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A VAN DER CORPUT TYPE LEMMA FOR OSCILLATORY INTEGRALS WITH HÖLDER AMPLITUDES AND ITS APPLICATIONS

Hussain Al-Qassem, Leslie Cheng, and Yibiao Pan

Abstract. We prove a decay estimate for oscillatory integrals with Hölder amplitudes and polynomial phases. The estimate allows us to answer certain questions concerning the uniform boundedness of oscillatory singular integrals on various spaces.

1. Introduction

In the study of oscillatory integrals, a well-known result which is of fundamental importance is the van der Corput's lemma. It has found applications throughout many branches of mathematics such as harmonic analysis, functional analysis, number theory, differential equations, probability theory, to name just a few ([3, 6, 7, 9, 11–14, 16, 18, 20, 24, 25]). The result can be stated as follows:

Theorem 1.1. (i) *Let ϕ be a real-valued C^k function on $[a, b]$ satisfying $|\phi^{(k)}(x)| \geq 1$ for every $x \in [a, b]$. Suppose that $k \geq 2$, or that $k = 1$ and ϕ' is monotone on $[a, b]$. Then there exists a positive constant c_k such that*

$$\left| \int_a^b e^{i\lambda\phi(x)} dx \right| \leq c_k |\lambda|^{-1/k} \tag{1}$$

for all $\lambda \in \mathbb{R}$. The constant c_k is independent of λ, a, b and ϕ.

(ii) *Let ϕ and c_k be the same as in* (i). *If $\psi \in C^1([a, b])$, then*

$$\left| \int_a^b e^{i\lambda\phi(x)} \psi(x) dx \right| \leq c_k |\lambda|^{-1/k} (\|\psi\|_{L^\infty([a,b])} + \|\psi'\|_{L^1([a,b])}) \tag{2}$$

holds for all $\lambda \in \mathbb{R}$.

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Part (i) of the above theorem is what is classically called van der Corput's lemma. Part (ii) (which follows from (i) by integration by parts) is the version one usually finds convenient to use in many applications.

On the other hand, there are problems in which one encounters oscillatory integrals whose amplitudes are not C^1 (or nearly C^1). The main purpose of this paper is to establish a version of (2) when the amplitude ψ is in $C^{0,\alpha}$, the Hölder classes.

Recall that, for any $0 < \alpha \le 1$ and $f : [a, b] \to \mathbb{C}$,

$$\|f\|_{C^{0,\alpha}([a,b])} = \sup\left\{\frac{|f(x) - f(y)|}{|x - y|^\alpha} : x, y \in [a, b] \text{ and } x \ne y\right\}$$

and

$$C^{0,\alpha}([a, b]) = \left\{f : \|f\|_{C^{0,\alpha}([a,b])} < \infty\right\}.$$

When the phase function ϕ is linear, the corresponding oscillatory integral with amplitude ψ can be treated essentially as a Fourier coefficient of ψ. It is known classically to behave as $O(|\lambda|^{-\alpha})$ when $|\lambda| \to \infty$, if $\psi \in C^{0,\alpha}$ (see page 36 of [8]).

We have the following:

Theorem 1.2. *Let $p(x)$ denote a real-valued monic polynomial of degree k and $I = [a, b]$. Suppose that $\psi \in C^{0,\alpha}(I)$ where $\alpha \in (0, 1/(k-1))$ when $k \ge 2$, or $\alpha \in (0, 1]$ when $k = 1$. Then there exists a $d_{k,\alpha} > 0$ such that*

$$\left|\int_I e^{i\lambda p(x)}\psi(x)dx\right| \le d_{k,\alpha}(|\lambda|^{-1/k}\|\psi\|_{L^\infty(I)} + |\lambda|^{-\alpha}|I|^{1-(k-1)\alpha}\|\psi\|_{C^{0,\alpha}(I)}) \tag{3}$$

for all $\lambda \in \mathbb{R}$. The constant $d_{k,\alpha}$ is independent of $\lambda, a, b, p(x)$ and $\psi(x)$.

For $1/k \le \alpha < 1/(k-1)$, (3) provides the decay rate of $|\lambda|^{-1/k}$ which is known to be optimal. But for $\alpha < 1/k$, it is unclear whether (3) still provides us with the optimal decay rate as $|\lambda| \to \infty$. Another interesting question for future investigation is whether (3) continues to hold for more general phase functions whose k-th derivative is bounded away from 0.

It should be pointed out that the case $\alpha = 1$ is not of primary concern here, because it falls under a trivial extension of (2) to oscillatory integrals with amplitudes having bounded variations.

