \subset \overline{\mathbf{M}}_{1,0}(\mathbb{P}^n, d)$$

to be the substack of all element

$$[u] \subset \overline{\mathbf{M}}_{1,0}(\mathbb{P}^n, d)$$

whose associated terminally weighted rooted trees is γ. Finally they showed that the space

$$\overline{\mathbf{M}}_{1,0}(\mathbb{P}^n, d)$$

has singularity type γ along the subspace

$$\overline{\mathbf{M}}_{1,0}(\mathbb{P}^n, d)_\gamma \,.$$

For the case $d = 3$, we have the following explicit description of the space

$$\widetilde{\mathbf{M}}_{1,0}(\mathbb{P}^n, 3)_0 \,.$$

The 1-tail locus $D_1 \subset \overline{\mathbf{M}}_{1,0}(\mathbb{P}^n, 3)_0$ and the 2-tail locus $D_2 \subset \overline{\mathbf{M}}_{1,0}(\mathbb{P}^n, 3)_0$ are smooth divisors. The 3-tail locus $D_3 \subset \overline{\mathbf{M}}_{1,0}(\mathbb{P}^n, 3)_0$ has description as follows. Let Z be the scheme defined by

$$\{(a_1, a_2, \dots, a_{n-1}, b_1, b_2, \dots, b_{n-1}, z_1, z_2) \in \mathbb{A}^{2n} \mid$$
$$a_1 z_1 - b_1 z_2 = a_2 z_1 - b_2 z_2 = \cdots = a_{n-1} z_1 - b_{n-1} z_2 = 0\}\,,$$

where $\mathbb{A}^n$ is an n-dimensional affine space. Denote by $Z^0 \subset Z$ the subscheme defined by

$$\{(a_1, a_2, \dots, a_{n-1}, b_1, b_2, \dots, b_{n-1}, z_1, z_2) \in Z : z_1 = z_2 = 0\}\,.$$

To each element $[u] \in D_3$, there is a scheme Y, a point $y \in Y$ and two smooth morphisms

$$q_1 : Y \to \overline{\mathbf{M}}_{1.0}(\mathbb{P}^n, d)_0 \,, q_2 : Y \to Z$$

with $q_1(y) = [u]$ and $q_2(y) \in Z^0$. Therefore we conclude

$$\widetilde{\mathbf{M}}_{1,0}(\mathbb{P}^n, 3)_0 \,=\, \mathrm{Bl}_{D_3} \overline{\mathbf{M}}_{1,0}(\mathbb{P}^n, 3)_0 \,.$$

2.4. Logarithmic stable maps

We briefly review the logarithmic stable maps defined in [4]. We refer the reader to [3] for an introduction to the log structure.

Definition 3. An algebraic space W over S is called a *Fulton-Macpherson* (*FM*) *type space* if

(1) $W \to S$ is a proper, flat morphism;
(2) for every closed point $s \in S$, etale locally there is an etale morphism

$$W_{\overline{s}} \to Spec(k(\overline{s})[x, y, z_1, z_2, \dots, z_{k-1}]/(xy))$$

where x, y and z_i are indeterminates.

Definition 4 ([4, 5.1.1])**.** A triple $((C/S, \mathbf{p}), W/S, f : C \longrightarrow W)$ is called an n-pointed, genus g, *admissible map* to a FM type space W/S if

(1) $(C/S, \mathbf{p} = (p_1, \dots, p_n))$ is an n-pointed, genus g, prestable curve over S.
(2) W/S is an FM type space.
(3) $f : C \longrightarrow W$ is a map over S.
(4) (*Admissibility*) If a point $p \in C$ is mapped into the relatively singular locus $(W/S)^{\mathrm{sing}}$ of W/S, then étale locally at $\bar{p}$, f is factorized as

$$\begin{tikzcd}
C \arrow[dd, "f"'] \arrow[dr] & & U \arrow[ll] \arrow[rr] \arrow[dd] \arrow[dr] & & \mathrm{Spec}(A[u,v]/(uv-t)) \arrow[dl] \arrow[dd] \\
& S & & \mathrm{Spec}A \arrow[ll] & \\
W \arrow[ur] & & V \arrow[ll] \arrow[rr] \arrow[ur] & & \mathrm{Spec}A[x,y,z_1,\dots,z_{r-1}]/(xy-\tau) \arrow[ul]
\end{tikzcd}$$

where all 5 horizontal maps are formally étale; u, v, x, y, z_i are indeterminates; $x = u^l$, $y = v^l$ under the far right vertical map for some positive integer l; t, τ are elements in the maximal ideal $\mathfrak{m}_A$ of the local ring A; and $\bar{p}$ is mapped to the point defined by the ideal $(u, v, \mathfrak{m}_A)$.

A log morphism $(W, M_W)/(S, N)$ is called an extended log twisted FM type space if $W \to S$ is FM type space and M_W, N are log structures on W, S satisfying some conditions.

Definition 5 ([4, 5.2.2])**.** A log morphism

$$(f : (C, M_C, \mathbf{p}) \longrightarrow (W, M_W))\, /(S, N)$$

is called a (g, n) *logarithmic prestable map over* (S, N) if

(1) $((C, M)/(S, N), \mathbf{p})$ is an n-pointed, genus g, minimal log prestable curve.
(2) $(W, M_W)/(S, N)$ is an extended log twisted FM type space.
(3) (*Corank = # Nondistinguished Nodes Condition*) For every $s \in S$, the rank of $\mathrm{Coker}(N_{\bar{s}}^{W/S} \longrightarrow N_{\bar{s}})$ coincides with the number of nondistinguished nodes on $C_{\bar{s}}$.
(4) $f : (C, M_C) \longrightarrow (W, M_W)$ is a log morphism over (S, N).
(5) (*Log Admissibility*) either of the following conditions, equivalent under the above four conditions, holds:
- $\underline{f}$ is admissible.
- $\overline{f}^b : f^* M_W \longrightarrow M_C$ is simple at every distinguished node.

Definition 6 ([4, Definition 8.1])**.** Let $\overline{\mathbf{M}}_{1,0}^{log,ch}(X, d)$ be the moduli stack of $(g = 1, n = 0, d \neq 0)$ logarithmic stable maps (f, C, W) satisfying the following conditions additional to those in Definition 3.0.2. For every $s \in S$,

(1) Every end component of $W_{\bar{s}}$ contains the entire image of the essential part of $C_{\bar{s}}$ under $f_{\bar{s}}$.

(2) The image of the essential part of $C_{\bar{s}}$ is nonconstant.

Here, it is possible that some of irreducible components in the essential part are mapped to points. Note that the dual graph of the target W_s must be a chain. Such a log stable map is called an elliptic log stable map to a chain type FM space W of the smooth projective variety X.

Theorem 7 ([4, Main Theorem B]). *The moduli stack $\overline{\mathbf{M}}_{1,0}^{log,ch}(X,d)$ of elliptic logarithmic stable maps to chain type FM spaces of X is a proper Delign-Mumford stack. When X is a projective space $\mathbb{P}^n$, the stack is smooth.*

We define the moduli space

$$\overline{\mathbf{M}}_{1,0}^{ch}(X,d)$$

of admissible stable maps to chain type FM spaces of X to parameterize the same data as $\overline{\mathbf{M}}_{1,0}^{log,ch}(X,d)$ without log structures. There is natural morphism

$$\overline{\mathbf{M}}_{1,0}^{log,ch}(X,d) \longrightarrow \overline{\mathbf{M}}_{1,0}^{ch}(X,d)$$

which forgets the log structure.

3. Description of degenerations

Here we study the local model of an admissible stable maps which will be useful for the precise description of

$$\overline{\mathbf{M}}_{1,0}^{log,ch}(\mathbb{P}^n,3)$$

in Section 6.

First we construct a family of elliptic stable maps over affine scheme

$$S = \mathbb{A}^2 .$$

Let k be an algebraically closed field. Denote by

$$R = k[t,a]$$

the coordinate ring of S, where t,a are indeterminates. Let

$$C' = Proj(R[x,y,z]/zy^2 - x^3 - z^2x - z^3) .$$

Let

$$f' : C' \dashrightarrow \mathbb{P}^2$$

be a family of rational maps given by

$$[t^3y, at^2x, z] .$$

It is a well defined family of elliptic stable maps except at

$$\{(t,a) : t = 0\} \subset S .$$

If we blow up C' along the ideal (t,x,z), it extends to a family of elliptic stable maps on whole S. Denote by

$$f : C = \mathrm{Bl}_{(x,y,z)}C' \longrightarrow \mathbb{P}^2$$

this family of stable maps which extend f' over S. For $t \neq 0$, the domain curves of f are smooth. For $t = 0$, the domain curve consists of the elliptic component whose degree of map is 0 and one rational component on which the map is given by

$$[s^3, as^2, 1]\,,$$

where s is the local coordinate of the rational component such that $\{s = 0\}$ is the intersection point with the elliptic component.

Now we construct a family of elliptic stable admissible maps over $\widetilde{S} = Bl_{(t,a)}S$ as follows.

Proposition 8. *Let R, C', C, f be as above. Let $\widetilde{S}$ be the blow up of S at the origin and let E be the exceptional divisor. Let D be the proper transform of subscheme defined by*

$$(t) \subset R\,.$$

Let C'' be a pullback of C' along

$$\widetilde{S} \longrightarrow S$$

and $\widetilde{C}$ be the blow up of C'' along ideals

$$(D, x, z)\,, (E, x, z)\,.$$

Here we first blow up along (D, x, z) and next blow up along the proper transform of (E, x, z). Let $\widetilde{W}$ be the blow up of

$$\widetilde{S} \times \mathbb{P}^2$$

along ideals

$$(E^3, x_0, x_1), (D^2, x_0, x_1)\,,$$

where x_0, x_1, x_2 are coordinates of $\mathbb{P}^2$. Then the family of maps

$$f : C \longrightarrow \mathbb{P}^2$$

extends to a family of maps

$$\widetilde{f} : \widetilde{C} \longrightarrow \widetilde{W}$$

and define a family of admissible stable maps.

Proof. we can choose a local coordinate of $\widetilde{S}$ as

$$\{(t, a)\} \simeq \mathbb{A}^2$$

so that the morphism

$$\widetilde{S} \longrightarrow S$$

is given by

$$(t, a) \mapsto (ta, a)\,.$$

Then the induced morphism is given by

$$[t^3a^3y : t^2a_3x, z]\,.$$

Since we only need to consider a neighborhood of $\{[x : y : z] = [0, 1, 0]\}$ which is smooth point, the problem is reduced to the following lemma.

Lemma 9. *Suppose the map*

$$f : C = \mathbb{A}^1 \times \mathbb{A}^2 = Spec(k[x,t,a]) \longrightarrow W = \mathbb{A}^2 \times \mathbb{A}^2 = Spec(k[X,Y,t,a])$$

is given by

$$(x,t,a) \mapsto \Big(\frac{t^3a^3}{z}, \frac{t^2a^3x}{z}, t, a\Big),$$

where z is a function of x such that the vanishing order of z at $x = 0$ is 3. If we let $\widetilde{C}$ be the blow up of C along ideals

$$(x,t)\,, (x,a)$$

and $\widetilde{W}$ be the blow up of W along ideals

$$(X,Y,a^3)\,, (X,Y,t^2),$$

then the map

$$f : C \longrightarrow W$$

extends to the map

$$\widetilde{f} : \widetilde{C} \longrightarrow \widetilde{W}\,.$$

Proof. Using the universal property of blowing ups, we need to check that the inverse image sheaves of (X,Y,a^3) and (X,Y,t^3) are invertible. At an open set $U \subset C$ given by $\{(x,t,a) : x \neq 0\}$, the inverse image sheaves of (X,Y,a^3) and (X,Y,t^3) are

$$(\frac{t^3a^3}{z}, \frac{t^2a^3x}{z}, a_3) = (a^3)$$

and

$$(\frac{t^3a^3}{z}, \frac{t^2a^3x}{z}, t^2) = (t^2)$$

respectively, which are invertible sheaves. □

We can easily check that the map $\widetilde{f}$ in the above lemma satisfies admissible conditions. Similarly we can check the other open sets of $\widetilde{S}$. □

Remark 10. The origin of S parameterizes a stable map whose domain curve consist of an elliptic component of degree 0 and one rational component whose morphism has ramification order 3 at the intersection point with the elliptic component. i.e. it is an element of $\sum_1$.

Remark 11. In the proof, we can describe an element of admissible stable map explicitly. For example, over

$$\{(t,a) : a = 0, t \neq 0\} \subset \widetilde{S}\,,$$

the domain curve $\widetilde{C}$ is of the type $E[C_1]$ and $\widetilde{W} = \mathbb{P}^2(1)$. The restriction of the map to the component E

$$\widetilde{f}|_E : E \longrightarrow \mathbb{P}^2_1$$

is given by

$$[X_0, X_1, X_2] = [t^3y : t^2x : z] = [ty : x : z],$$

where X_0, X_1, X_2 are the coordinates of $\mathbb{P}^2_1$ such that $D(1)$ is given by

$$\{[X_1, X_2, X_3] : X_2 = 0\}.$$

The last equality is due to the existence of an automorphism of $\mathbb{P}^2_1$ fixing $D(1)$.

4. The description of fiber in the moduli space of elliptic admissible stable maps

Denote the natural forgetting morphism by

$$\phi : \overline{\mathbf{M}}^{ch}_{1,0}(\mathbb{P}^n, d) \longrightarrow \overline{\mathbf{M}}_{1,0}(\mathbb{P}^n, d).$$

We describe the set theoretic fibers of ϕ for $d = 3$. The results in this section will play important role in the proof of the main theorem in Section 6. If the essential part is not contracted to a point, the corresponding target is not expanded. Therefore the fiber of ϕ is a point in this case. For the case of the contracted essential part, we have the following results.