We shall present the proof of Theorem 1.2 in Section 2. Applications will be discussed in Section 3. There we will consider oscillatory singular integral operators with polynomial phases and Calderón–Zygmund kernels in the Hölder classes and explore their boundedness on various classical spaces. For L^p spaces, our work extends the well-known results of Ricci and Stein in [19] for the family of C^1 Calderón–Zygmund kernels to the family of $C^{0,\alpha}$ Calderón–Zygmund kernels. This particular theme of research began in [1] and [2]. As one will see in Section 3, the questions raised in [2] are now answered in the affirmative.

2. Proof of Theorem 1.2

By a change of variable if necessary, it suffices to show that there exists a constant $c_{k,\alpha}$, which depends on k and α only, such that

$$\left|\int_0^1 e^{i\lambda p(x)}\psi(x)dx\right| \le c_{k,\alpha}(\lambda^{-1/k}\|\psi\|_{L^\infty([0,1])} + \lambda^{-\alpha}\|\psi\|_{C^{0,\alpha}([0,1])}) \tag{4}$$

holds for all $\lambda > 2$. Also, we will assume that $k \ge 2$ and leave out the easier case of $k = 1$. Below we shall use $A \lesssim B$ to mean that $A \le cB$ for a certain constant c which depends on some essential parameters only (those being k and α in this instance).

Let $Q(x) = p'(x)$. There exist $\zeta_1, \dots, \zeta_{k-1} \in \mathbb{C}$ (not necessarily distinct) such that

$$Q(x) = k\prod_{j=1}^{k-1}(x - \zeta_j).$$

Let

$$\Omega = [0,1] \cap \left(\bigcup_{j=1}^{k-1}[|\zeta_j| - 2\lambda^{-1/k}, |\zeta_j| + 2\lambda^{-1/k}]\right).$$

Trivially we have

$$\left|\int_\Omega e^{i\lambda p(x)}\psi(x)dx\right| \lesssim \lambda^{-1/k}\|\psi\|_{L^\infty([0,1])}. \tag{5}$$

The set $[0,1]\backslash\Omega$ is the union of at most k intervals. Let $J = [u, v]$ be one such interval. For $x \in J \subseteq [0,1]\backslash\Omega$ and $1 \le j \le k-1$,

$$|x - \zeta_j| \ge ||x| - |\zeta_j|| = |x - |\zeta_j|| > 2\lambda^{-1/k}.$$

Thus we have

$$|Q(x)| = k\prod_{j=1}^{k-1}|x - \zeta_j| \ge 2^{k-1}k\lambda^{-1+1/k}$$

for $x \in J$. Without loss of generality, we may assume that $Q(x) > 0$ for all $x \in J$. It follows that $p|_J$ has a strictly increasing inverse function which we shall denote by $h(\cdot)$.

If $p(v) - p(u) \le 2\pi\lambda^{-1}$, then

$$\begin{aligned}\left|\int_J e^{i\lambda p(x)}\psi(x)dx\right| &\le (v-u)\|\psi\|_{L^\infty([0,1])}\\ &\le (\sup_{x\in J}|Q(x)|^{-1})(p(v)-p(u))\|\psi\|_{L^\infty([0,1])}\\ &\lesssim \lambda^{-1/k}\|\psi\|_{L^\infty([0,1])}.\end{aligned}$$

We may now assume that $p(v) - p(u) > 2\pi\lambda^{-1}$. By letting $\gamma = h(p(u) + \pi\lambda^{-1})$ and $\eta = h(p(v) - \pi\lambda^{-1})$, we have $p(\gamma) - p(u) = \pi\lambda^{-1}$ and $p(v) - p(\eta) = \pi\lambda^{-1}$.

It follows from the preceding argument that

$$\left|\int_u^\gamma e^{i\lambda p(x)}\psi(x)dx\right| \lesssim \lambda^{-1/k}\|\psi\|_{L^\infty([0,1])}; \tag{6}$$

$$\left|\int_\eta^v e^{i\lambda p(x)}\psi(x)dx\right| \lesssim \lambda^{-1/k}\|\psi\|_{L^\infty([0,1])}. \tag{7}$$