Lemma 12. *Let $f : C \longrightarrow \mathbb{P}^n$ be an element of the main component of the moduli space of elliptic stable maps and the domain curve C is of the type $E(0)[C_1(3)]$.*

(1) *If f has ramification order 2 at c_1 and there is no smooth point $q_1 \in C_1$ such that $f(c_1) = f(q_1)$, then the fiber of ϕ at f is equal to a point, set theoretically.*

(2) *If f has ramification order 2 at c_1 and there is a smooth point $q_1 \in C_1$ such that $f(c_1) = f(q_1)$, then the fiber of ϕ at f is equal to $\mathbb{P}^{n-1}$, set theoretically.*

(3) *If f has ramification order 3 at c_1, then the fiber of ϕ at f is equal to $Bl_{pt}\mathbb{P}^n$, set theoretically.*

Proof. (1) • **point:** The domain curve $\tilde{C}$ is of the type $E[C_1]$ and $W = \mathbb{P}^n(1)$. $\tilde{f} : C_1 \longrightarrow \mathbb{P}^n_0$ is already given. $\tilde{f} : E \longrightarrow \mathbb{P}^n_1$ is given by $[X_0 : X_1 : \cdots : X_n] = [x : 0 : \cdots : 0 : z]$, where E are given by $\{[x, y, z] : zy^2 = x^3 + z^2x + Az^3\}$ and X_0, X_1, …, X_n are coordinates of $\mathbb{P}^n$. Here A is a formal variable parameterizing complex structure of the elliptic curve. c_1 is given by $\{[x : y : z] : z = x = 0\}$ and $D(1)$ is given by $\{[X_1, X_2, \ldots, X_n] : X_n = 0\}$.

(2) • $\mathbb{A}^{n-1}$ **with parameter** $\{[\boldsymbol{\alpha_0} : \boldsymbol{\alpha_1} : \cdots : \boldsymbol{\alpha_{n-1}}],\ \boldsymbol{\alpha_{n-1}} \neq \mathbf{0}\ \}$**:** The domain curve $\tilde{C}$ is of the type $E[C_1[A_1]]$ and $W = \mathbb{P}^n(1)$. $\tilde{f}|_{C_1} : C_1 \longrightarrow \mathbb{P}^n_0$ is already given. $\tilde{f}|_E : E \longrightarrow \mathbb{P}^n_1$ is given by $[X_0 : X_1 : \cdots : X_n] = [x : 0 : \cdots : 0 : z]$. $\tilde{f}|_{A_1} : A_1 \longrightarrow \mathbb{P}^n(1)$ is given by $[1 : \alpha_0 t : \alpha_1 t : \cdots : \alpha_{n-1} t]$, where t are a local parameter of A_1 such that a_1 is given by $\{t = 0\}$ and $D(1)$ is given by $\{[X_1, X_2, \ldots, X_n] : X_n = 0\}$. Here a_1 is the intersection point of A_1 and C_1.

• $\mathbb{P}^{n-2}$ **with parameter** $\{[\boldsymbol{\alpha_0} : \boldsymbol{\alpha_1} : \cdots : \boldsymbol{\alpha_{n-2}}]\}$**:** The domain curve $\tilde{C}$ is of the type $E[C_1'[C_1[A_1]]]$ and $W = \mathbb{P}^n(2)$. $\tilde{f}|_{C_1} : C_1 \longrightarrow \mathbb{P}^n_0$ already given. $\tilde{f}|_{C_1'} : C_1' \longrightarrow \mathbb{P}^n_1$ is given by $[t^2 : 0 : \cdots : 0 : 1]$ where t is a local parameter of C_1' such that c_1 are given by $\{t = 0\}$ and $D(1)$ is given by

$\{[X_1, X_2, \ldots, X_n] : X_n = 0\}$. $\tilde{f}|_E : E \longrightarrow \mathbb{P}^n_2$ is given by $[x : 0 : \cdots : 0 : z]$ where $D(2)$ are given by $\{[X_1, X_2, \ldots, X_n] : X_n = 0\}$. $\tilde{f}|_{A_1} : A_1 \longrightarrow \mathbb{P}^2_1$ is given by $[1 : \alpha_0 s : \alpha_1 s : \cdots : \alpha_{n-2}s : s]$ where s is a local parameter of A_1 such that a_1 is given by $\{s = 0\}$.

(3) • **$\mathbb{A}^n$ with parameter $\{[\alpha_0 : \alpha_1 : \cdots : \alpha_n], \alpha_n \neq 0\}$:** The domain curve $\tilde{C}$ is of the type $E[C_1]$ and $W = \mathbb{P}^n(1)$. $\tilde{f}|_{C_1} : C_1 \longrightarrow \mathbb{P}^n_0$ already given. $\tilde{f}|_E : E \longrightarrow \mathbb{P}^n_1$ is given by $[X_0 : X_1 : \cdots : X_n] = [\alpha_0 x + \alpha_n y : \alpha_1 x : \alpha_2 : \cdots : \alpha_{n-1} : z]$, where $D(1)$ is given by $\{[X_1, X_2, \ldots, X_n] : X_n = 0\}$.

• **$\mathbb{P}^{n-1}$ \pt with parameter $\{[\alpha_0 : \alpha_1 : \cdots : \alpha_{n-1}]$, not all α_k are 0 for $1 \leqslant k \leqslant n-1\}$:** The domain curve $\tilde{C}$ is of the type $E[C_1'[C_1]]$ and $W = \mathbb{P}^n(2)$. $\tilde{f}|_{C_1} : C_1 \longrightarrow \mathbb{P}^n_0$ is already given. $\tilde{f}|_{C_1'} : C_1' \longrightarrow \mathbb{P}^n_1$ is given by $[1 - \alpha_0 t : \alpha_1 t : \alpha_2 t : \cdots : \alpha_{n-1}t : t^3]$ where t is a local parameter of C_1' such that c_1 is given by $\{t = 0\}$ and $D(1)$ is given by $\{[X_1, X_2, \ldots, X_n] : X_n = 0\}$. $\tilde{f}|_E : E \longrightarrow \mathbb{P}^n_2$ is given by $[x : 0 : \cdots : 0 : z]$ where $D(2)$ are given by $\{[X_1, X_2, \ldots, X_n] : X_n = 0\}$.

• **$\mathbb{A}^{n-1}$ with parameter $\{[\alpha_0 : \alpha_1 : \cdots : \alpha_{n-1}], \alpha_{n-1} \neq 0\}$:** The domain curve $\tilde{C}$ is of the type $E[C_1'[C_1, A_1]]$ and $W = \mathbb{P}^n_2$. $\tilde{f}|_{C_1} : C_1 \longrightarrow \mathbb{P}^n_0$ is already given. $\tilde{f}|_{C_1'} : C_1' \longrightarrow \mathbb{P}^n_1$ is given by $[1 - t : 0 : \cdots : 0 : t^3]$ where t is a local parameter of C_1' such that c_1 is given by $\{t = 0\}$ and a_1 is given by $\{t = 1\}$ and $D(1)$ is given by $\{[X_1, X_2, \ldots, X_n] : X_n = 0\}$. $\tilde{f}|_E : E \longrightarrow \mathbb{P}^n_2$ is given by $[x : 0 : \cdots : 0 : z]$. $\tilde{f}|_{A_1} : A_1 \longrightarrow \mathbb{P}^n_2$ is given by $[1 : \alpha_0 : \alpha_1 : \cdots : \alpha_{n-1}s]$, where s is a local parameter of A_1 such that a_1 is given by $\{s = 0\}$ and $D(2)$ is given by $\{[X_1, X_2, \ldots, X_n] : X_n = 0\}$.