Let $\Phi(t) = \psi(h(t))h'(t)$. Then we have

$$\begin{aligned}
&\int_J e^{i\lambda p(x)}\psi(x)dx\\
&= \int_u^v e^{i\lambda p(x)}\psi(x)dx\\
&= \left(\frac{1}{2}\right)\left[\int_u^\gamma e^{i\lambda p(x)}\psi(x)dx + \int_\eta^v e^{i\lambda p(x)}\psi(x)dx\right]\\
&\quad + \left(\frac{1}{2}\right)\left[\int_u^\eta e^{i\lambda p(x)}\psi(x)dx + \int_\gamma^v e^{i\lambda p(x)}\psi(x)dx\right]\\
&= \left(\frac{1}{2}\right)\left[\int_u^\gamma e^{i\lambda p(x)}\psi(x)dx + \int_\eta^v e^{i\lambda p(x)}\psi(x)dx\right]\\
&\quad + \left(\frac{1}{2}\right)\left[\int_{p(u)}^{p(\eta)} e^{i\lambda t}\Phi(t)dt + \int_{p(\gamma)}^{p(v)} e^{i\lambda t}\Phi(t)dt\right]\\
&= \left(\frac{1}{2}\right)\left[\int_u^\gamma e^{i\lambda p(x)}\psi(x)dx + \int_\eta^v e^{i\lambda p(x)}\psi(x)dx\right]\\
&\quad + \left(\frac{1}{2}\right)\left[\int_{p(u)}^{p(\eta)} e^{i\lambda t}\Phi(t)dt + \int_{p(\gamma)-\pi\lambda^{-1}}^{p(v)-\pi\lambda^{-1}} e^{i\lambda(t+\pi\lambda^{-1})}\Phi(t+\pi\lambda^{-1})dt\right]\\
&= \left(\frac{1}{2}\right)\left\{\int_{[u,\gamma]\cup[\eta,v]} e^{i\lambda p(x)}\psi(x)dx + \int_{p(u)}^{p(\eta)} e^{i\lambda t}[\Phi(t) - \Phi(t+\pi\lambda^{-1})]dt\right\}.
\end{aligned}$$

In light of (6)-(7) and the following inequality:

$$\begin{aligned}
&|\Phi(t) - \Phi(t+\pi\lambda^{-1})|\\
&\le |\psi(h(t)) - \psi(h(t+\pi\lambda^{-1}))|h'(t) + |\psi(h(t+\pi\lambda^{-1}))||h'(t) - h'(t+\pi\lambda^{-1})|\\
&\le \|\psi\|_{C^{0,\alpha}}|h(t) - h(t+\pi\lambda^{-1})|^\alpha h'(t) + \|\psi\|_\infty|h'(t) - h'(t+\pi\lambda^{-1})|,
\end{aligned}$$

it suffices to prove that

$$\int_{p(u)}^{p(\eta)} |h(t) - h(t+\pi\lambda^{-1})|^\alpha h'(t)dt \lesssim \lambda^{-\alpha} \tag{8}$$

and

$$\int_{p(u)}^{p(\eta)} |h'(t) - h'(t+\pi\lambda^{-1})|dt \lesssim \lambda^{-1/k}. \tag{9}$$

For any $t \in [p(u), p(\eta)]$ and $t \leq s \leq t + \pi\lambda^{-1}$, we have

$$\begin{aligned}|h(s) - h(t)| &\leq \left(\sup_{x \in J} |Q(x)|^{-1}\right) |s - t| \\ &\leq 2^{1-k} k^{-1} \lambda^{1-1/k} (\pi\lambda^{-1}) \leq (\pi/4)\lambda^{-1/k} < \lambda^{-1/k},\end{aligned}$$

and, for $1 \leq j \leq k-1$,

$$|h(s) - \zeta_j| \geq |h(t) - \zeta_j| - |h(s) - h(t)| \geq |h(t) - \zeta_j| - \lambda^{-1/k}.$$

Since $h(t) \in [u, \eta] \subset J$,

$$|h(t) - \zeta_j| \geq |h(t) - |\zeta_j|| \geq 2\lambda^{-1/k},$$

which implies that

$$|h(s) - \zeta_j| \geq (1/2)|h(t) - \zeta_j|. \tag{10}$$

Thus, for each $t \in [p(u), p(\eta)]$, there exists a $\tau \in [t, t + \pi\lambda^{-1}]$ such that

$$\begin{aligned}|h(t) - h(t + \pi\lambda^{-1})| &= \pi\lambda^{-1} |Q(h(\tau))|^{-1} \\ &= \pi\lambda^{-1} k^{-1} \left(\prod_{j=1}^{k-1} |h(\tau) - \zeta_j|\right)^{-1} \\ &\leq 2^{k-1} k^{-1} \pi\lambda^{-1} \left(\prod_{j=1}^{k-1} |h(t) - \zeta_j|\right)^{-1} \\ &\leq 2^{k-1} k^{-1} \pi\lambda^{-1} \left(\prod_{j=1}^{k-1} |h(t) - |\zeta_j||\right)^{-1}.\end{aligned}$$