• **$\mathbb{P}^{n-2}$ with parameter $\{[\alpha_0 : \alpha_1 : \cdots : \alpha_{n-2}]\}$:** The domain curve $\tilde{C}$ is of the type $E[C_1''[C_1'[C_1, A_1]]]$ and $W = \mathbb{P}^n(3)$. $\tilde{f}|_{C_1} : C_1 \longrightarrow \mathbb{P}^n_0$ is already given. $\tilde{f}|_{C_1'} : C_1' \longrightarrow \mathbb{P}^n_1$ is given by $[1 - t : 0 : \cdots : 0 : t^3]$ where t is a local parameter of C_1' such that c_1 is given by $\{t = 0\}$ and a_1 is given by $\{t = 1\}$ and $D(1)$ is given by $\{[X_1, X_2, \ldots, X_n] : X_n = 0\}$. $\tilde{f}|_{C_1''} : C_1'' \longrightarrow \mathbb{P}^n_2$ is given by $[1 : 0 : \cdots : 0 : s^2]$ where s is parameter of C_1'' such that c_1' is given by $\{s = 0\}$. $\tilde{f}|_{A_1} : A_1 \longrightarrow \mathbb{P}^n_2$ is given by $[1 : \alpha_0 u : \alpha_1 u : \cdots : \alpha_{n-2}u : u]$ where u is a local parameter of A_1 such that a_1 is given by $\{u = 0\}$ and $D(2)$ is given by $\{[X_1, X_2, \ldots, X_n] : X_n = 0\}$. $\tilde{f}|_E : E \longrightarrow \mathbb{P}^n_3$ is given by $[x : 0 : \cdots : 0 : z]$ where $D(3)$ is given by $\{[X_1, X_2, \ldots, X_n] : X_n = 0\}$. □

Similarly, we have the following results for the domain curves of other types.

Lemma 13. *Let $f : C \longrightarrow \mathbb{P}^n$ be an element of the main component of the moduli space of stable maps and C is of the type $E(0)[C_1(1), C_2(2)]$.*

(1) *If f has ramification order 1 at c_2 and there is no smooth point $q_2 \in C_2$ such that $f(c_2) = f(q_2)$, then the fiber of ϕ is equal to a point, set theoretically.*

(2) *If f has ramification order 1 at c_2 and there is a smooth point $q_2 \in C_2$ such that $f(c_2) = f(q_2)$, then the fiber of ϕ is equal to $\mathbb{P}^n$, set theoretically.*
(3) *If f has ramification order 2 at c_2 and images of C_1 and C_2 are different lines, then fiber of ϕ is equal to $\mathbb{P}^1$, set theoretically.*
(4) *If f has ramification order 2 at c_2 and images of C_1 and C_2 are same lines, then the fiber of ϕ is equal to $\mathbb{P}^{n-1} \bigcup \mathbb{P}^1$ glued at one point, set theoretically.*

Lemma 14. *Let $f : C \longrightarrow \mathbb{P}^n$ be an element of the main component of the moduli space of stable maps and C is of the type $E(0)[C_1(1), C_2(1), C_3(1)]$.*
(1) *If images of C_1 and C_2 and C_3 are distinct lines, then the fiber of ϕ is equal to a point, set theoretically.*
(2) *If images of C_1 and C_2 are same lines and the image of C_3 is the distinct line, then the fiber of ϕ is equal to a point, set theoretically.*
(3) *If images of C_1 and C_2 and C_3 are all same lines, then the fiber of ϕ is equal to $\mathbb{P}^1$, set theoretically.*

Lemma 15. *Let $f : C \longrightarrow \mathbb{P}^n$ be an element of the main component of the moduli space of stable maps satisfying and C is of the type $E(0)[B_1(0)[C_1(1), C_2(2)]]$.*
(1) *If f has ramification order 1 at c_2 and there is no smooth point $q_2 \in C_2$ such that $f(c_2) = f(q_2)$, then the fiber of ϕ is equal to a point, set theoretically.*
(2) *If f has ramification order 1 at c_2 and there is a smooth point $q_2 \in C_2$ such that $f(c_2) = f(q_2)$, then the fiber of ϕ is equal to $\mathbb{P}^{n-1} \bigcup \mathbb{P}^{n-1}$ glued along $\mathbb{P}^{n-2}$, set theoretically.*
(3) *If f has ramification order 2 at c_2 and the tangent lines of images of C_1 and C_2 are independent, then the fiber of ϕ is equal to $\mathbb{P}^1$, set theoretically.*
(4) *If the tangent lines of images of C_1 and C_2 are dependent, then the fiber of ϕ is equal to $Bl_{pt}\mathbb{P}^n \bigcup (\mathbb{P}^1 \times \mathbb{P}^{n-1}) \bigcup Bl_{pt}\mathbb{P}^n$ glued along $\mathbb{P}^{n-1}$, $\mathbb{P}^{n-2}$, set theoretically.*

Lemma 16. *Let $f : C \longrightarrow \mathbb{P}^n$ be an element of the main component of the moduli space of stable maps and C is of the type $E(0)[B_1(0)[C_1(1), C_2(1), C_3(1)]]$.*
(1) *If the images of C_1 and C_2 and C_3 are distinct lines, then the fiber of ϕ is equal to a point, set theoretically.*
(2) *If the images of C_1 and C_2 are same line and the image of C_3 is distinct line, then the fiber of ϕ is equal to a point, set theoretically.*
(3) *If the images of C_1 and C_2 and C_3 are all same lines, then the fiber of ϕ is equal to $(\mathbb{P}^1 \times \mathbb{P}^{n-1}) \bigcup Bl_{pt}\mathbb{P}^n$ glued along $\mathbb{P}^{n-1}$, set theoretically.*

Lemma 17. *Let $f : C \longrightarrow \mathbb{P}^n$ be an element of the main component of the moduli space of stable maps and C is of the type $E(0)[B_1(0)[C_1(1), C_2(1)], C_3(1)]]$.*
(1) *If the images of C_1 and C_2 are distinct lines, then fiber of ϕ is equal to a point, set theoretically.*

(2) *If the images of C_1 and C_2 are same lines and the image of C_3 is distinct line, then the fiber of ϕ is equal to $\mathbb{P}^{n-1}$, set theoretically.*
(3) *If the images of C_1 and C_2 and C_3 are all same lines, then the fiber of ϕ is equal to $(\mathbb{P}^1 \times \mathbb{P}^{n-1}) \bigcup \mathbb{P}^{n-1}$ glued along $\mathbb{P}^{n-2}$, set theoretically.*

Lemma 18. *Let $f : C \longrightarrow \mathbb{P}^n$ be an element of the main component of the moduli space of stable maps and C is of the type $E(0)[B_1(0)[B_2(0)[C_1(1), C_2(1)], C_3(1)]]$.*

(1) *If the images of C_1 and C_2 are distinct lines, then the fiber of ϕ is equal to point, set theoretically.*
(2) *If the images of C_1 and C_2 are same lines and the image of C_3 is the distinct line, then the fiber of ϕ is equal to $\mathbb{P}^{n-1}$, set theoretically.*
(3) *If the images of C_1 and C_2 and C_3 are all same lines, then the fiber of ϕ is equal to $Bl_{pt}\mathbb{P}^n \bigcup Bl_{pt}(\mathbb{P}^{n-1} \times \mathbb{P}^1) \bigcup(\mathbb{P}^{n-1} \times \mathbb{P}^1) \bigcup(\mathbb{P}^{n-1} \times \mathbb{P}^1)$, set theoretically.*

5. The case of the degree 2

5.1. Overview

In this section, we show that for $d = 2$,

$$\widetilde{\mathbf{M}}_{1,0}(\mathbb{P}^n, 2)_0 = \overline{\mathbf{M}}_{1,0}^{log,ch}(\mathbb{P}^n, 2)\,.$$

Note that if the degree is 2, we have

$$\widetilde{\mathbf{M}}_{1,0}(\mathbb{P}^n, 2)_0 = \overline{\mathbf{M}}_{1,0}(\mathbb{P}^n, 2)_0\,,$$

since the blowing-up locus in $\overline{\mathbf{M}}_{1,0}(\mathbb{P}^n, 2)_0$ of the Vakil-Zinger's desingularization is a smooth divisor. By analyzing each fiber, we can easily check that the map

$$\phi : \overline{\mathbf{M}}_{1,0}^{ch}(\mathbb{P}^n, 2) \longrightarrow \widetilde{\mathbf{M}}_{1,0}(\mathbb{P}^n, 2)_0$$

is bijective. Therefore we have

$$\widetilde{\mathbf{M}}_{1,0}(\mathbb{P}^n, 2)_0 = \overline{\mathbf{M}}_{1,0}^{log,ch}(\mathbb{P}^n, 2)$$

by the Zariski's main theorem. Now we construct the morphism ϕ explicitly for $n = 1$. The argument of the construction here will be used in Section 6.

Note that when the essential part is not contracted to a point, two moduli spaces are naturally isomorphic. Therefore we only need to consider the neighborhoods of points where the essential part is contracted to a point.

We describe an etale atlas of $\overline{\mathbf{M}}_{1,0}(\mathbb{P}^1, 2)_0$. Because of the stackness of the moduli space of elliptic curves, we need to separate the case according to j-invariant of the essential part of the domain curve.

5.1.1. *When essential part is smooth elliptic curve with $j \neq 0$.* Let k be an algebraically closed field and t, α, γ, c, A be indeterminates. Motivated by the local descriptions in [2, Section 5.2], we consider the ring

$$R = k[t, \alpha, \gamma, c, A]/(\gamma - \alpha^3 - \gamma^2\alpha - A\gamma^3)\,. \tag{1}$$

Denote by D_1,D_2 the subschemes defined by ideals (α, γ),(t) respectively. Let $S = Spec(R)\backslash V$ where $V \subset Spec(R)$ is a subscheme defined by an ideal $(4 + 27A^2)$ and

$$C' = Proj(R[x, y, z]/zy^2 - x^3 - z^2x - Az^3)\,.$$

If we apply the argument in [2, Section 5.2] to our specific situation, the rational map

$$f' : C' \dashrightarrow \mathbb{P}^1$$

defined by

$$[t\gamma(x + \alpha y) + c(\gamma x - \alpha z), \gamma x - \alpha z]$$

define the family of elliptic stable maps except at D_1 and D_2. Denote by C the blowing up of C' along

$$(D_1, x, z)\,, (D_2, x - \alpha y, z - \gamma y)\,, (D_2, x, z)\,,$$

we can easily check that

$$f' : C' \dashrightarrow \mathbb{P}^1$$

extends to

$$f : C \longrightarrow \mathbb{P}^1$$

and f define a family of elliptic stable maps over entire S.

Moreover we can describe every element of the family over S explicitly as follows:

- Over $\{\gamma = 0, t \neq 0\}$, the domain curves are of the type $E[C_1]$ and
$$f|_{C_1} : C_1 \to \mathbb{P}^1$$
is given by
$$[ts^2 + c(s - 1) : s - 1]\,,$$
where s is a local parameter of C_1 such that c_1 is given by $\{s = 0\}$.
- Over $\{t = 0, \gamma \neq 0\}$, the domain curves are of the type $E[C_1, C_2]$ and c_1 and c_2 are given by $(z = 0)$, $(x = \alpha y, z = \gamma y)$ in
$$E = \{[x; y; z] : zy^2 = x^3 + z^2x + Az^3\}$$
and
$$f|_{C_i} : C_i \to \mathbb{P}^1$$
is given by
$$[s_i + c, 1]\,,$$
where c_i is a local parameter of C_i such that c_i is given by $(s_i = 0)$ for $i = 1, 2$.

- Over $\{\gamma = 0, t = 0\}$, the domain curves are of the type $E[B_1[C_1, C_2]]$ and
$$f|_{C_i} : C_i \to \mathbb{P}^1$$
is given by
$$[s_i + c, 1]\,,$$
where s_i local parameter of C_i such that c_i is given by $(s_i = 0)$ for $i = 1, 2$.

We conclude that S is an etale atlas of $\overline{\mathbf{M}}_{1,0}(\mathbb{P}^1, 2)_0$. Now the following result follow easily from the above descriptions.

Proposition 19. *If we let W be the blow up of $S \times \mathbb{P}^1$ along*
$$(D_2, x_0 - cx_1)\,, (D_1^2, x_0 - cx_1),$$
where x_0, x_1 are coordinates of $\mathbb{P}^1$, then
$$f : C \longrightarrow \mathbb{P}^1$$
extends to
$$\widetilde{f} : C \longrightarrow W$$
and $\widetilde{f}$ define a family of elliptic admissible maps over S.

Remark 20. We can describe every element over S explicitly. Here we only describe the type of domain curves and target. Over $\{t \neq 0, \gamma \neq 0\}$, the domain curve C is of type E and the target $W = \mathbb{P}^1$. Over $\{t = 0, \gamma \neq 0\}$, the domain curve C is of the type $E[C_1, C_2]$ and the target $W = \mathbb{P}^1(1)$. Over $\{t \neq 0, \gamma = 0\}$, the domain curve C is of the type $E[C_1]$ and the target $W = \mathbb{P}^1(1)$. Over $\{t = 0, \gamma = 0\}$, domain $C = E[B_1[C_1, C_2]]$ and the target $W = \mathbb{P}^1(2)$.

5.1.2. *When essential part is smooth elliptic curve with $j \neq 1728$.* If we change the equation of elliptic curves
$$\gamma - \alpha^3 - \gamma^2\alpha - A\gamma^3$$
in (1) to
$$\gamma - \alpha^3 - A\gamma^2\alpha - \gamma^3\,,$$
we have similar result by the argument in Section 5.1.1.

5.1.3. *When essential part is singular curve.* Let
$$R = k[t, \alpha, \beta, c, A]/(\beta^2 - \alpha^3 - \alpha^2 - A)\,.$$
Let D_2 be a subscheme defined by an ideal (t). Let $S = Spec(R)$ and
$$C' = Proj(R[x, y, z]/(zy^2 - x^3 - zx^2 - Az^3))\,.$$
Then the rational map
$$f' : C' \dashrightarrow \mathbb{P}^1$$
defined by
$$[t(y + \beta z) + c(x - \alpha z), x - \alpha z]$$

defines a family of elliptic stable maps except at D_2 and $\{\alpha = \beta = 0\}$. Denote by C be the blowing up of C' along ideals

$$\left(y - x - \frac{x^2 + \alpha x + \alpha^2}{2}, \beta - \alpha - \frac{x^2 + \alpha x + \alpha^2}{2}\right), (D_2, x, z), (D_2, x - \alpha z, y - \beta z).$$

The rational map

$$f' : C' \dashrightarrow \mathbb{P}^1$$

extends to

$$f : C \longrightarrow \mathbb{P}^1$$

and f defines a family of elliptic stable maps over entire S. Note that the effect of blowing up along ideal

$$\left(y - x - \frac{x^2 + \alpha x + \alpha^2}{2}, \beta - \alpha - \frac{x^2 + \alpha x + \alpha^2}{2}\right)$$

is inserting a rational component at singular point in the rational nodal curve at $\{\alpha = \beta = 0\}$. Now from the above description, we have the following results.