It follows that

$$\int_{p(u)}^{p(\eta)} |h(t) - h(t + \pi\lambda^{-1})|^{\alpha} h'(t) dt \lesssim \lambda^{-\alpha} \int_u^{\eta} \left(\prod_{j=1}^{k-1} |x - |\zeta_j||\right)^{-\alpha} dx.$$

For each $j \in \{1, \ldots, k-1\}$, let $\xi_j = \min\{|\zeta_j|, 1\}$. Then, $\forall x \in [u, \eta] \subseteq [0, 1]$, we have $|x - \xi_j| \in [0, 1]$, $|x - |\zeta_j|| \geq |x - \xi_j|$ and

$$\begin{aligned}\left(\prod_{j=1}^{k-1} |x - |\zeta_j||\right)^{-\alpha} &\leq \left(\min_{1 \leq j \leq k-1} |x - |\zeta_j||\right)^{-\alpha(k-1)} \\ &\leq \sum_{j=1}^{k-1} |x - |\zeta_j||^{-\alpha(k-1)} \leq \sum_{j=1}^{k-1} |x - \xi_j|^{-\alpha(k-1)}.\end{aligned}$$

Since $\alpha < 1/(k-1)$, we thus have

$$\int_{p(u)}^{p(\eta)} |h(t) - h(t + \pi\lambda^{-1})|^{\alpha} h'(t) dt \lesssim \lambda^{-\alpha} \left(\sum_{j=1}^{k-1} \int_0^1 |x - \xi_j|^{-\alpha(k-1)} dx\right)$$

$$\lesssim \lambda^{-\alpha} \int_0^1 \omega^{-\alpha(k-1)} d\omega \lesssim \lambda^{-\alpha},$$

which proves (8).

Also, for each $t \in [p(u), p(\eta)]$, there exists a $\tilde{\tau} \in [t, t + \pi\lambda^{-1}]$ such that

$$|h'(t) - h'(t + \pi\lambda^{-1})| = \left(\frac{\pi}{\lambda}\right) \frac{|Q'(h(\tilde{\tau}))|}{|Q(h(\tilde{\tau}))|^3}.$$

By letting $s = \tilde{\tau}$ in (10), we get

$$|h(\tilde{\tau}) - \zeta_j|^{-1} \le 2|h(t) - \zeta_j|^{-1}$$

for $1 \le j \le k-1$. Thus,

$$\begin{aligned}
|h'(t) - h'(t + \pi\lambda^{-1})| &\lesssim \lambda^{-1} \left(\sum_{j=1}^{k-1} \frac{1}{|h(\tilde{\tau}) - \zeta_j|} \right) \left(\frac{1}{|Q(h(\tilde{\tau}))|^2} \right) \\
&= \lambda^{-1} \left(\sum_{j=1}^{k-1} \frac{1}{|h(\tilde{\tau}) - \zeta_j|} \right) \left(k^{-1} \prod_{j=1}^{k-1} |h(\tilde{\tau}) - \zeta_j|^{-1} \right)^2 \\
&\lesssim \lambda^{-1} \left(\sum_{j=1}^{k-1} \frac{1}{|h(t) - \zeta_j|} \right) \left(k^{-1} \prod_{j=1}^{k-1} |h(t) - \zeta_j|^{-1} \right)^2 \\
&= \lambda^{-1} \left(\sum_{j=1}^{k-1} \frac{1}{|h(t) - \zeta_j|} \right) \left(\frac{1}{|Q(h(t))|^2} \right) \\
&= \lambda^{-1} h'(t) \left(\sum_{j=1}^{k-1} \frac{1}{|h(t) - \zeta_j|} \right) \left(\frac{1}{|Q(h(t))|} \right).
\end{aligned}$$

Thus

$$\int_{p(u)}^{p(\eta)} |h'(t) - h'(t + \pi\lambda^{-1})| dt \lesssim \lambda^{-1} \int_u^\eta \left(\sum_{j=1}^{k-1} \frac{1}{|x - \zeta_j|} \right) \left(\frac{1}{|Q(x)|} \right) dx.$$