Proposition 21. *If we let W be the blow up of $S \times \mathbb{P}^1$ along ideal $(D_2, x_0 - cx_1)$ where x_0,x_1 are coordinates of $\mathbb{P}^1$, then*

$$f : C \longrightarrow \mathbb{P}^1$$

extends to

$$\widetilde{f} : C \longrightarrow W$$

and $\widetilde{f}$ defines a family of elliptic admissible maps over S.

5.2. Conclusion

By the results in Section 5.1, we get a morphism

$$\phi : \widetilde{\mathbf{M}}_{1,0}(\mathbb{P}^1, 2)_0 \to \overline{\mathbf{M}}^{ch}_{1,0}(\mathbb{P}^1, 2)\,.$$

We recall here that

$$\overline{\mathbf{M}}^{ch}_{1,0}(\mathbb{P}^1, 2)$$

is the moduli space of elliptic admissible stable maps without log structures. From the construction in Section 5.1, it is easy to check that ϕ is bijective. On the other hand we also have a bijective morphism

$$\pi : \overline{\mathbf{M}}^{ch,log}_{1,0}(\mathbb{P}^1, 2) \to \overline{\mathbf{M}}^{ch}_{1,0}(\mathbb{P}^1, 2)$$

which is the natural map forgetting log structures. By the uniqueness of the normalization, we conclude that

$$\widetilde{\mathbf{M}}_{1,0}(\mathbb{P}^1, 2)_0 = \overline{\mathbf{M}}^{ch,log}_{1,0}(\mathbb{P}^1, 2)\,.$$

6. The case of the degree 3

6.1. Overview

In this section we describe local charts of

$$\overline{\mathbf{M}}_{1,0}(\mathbb{P}^n,3)_0 \text{ and } \overline{\mathbf{M}}_{1,0}^{log,ch}(\mathbb{P}^n,3)\,.$$

From these explicit description, we prove the main results of the paper. We separate the case according to j-invariant of the essential part of the domain curve.

6.2. Local chart of $\overline{\mathbf{M}}_{1,0}(\mathbb{P}^n,3)_0$

6.2.1. *When the essential part is smooth elliptic curve with $j \neq 0$.* Let k be an algebraically closed field and a_1, a_2, $\dots$, a_{n-1}, b_1, b_2, $\dots$, b_{n-1}, c_1, c_2, $\dots$, c_{n-1}, d_1, d_2, $\dots$, d_n, z_1, z_2, A, α, γ, α', γ' be indeterminates.

If we apply the argument in [2, Section 5.2], we obtain the local charts by modifying the following ring:

$$\begin{aligned}(2)\quad R = k[&a_1,a_2,\dots,a_{n-1},b_1,b_2,\dots,b_{n-1},c_1,c_2,\dots,c_{n-1},d_1,d_2,\dots,d_n,\\ &z_1,z_2,A,\alpha,\gamma,\alpha',\gamma']/(a_1z_1-b_1z_2,a_2z_1-b_2z_2,\dots,a_{n-1}z_1-b_{n-1}z_2,\\ &\qquad \gamma-\alpha^3-\gamma^2\alpha-A\gamma^3,\gamma'-\alpha'^3-\gamma'^2\alpha'-A\gamma'^3)\end{aligned}$$

as follows to construct the family of stable maps of elliptic curves.

Let $D_{2,\alpha}$, $D_{2,\alpha'}$, $D_{2,\alpha-\alpha'}$, D_3, F_α, $F_{\alpha'}$, $F_{\alpha-\alpha'}$, G be subschemes defined by following ideals respectively,

$$\begin{aligned}&(\alpha,\gamma),\\ &(\alpha',\gamma'),\\ &(\alpha-\alpha',\gamma-\gamma'),\\ &(z_1,z_2),\\ &(z_1,b_1,b_2,\dots,b_{n-1}),\\ &(z_2,a_1,a_2,\dots,a_{n-1}),\\ &(z_1-z_2,a_1-b_1,a_2-b_2,\dots,a_{n-1}-b_{n-1}),\\ &\left(\frac{\alpha'\gamma\frac{z_1}{z_2}(x+\alpha y)(\gamma' x-\alpha' z)-\alpha\gamma'(\gamma x-\alpha z)(x+\alpha' y)}{\alpha\alpha'(\alpha-\alpha')(\gamma x-\alpha z)(\gamma' x-\alpha' z)}\right).\end{aligned}$$

Let

$$\widehat{S} = \mathrm{Spec}(R)\setminus V \quad\text{and}\quad S = \mathrm{Bl}_{(\alpha,\alpha',\gamma,\gamma')}\widehat{S}\,,$$

where V is the subscheme defined by the ideal

$$(4+27A^2)(\alpha-\alpha',\gamma-\gamma')\,.$$

Let D_1 be the exceptional divisor. Let

$$\widehat{C} = \mathit{Proj}(R[x,y,z]/zy^2-x^3-z^2x-Az^3)\,,$$

and let C' be the pull-back of $\widehat{C}$ by

$$S \to \widehat{S}$$

and let C be the blow up of C' along the following ideals:
(D_1, x, z),
$(D_{2,\alpha}, x - \alpha' y, z - \gamma' y)$,$(D_{2,\alpha}, x, z)$,$(D_{2,\alpha'}, x - \alpha y, z - \gamma y)$,$(D_{2,\alpha'}, x, z)$,
(D_3, x, z),$(D_3, x - \alpha y, z - \gamma y)$,$(D_3, x - \alpha' y, z - \gamma' y)$,
$(F_\alpha, x - \alpha y, z - \gamma y)$,$(F_{\alpha'}, x - \alpha' y, z - \gamma' y)$,$(F_{\alpha-\alpha'}, x, z)$.
As in [2, Section 5.2], if we define the rational map $\widehat{f} : \widehat{C} \dashrightarrow \mathbb{P}^n$ by
$[\alpha'\gamma(a_1 + c_1)z_1(x + \alpha y)(\gamma' x - \alpha' z) - \alpha\gamma'(b_1 + c_1)z_2(x + \alpha' y)(\gamma x - \alpha z) + d_1(\alpha - \alpha')(\gamma x - \alpha z)(\gamma' z - \alpha' x), \alpha'\gamma(a_2 + c_2)z_1(x + \alpha y)(\gamma' x - \alpha' z) - \alpha\gamma'(b_2 + c_2)z_2(x + \alpha' y)(\gamma x - \alpha z) + d_2(\alpha - \alpha')(\gamma x - \alpha z)(\gamma' z - \alpha' x), \ldots, \alpha'\gamma(a_{n-1} + c_{n-1})z_1(x + \alpha y)(\gamma' x - \alpha' z) - \alpha\gamma'(b_{n-1} + c_{n-1})z_2(x + \alpha' y)(\gamma x - \alpha z) + d_{n-1}(\alpha - \alpha')(\gamma x - \alpha z)(\gamma' z - \alpha' x), \alpha'\gamma z_1(x + \alpha y)(\gamma' x - \alpha' z) - \alpha\gamma' z_2(x + \alpha' y)(\gamma x - \alpha z) + d_n(\alpha - \alpha')(\gamma x - \alpha z)(\gamma' z - \alpha' x), (\alpha - \alpha')(\gamma x - \alpha x)(\gamma' x - \alpha' z)]$,
then it extends to a morphism

$$f : C \longrightarrow \mathbb{P}^n .$$

This define a family of semi-stable maps of elliptic curves and after the stabilization we obtain a family of stable maps of elliptic curves over S.

6.2.2. *When essential part is smooth elliptic curve with* $j \neq 1728$. If we change the equation of elliptic curves

$$\gamma - \alpha^3 - \gamma^2\alpha - A\gamma^3$$

in (2) to

$$\gamma - \alpha^3 - A\gamma^2\alpha - \gamma^3 ,$$

we have similar result by the argument in Section 6.2.1.

6.2.3. *When essential part is singular curve.* Let k be an algebraically closed field and $a_1, a_2, \ldots, a_{n-1}, b_1, b_2, \ldots, b_{n-1}, c_1, c_2, \ldots, c_{n-1}, d_1, d_2, \ldots, d_n, z_1, z_2, A, \alpha, \beta, \alpha', \beta'$ be indeterminates. Let

$$\begin{aligned} R = k[&a_1, a_2, \ldots, a_{n-1}, b_1, b_2, \ldots, b_{n-1}, c_1, c_2, \ldots, c_{n-1}, d_1, d_2, \ldots, d_n, \\ &z_1, z_2, A, \alpha, \beta, \alpha', \beta']/(a_1 z_1 - b_1 z_2, a_2 z_1 - b_2 z_2, \ldots, a_{n-1} z_1 - b_{n-1} z_2, \\ &\beta^2 - \alpha^3 - \alpha - A, \beta' - \alpha'^3 - \alpha' - A). \end{aligned}$$

Let D_2, D_3, F_α, $F_{\alpha'}$, $F_{\alpha-\alpha'}$, G be the subscheme defined by the following ideals respectively,

$$\begin{aligned} &(\alpha - \alpha', \beta - \beta'), \\ &(z_1, z_2), \\ &(z_1, b_1, b_2, \ldots, b_{n-1}), \\ &(z_2, a_1, a_2, \ldots, a_{n-1}), \\ &(z_1 - z_2, a_1 - b_1, a_2 - b_2, \ldots, a_{n-1} - b_{n-1}), \end{aligned}$$

$$(\frac{z_1}{z_2}(y+\beta)(x-\alpha')-(y+\beta')(x-\alpha)).$$

Let

$$\widehat{S}=\mathrm{Spec}(R) \quad\text{and}\quad S=\mathrm{Bl}_{(\beta-\alpha-\frac{\alpha^2+\alpha\alpha'+\alpha'^2}{2},\beta'-\alpha'-\frac{\alpha^2+\alpha\alpha'+\alpha'^2}{2})}\widehat{S}\,.$$

Let

$$\widehat{C}=\mathrm{Proj}(R[x,y,z]/zy^2-x^3-x^2z-Az^3)\,,$$

let C' be a pull back of $\widehat{C}$, and let C be the blow up of C' along the ideals:
$\left(y-x-\frac{x^2+\alpha x+\alpha^2}{2},\beta-\alpha-\frac{x^2+\alpha x+\alpha^2}{2}\right)$,
$\left(y-x-\frac{x^2+\alpha' x+\alpha'^2}{2},\beta'-\alpha'-\frac{x^2+\alpha' x+\alpha'^2}{2}\right)$,
$(D_2,x-\alpha z,y-\beta z)$,$(D_2,x,z)$,
(D_3,x,z),$(D_3,x-\alpha z,y-\beta z)$,$(D_3,x-\alpha' z,y-\beta')$,
$(F_\alpha,x-\alpha z,y-\beta z)$,$(F_{\alpha'},x-\alpha' z,y-\beta' z)$,$(F_{\alpha-\alpha'},x,z)$.
Then the rational map $\widehat{f}:\widehat{C}\dashrightarrow\mathbb{P}^n$ defined by
$[(\alpha-\alpha')(a_1+c_1)z_1(y+\beta z)(x-\alpha' z)-(\alpha-\alpha')(b_1+c_1)z_2(y+\beta' z)(x-\alpha z)+d_1(\beta-\beta')(x-\alpha z)(x-\alpha' z),(\alpha-\alpha')(a_2+c_2)z_1(y+\beta z)(x-\alpha' z)-(\alpha-\alpha')(b_2+c_2)z_2(y+\beta' z)(x-\alpha z)+d_1(\beta-\beta')(x-\alpha z)(x-\alpha' z),\ldots,(\alpha-\alpha')(a_{n-1}+c_{n-1})z_1(y+\beta z)(x-\alpha' z)-(\alpha-\alpha')(b_{n-1}+c_{n-1})z_2(y+\beta' z)(x-\alpha z)+d_{n-1}(\beta-\beta')(x-\alpha z)(x-\alpha' z),(\alpha-\alpha')z_1(y+\beta z)(x-\alpha' z)-(\alpha-\alpha')z_2(y+\beta' z)(x-\alpha z)+d_n(\beta-\beta')(x-\alpha z)(x-\alpha' z),(\beta-\beta')(x-\alpha z)(x-\alpha' z)]$
extends to a morphism

$$f:C\longrightarrow\mathbb{P}^n\,.$$

This defines a family of semi-stable maps of elliptic curves and after the stabilization we obtain a family of stable map of elliptic curves over S.

6.3. Local chart of $\overline{\mathbf{M}}^{log,ch}_{1,0}(\mathbb{P}^n,3)$

6.3.1. *When essential part is smooth elliptic curve with $j\neq 0$.* In the previous local chart S of $\overline{\mathbf{M}}_{1,0}(\mathbb{P}^2,3)_0$, the blow-up center of Vakil-Zinger desingularization is given by D_3. And $\sum_1$, $\sum_2$, Γ_1, Γ_2 are given by proper transforms of
$(D_1,a_1,a_2,\ldots,a_{n-1},b_1,b_2,\ldots,b_{n-1},\alpha z_1+\alpha' z_2)$,
$(D_{2,\alpha},z_1,b_1,b_2,\ldots,b_{n-1})(D_{2,\alpha'},z_2,a_1,a_2,\ldots,a_{n-1})$,
$(D_1,a_1,a_2,\ldots,a_{n-1},b_1,b_2,\ldots,b_{n-1})$,
$(D_{2,\alpha},a_1,a_2,\ldots,a_{n-1},b_1,b_2,\ldots,b_{n-1})$
$(D_{2,\alpha'},a_1,a_2,\ldots,a_{n-1},b_1,b_2,\ldots,b_{n-1})$.
Let $\widetilde{S}$ be the blow up of S along D_3, $\sum_2$, Γ_2, $\sum_1$, Γ_1 and let E_1, $E_{2,\alpha}\bigcup E_{2,\alpha'}$, L_1, $L_{2,\alpha}\bigcup L_{2,\alpha'}$ be the exceptional divisors corresponding to $\sum_1$, $\sum_2$, Γ_1, Γ_2. Note that after blowing up along $\sum_2$, $\sum_1$ and Γ_2 are separated. Now let C'' be the pull back of $\widehat{C}$ along $\widetilde{S}$ and let $\widetilde{C}$ be the blow up of C'' along ideals:
(D_1,x,z), (L_1,x,z), (E_1,x,z),
$(D_{2,\alpha},x-\alpha' y,z-\gamma' y)$, $(L_{2,\alpha},x-\alpha' y,z-\gamma' y)$, $(E_{2,\alpha},x-\alpha' y,z-\gamma' y)$,
$(D_{2,\alpha},x,z)$, $(L_{2,\alpha},x,z)$, $(E_{2,\alpha},x,z)$,