For $x \in [u, \eta]$ and $1 \le j \le k-1$, by combining $|x - \zeta_j| \ge |x - |\zeta_j|| \ge |x - \xi_j|$ and $|x - |\zeta_j|| \ge 2\lambda^{-1}$, we have

$$|x - \zeta_j| \ge (1/2)|x - \xi_j| + \lambda^{-1},$$

which implies that

$$\begin{aligned}
\left(\sum_{j=1}^{k-1} \frac{1}{|x - \zeta_j|} \right) \left(\frac{1}{|Q(x)|} \right) &\le (k-1) \left(\min_{1 \le j \le k-1} |x - \zeta_j| \right)^{-1} |Q(x)|^{-1} \\
&\le \left(\frac{k-1}{k} \right) \left(\min_{1 \le j \le k-1} |x - \zeta_j| \right)^{-k} \\
&\le \left(\frac{k-1}{k} \right) \sum_{j=1}^{k-1} \frac{1}{|x - \zeta_j|^k}
\end{aligned}$$

$$\leq \left(\frac{k-1}{k}\right) \sum_{j=1}^{k-1} \frac{1}{[(1/2)|x-\xi_j| + \lambda^{-1}]^k}.$$

Since $x, \xi_j \in [0,1]$ and $0 < \lambda^{-1} < 1/2$, we also have $(1/2)|x-\xi_j| + \lambda^{-1} \in [\lambda^{-1}, 1]$. Thus,

$$\int_{p(u)}^{p(\eta)} |h'(t) - h'(t+\pi\lambda^{-1})| dt \lesssim \lambda^{-1} \left(\sum_{j=1}^{k-1} \int_u^\eta \frac{dx}{[(1/2)|x-\xi_j| + \lambda^{-1}]^k} \right)$$
$$\lesssim \lambda^{-1} \int_{\lambda^{-1/k}}^1 \frac{d\omega}{\omega^k} \lesssim \lambda^{-1/k},$$

which proves (9). The proof of Theorem 1.2 is now complete.

3. Oscillatory singular integrals

Let $n \in \mathbb{N}$, $P(x,y)$ be a real-valued polynomial in $x, y \in \mathbb{R}^n$ and $K(x,y)$ be a singular kernel. Consider the following oscillatory singular integral operator:

$$T_{P,K} : f \to \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x,y)} K(x,y) f(y) dy. \tag{11}$$

The standard conditions used to define Calderón–Zygmund kernels which have C^1 smoothness away from their singularities are as follows:

There exists a $B > 0$ such that

(i) For all $(x,y) \in (\mathbb{R}^n \times \mathbb{R}^n) \backslash \Delta$ where $\Delta = \{(x,x) : x \in \mathbb{R}^n\}$,

$$|K(x,y)| \leq \frac{B}{|x-y|^n}; \tag{12}$$

(ii) $K(x,y) \in C^1((\mathbb{R}^n \times \mathbb{R}^n)\backslash\Delta)$, and for $(x,y) \in (\mathbb{R}^n \times \mathbb{R}^n)\backslash\Delta$

$$|\nabla_x K(x,y)| + |\nabla_y K(x,y)| \leq \frac{B}{|x-y|^{n+1}}; \tag{13}$$

(iii)

$$\|T_o\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} \leq B, \tag{14}$$

where

$$T_o f(x) = \text{p.v.} \int_{\mathbb{R}^n} K(x,y) f(y) dy. \tag{15}$$

Let's begin with the following result of F. Ricci and E. M. Stein concerning L^p boundedness.

Theorem 3.1 ([19]). *Suppose that conditions* (i), (ii), (iii) *are satisfied. Then, for $1 < p < \infty$, there exists a $C_p > 0$ such that*

$$\|T_{P,K} f\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)} \tag{16}$$

for all $f \in L^p(\mathbb{R}^n)$. The constant C_p may depend on p, n, B and $\deg(P)$ *but is independent of the coefficients of P.*

The above theorem is a strengthened form of Theorem 1 in [19], as described in Section 5 of [19]. For earlier results on operators with bilinear phases, see [17].

It is well-known that Calderón–Zygmund singular integrals are bounded on L^p spaces even when the C^1 condition (ii) is replaced by various weaker conditions ([10, 23]). One such family of Calderón–Zygmund kernels are those $K(x, y)$ which satisfy conditions (i), (ii)$'$, (iii), where (i) and (iii) are given as above while (ii)$'$ is the following Hölder type condition:

(ii)$'$ There exists a $\delta > 0$ such that

$$|K(x, y) - K(x', y)| \leq \frac{B|x - x'|^\delta}{(|x - y| + |x' - y|)^{n+\delta}} \tag{17}$$

whenever $|x - x'| < (1/2) \max\{|x - y|, |x' - y|\}$ and

$$|K(x, y) - K(x, y')| \leq \frac{B|y - y'|^\delta}{(|x - y| + |x - y'|)^{n+\delta}} \tag{18}$$

whenever $|y - y'| < (1/2) \max\{|x - y|, |x - y'|\}$.