$(D_{2,\alpha'}, x-\alpha y, z-\gamma y)$, $(L_{2,\alpha'}, x-\alpha y, z-\gamma y)$, $(E_{2,\alpha'}, x-\alpha y, z-\gamma y)$,
$(D_{2,\alpha'}, x, z)$, $(L_{2,\alpha'}, x, z)$, $(E_{2,\alpha'}, x, z)$,
(D_3, x, z), $(D_3, x-\alpha y, z-\gamma y)$, $(D_3, x-\alpha' y, z-\gamma' y)$,
$(F_\alpha, x-\alpha y, z-\gamma y)$, $(F_{\alpha'}, x-\alpha' y, z-\gamma' y)$, $(F_{\alpha-\alpha'}, x, z)$,
$(\widetilde{L}_1^2, G)$, $(\widetilde{L}_{2,\alpha}, G)$, $(\widetilde{L}_{2,\alpha'}, G)$,
where $\widetilde{L}_1$, $\widetilde{L}_{2,\alpha}$, $\widetilde{L}_{2,\alpha}$ are exceptional divisor of (L_1, x, z), $(L_{2,\alpha}, x, z)$, $(L_{2,\alpha'}, x, z)$.
Let $\widetilde{W}$ be the blow-up of $\widetilde{S} \times \mathbb{P}^2$ along ideals:
$(D_3, x_0 - d_1x_n, x_1 - d_2x_n, \dots, x_{n-1} - d_nx_n)$,
$(E_2^2, x_0 - d_1x_n, x_1 - d_2x_n, \dots, x_{n-1} - d_nx_n)$,
$(L_2, x_0 - d_1x_n, x_1 - d_2x_n, \dots, x_{n-1} - d_nx_n)$,
$(D_2, x_0 - d_1x_n, x_1 - d_2x_n, \dots, x_{n-1} - d_nx_n)$,
$(E_1^3, x_0 - d_1x_n, x_1 - d_2x_n, \dots, x_{n-1} - d_nx_n)$,
$(L_1^2, x_0 - d_1x_n, x_1 - d_2x_n, \dots, x_{n-1} - d_nx_n)$,
$(D_1^2, x_0 - d_1x_n, x_1 - d_2x_n, \dots, x_{n-1} - d_nx_n)$,
where $x_0, x_1, \dots, x_n$ are coordinates of $\mathbb{P}^n$.
Then the rational map

$$\widehat{f} : \widehat{C} \dashrightarrow \mathbb{P}^n$$

extends to

$$\widetilde{f} : \widetilde{C} \longrightarrow \widetilde{W}$$

and we obtain a family of admissible maps over $\widetilde{S}$.

6.3.2. *When the essential part is smooth elliptic curve with $j \neq 1728$.* We obtain a similar result, if we change the equation of elliptic curve $\gamma - \alpha^3 - \gamma^2\alpha - A\gamma^3$ in Section 6.3.1 to $\gamma - \alpha^3 - A\gamma^2\alpha - \gamma^3$.

6.3.3. *When the essential part is singular curve.* In the previous local chart S of $\overline{\mathbf{M}}_{1,0}(\mathbb{P}^2, 3)_0$, the blow up center of Vakil-Zinger desingularization is given by D_3. And $\sum_2$, Γ_2 are given by proper transforms of $(D_2, z_1 - z_2, a_1 - b_1)$, (D_2, a_1, b_1).

Let $\widetilde{S}$ be the blow up of S along D_3, $\sum_2$, Γ_2 and let E_2, L_2 be the exceptional divisors corresponding to $\sum_2$, Γ_2. Denote by C'' the pull back of C' along $\widetilde{S}$ and let $\widetilde{C}$ be the blow up of C'' along ideals:
$(D_2, x-\alpha z, y-\beta z)$, (D_2, x, z), $(L_2, x-\alpha z, y-\beta z)$, (L_2, x, z),
$(E_2, x-\alpha z, y-\beta z)$, (E_2, x, z),
(D_3, x, z), $(D_3, x-\alpha z, y-\beta z)$, $(D_3, x-\alpha' z, y-\beta' z)$,
$(F_\alpha, x-\alpha z, y-\beta z)$, $(F_{\alpha'}, x-\alpha' z, y-\beta' z)$, $(F_{\alpha-\alpha'}, x, z)$, $(\widetilde{L}_2, G)$,
where $\widetilde{L}_2$ is exceptional divisor of (L_2, x, z).
Let $\widetilde{W}$ be blow-up of $\widetilde{S} \times \mathbb{P}^2$ along ideal:
$(D_3, x_0 - d_1x_n, x_1 - d_2x_n, \dots, x_{n-1} - d_nx_n)$,
$(E_2^2, x_0 - d_1x_n, x_1 - d_2x_n, \dots, x_{n-1} - d_nx_n)$,
$(L_2, x_0 - d_1x_n, x_1 - d_2x_n, \dots, x_{n-1} - d_nx_n)$,

$(D_2, x_0 - d_1 x_n, x_1 - d_2 x_n, \ldots, x_{n-1} - d_n x_n)$.
Then the rational map

$$\widehat{f} : \widehat{C} \dashrightarrow \mathbb{P}^n$$

extends to

$$\widetilde{f} : \widetilde{C} \longrightarrow \widetilde{W}$$

and we get the family of admissible maps over $\widetilde{S}$.

6.4. Proof of Theorem 1

Let $\widehat{\mathbf{M}}$ be the blowing up of

$$\widetilde{\mathbf{M}}_{1,0}(\mathbf{P}^n, 3)_0$$

along the ideals $\sum_2, \Gamma_2, \sum_1, \Gamma_1$. By the results from the previous subsections, we obtain a morphism from $\widehat{\mathbf{M}}$ to $\overline{\mathbf{M}}^{ch}_{1,0}(\mathbb{P}^n, 3)$. Here $\overline{\mathbf{M}}^{ch}_{1,0}(\mathbb{P}^n, 3)$ is moduli space of admissible stable maps of chain type without log structures. One can check that this morphism is finite and surjective by the result of section 5. Actually it is one to one morphism. On the other hand, we also have a finite surjective map

$$\pi : \overline{\mathbf{M}}^{log,ch}_{1,0}(\mathbb{P}^n, 3) \longrightarrow \overline{\mathbf{M}}^{ch}_{1,0}(\mathbb{P}^n, 3)$$

which is the natural map forgetting log structures. The proof of the theorem follows from the uniqueness of the normalization.

Remark 22. In the proof above, we only used the fact that the forgetting morphism

$$\pi : \overline{\mathbf{M}}^{log,ch}_{1,0}(\mathbb{P}^n, 3) \longrightarrow \overline{\mathbf{M}}^{ch}_{1,0}(\mathbb{P}^n, 3)$$

is a finite morphism. From the proof above, we conclude that π is bijective. We can also obtain this fact by explicitly calculating possible log structures. Therefore, when $d \leqslant 3$, there exists unique log structure on each admissible stable map. If $d \geqslant 4$, there could be more than one log structures on each admissible stable map.

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