A natural question is whether the corresponding oscillatory singular integrals with polynomial phases remain bounded on L^p spaces. As an application of Theorem 1.2, we obtain the following extension of Theorem 3.1 in which the Calderón–Zygmund kernels $K(x, y)$ are allowed to be in any Hölder class while the L^p spaces can be the usual L^p spaces with Lebesgue measures or weighted L^p spaces with any Muckenhoupt A_p weights.

Theorem 3.2. *Let $B, \delta > 0$ and $T_{P,K}$ be given as in* (11)*. Suppose that $1 < p < \infty$, $w \in A_p(\mathbb{R}^n)$ and $K(x, y)$ satisfies* (i), (ii)$'$ *and* (iii)*. Then there exists a positive C_p which may depend on p, n, δ, B, $\deg(P)$ and the A_p constant of w, but is independent of the coefficients of P, such that*

$$\|T_{P,K} f\|_{L^p_w(\mathbb{R}^n)} \leq C_p \|f\|_{L^p_w(\mathbb{R}^n)} \tag{19}$$

for all $f \in L^p_w(\mathbb{R}^n)$.

Proof. The special case of Theorem 3.2 in which $P(x, y)$ is assumed to be a bilinear form was proved in [1]. In what follows we shall concentrate on showing how our new van der Corput type lemma allows us to treat the general case of arbitrary polynomial phases. We therefore will provide details for the aforementioned key steps only and refer the readers to [1] and [19] for the other technical components needed for a complete proof.

For any nontrivial polynomial $P(x, y)$, one can write

$$P(x, y) = \sum_{|\alpha|=k} x^\alpha Q_\alpha(y) + R(x, y), \tag{20}$$

where $k \geq 1$, $R(x,y)$ is a polynomial whose terms do not contain any x^γ with $|\gamma| \geq k$, and for each α satisfying $|\alpha| = k$,

$$Q_\alpha(y) = \sum_{|\beta|=m} a_{\alpha\beta} y^\beta + q_\alpha(y) \tag{21}$$

with $\deg(q_\alpha) < m$. By using a rotation (i.e., an orthogonal matrix H and considering $P(xH, yH)$ instead of $P(x,y)$), we may assume that for $\alpha_0 = (k, 0, \ldots, 0)$ and a certain β_0 satisfying $|\beta_0| = m$,

$$|a_{\alpha\beta}| \lesssim |a_{\alpha_0\beta_0}| \tag{22}$$

for all $|\alpha| = k$ and $|\beta| = m$ (see [22]). By using a dilation $(x,y) \to (tx, ty)$ if necessary, we may further assume that $|a_{\alpha_0\beta_0}| = 1$. To see that there is no loss of generality in doing so, we point out that the kernels $t^n K(txH, tyH)$ satisfy (i), (ii)$'$ and (iii) with the same positive constants B and δ.

By using a smooth dyadic decomposition and a rescaling for each piece in the decomposition, the proof would eventually come to rest on obtaining a decay estimate $O(|\lambda|^{-\sigma_p})$, $\sigma_p > 0$, for the $L^p_w \to L^p_w$ norm of the operators

$$S_\lambda : f \to \int_{\mathbb{R}^n} e^{i\lambda P(x,y)} K(x,y) \phi(|x-y|) f(y) dy$$

as $\lambda \to \infty$, where

$$\phi \in C^\infty(\mathbb{R}), \quad \mathrm{supp}(\phi) \subseteq [1/4,\ 4], \quad \text{and} \quad 0 \leq \phi \leq 1$$

(here, in order to avoid unnecessary complications of notations, we continue to use P and K for the new phase and kernel functions even after the rescalings). Let $L_\lambda(x,y)$ denote the kernel of $S^*_\lambda S_\lambda$. Then

$$L_\lambda(x,y) = \int_{\mathbb{R}^n} e^{i\lambda[P(z,x)-P(z,y)]} K(z,x)\overline{K(z,y)} \phi(|z-x|)\phi(|z-y|) dz. \tag{23}$$

For any $x \in \mathbb{R}^n$, we write $x = (x_1, \tilde{x})$ where $\tilde{x}$ denotes the point $(x_2, \ldots, x_n)$ in $\mathbb{R}^{n-1}$. For $x, y, z \in \mathbb{R}^n$, let

$$G_{x,y,\tilde{z}}(z_1) = K(z,x)\overline{K(z,y)} \phi(|z-x|)\phi(|z-y|).$$

By (12) we have

$$\|G_{x,y,\tilde{z}}(\cdot)\|_{L^\infty(\mathbb{R})} \leq B^2. \tag{24}$$

Also, it follows from (12) and (17) that there exists a $C > 0$ independent of x, y and $\tilde{z}$ such that

$$\|G_{x,y,\tilde{z}}(\cdot)\|_{C^{0,\delta}(\mathbb{R})} \leq C \tag{25}$$

(for a proof of (25), see pages 2415–2416 of [1]).

For $s \in \mathbb{N}$, $a \in \mathbb{R}^s$ and $r > 0$, let $B_s(a,r)$ denote the ball centered at a in $\mathbb{R}^s$ with radius r. Let

$$\nu = \min\left\{\delta,\ \frac{1}{k}\right\}.$$

By (23), we have

$$|L_\lambda(x,y)| \lesssim \chi_{B_n(0,8)}(x-y) \times \int_{B_{n-1}(\tilde{x},4)\cap B_{n-1}(\tilde{y},4)}$$

$$\left| \int_{B_1(x_1,4)\cap B_1(y_1,4)} e^{i\lambda[z_1^k(Q_{\alpha_0}(x)-Q_{\alpha_0}(y))+U_{x,y,\tilde{z}}(z_1)]} G_{x,y,\tilde{z}}(z_1)dz_1 \right| d\tilde{z},$$

where $U_{x,y,\tilde{z}}(\cdot)$ is a polynomial of degree $\le k-1$. It follows from (24)-(25) and Theorem 1.2 that

$$|L_\lambda(x,y)| \lesssim \lambda^{-\nu}\chi_{B_n(0,8)}(x-y)|Q_{\alpha_0}(x)-Q_{\alpha_0}(y)|^{-\nu}. \tag{26}$$

Let $\mu = \min\{\nu, \frac{1}{2m}\}$. By using (26) and $\|L_\lambda(\cdot,\cdot)\|_\infty \lesssim 1$ (if necessary), we have

$$|L_\lambda(x,y)| \lesssim \lambda^{-\mu}\chi_{B_n(0,8)}(x-y)|Q_{\alpha_0}(x)-Q_{\alpha_0}(y)|^{-\mu}.$$

Since $0<\mu<\frac{1}{m}$ and $|a_{\alpha_0\beta_0}|=1$, it follows from the proposition on page 182 of [19] that

$$\int_{\mathbb{R}^n} |L_\lambda(x,y)|dx \lesssim \lambda^{-\mu}\Big(\sum_{|\beta|=m}|a_{\alpha_0\beta}|\Big)^{-\mu} \lesssim \lambda^{-\mu}$$

for all $y\in\mathbb{R}^n$. Similarly,

$$\int_{\mathbb{R}^n} |L_\lambda(x,y)|dy \lesssim \lambda^{-\mu}$$

for all $x\in\mathbb{R}^n$. By Schur's test,

$$\|S_\lambda\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \lesssim \lambda^{-\mu/2}. \tag{27}$$

By (12) and $\phi\in C_c^\infty(\mathbb{R}^n)$, we have

$$\|S_\lambda\|_{L^1(\mathbb{R}^n)\to L^1(\mathbb{R}^n)} + \|S_\lambda\|_{L^\infty(\mathbb{R}^n)\to L^\infty(\mathbb{R}^n)} \lesssim 1$$

and

$$|S_\lambda f| \lesssim M_{HL} f, \tag{28}$$

where M_{HL} is the Hardy–Littlewood maximal operator on $\mathbb{R}^n$. By interpolation we get

$$\|S_\lambda\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \lesssim \lambda^{-\mu(1-2|1/p-1/2|)/2}. \tag{29}$$

By a well-known property of A_p weights, there exists a $\theta>0$ such that $w^{1+\theta}\in A_p(\mathbb{R}^n)$. Both θ and the A_p constant of $w^{1+\theta}$ depend on n,p and the A_p constant of w only. It follows from (28) that

$$\|S_\lambda\|_{L^p_{w^{1+\theta}}(\mathbb{R}^n)\to L^p_{w^{1+\theta}}(\mathbb{R}^n)} \lesssim 1 \tag{30}$$

(see [5]). It then follows from (29)-(30) and Theorem 2 of [21] that

$$\|S_\lambda\|_{L^p_w(\mathbb{R}^n)\to L^p_w(\mathbb{R}^n)} \lesssim |\lambda|^{-\sigma_p} \tag{31}$$

with $\sigma_p = \mu\theta(1 - 2|1/p - 1/2|)/(2(1+\theta)) > 0$. By (22), we also have

$$\sum_{|\alpha|=k} \sum_{|\beta|=m} |a_{\alpha\beta}| \lesssim |a_{\alpha_0\beta_0}| = 1. \tag{32}$$

By using (31), (32) and a well-established procedure in the literature (see [1, 19]), one can obtain the L^p_w boundedness of $T_{P,K}$. Details of the remaining steps are omitted. □

In the endpoint case $p = 1$, for C^1 Calderón–Zygmund kernels, $T_{P,K}$ were known to be bounded from $L^{1,\infty}$ to L^1 (by Chanillo and Christ in [4]) and also from the Hardy type space H^1_E (see definition below) to L^1 (by Pan in [15]). We are now able to extend both results to the family of Hölder class Calderón–Zygmund kernels. We shall begin with the weak type (1, 1) result.

Theorem 3.3. *Let $B, \delta > 0$ and $T_{P,K}$ be given as in Theorem 3.2. Suppose that $K(x, y)$ satisfies* (i), (ii)$'$ *and* (iii). *Then $T_{P,K}$ is of weak type* $(1, 1)$*, i.e., there exists a positive C such that*

$$|\{x \in \mathbb{R}^n : |T_{P,K}f(x)| > \lambda\}| \le C\lambda^{-1}\|f\|_{L^1(\mathbb{R}^n)} \tag{33}$$

for all $f \in L^1(\mathbb{R}^n)$ and $\lambda > 0$. Moreover, while the constant C in (33) *may depend on n, δ, B and* $\deg(P)$*, it is otherwise independent of $K(\cdot,\cdot)$ and the coefficients of P.*

In order to describe the Hardy space result, we begin by recalling the definition of the space H^1_E, which is a variant of the standard Hardy space H^1 first introduced by Phong and Stein in [17] for bilinear phases and subsequently for polynomial phases in [15].

Definition. A measurable function $a(\cdot)$ on $\mathbb{R}^n$ is called an *atom* if there exists a cube Q such that $\mathrm{supp}(a) \subseteq Q$, $\|a\|_\infty \le |Q|^{-1}$ and

$$\int_Q e^{iP(x_Q,y)} a(y)dy = 0.$$

A function f is in $H^1_E(\mathbb{R}^n)$ if there exist a sequence $\{\lambda_j\}$ in $\mathbb{C}$ and a sequence of atoms $\{a_j\}$ such that

$$f = \sum_j \lambda_j a_j. \tag{34}$$

The H^1_E norm of f is the infimum of $\sum_j |\lambda_j|$ over all possible expressions of f described in (34).

We end the paper with the following theorem which extends Theorem 3 of [2] from bilinear phases to polynomial phases:

Theorem 3.4. *Let $B, \delta > 0$ and $T_{P,K}$ be given as in Theorem 3.2. Suppose that $K(x, y)$ satisfies* (i), (ii)$'$ *and* (iii). *Then there exists a positive C such that*

$$\|T_{P,K}f\|_{L^1(\mathbb{R}^n)} \le C\|f\|_{H^1_E(\mathbb{R}^n)} \tag{35}$$

for all $f \in H_E^1(\mathbb{R}^n)$*. Moreover, the constant* C *in* (35) *may depend on* n*,* δ*,* B *and* $\deg(P)$*, but is otherwise independent of* $K(\cdot,\cdot)$ *and the coefficients of* P*.*

In each of the proofs of Theorems 3.3 and 3.4, there is a "local" part which follows from the L^p boundedness obtained in Theorem 3.2 which in turn relied on Theorem 1.2, among other things. The "non-local" part of the weak type (1,1) proof is more complicated than that of the $H_E^1 \to L^1$ result, requiring an improved version of Lemma 4 in [2] in which bilinear forms are replaced by general polynomials. This is where, once again, one uses Theorem 1.2. For the sake of succinctness, details are omitted.

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References

[1] H. Al-Qassem, L. Cheng, and Y. Pan, *Oscillatory singular integral operators with Hölder class kernels*, J. Fourier Anal. Appl. **25** (2019), no. 4, 2141–2149. https://doi.org/10.1007/s00041-018-09660-y

[2] ———, *Endpoint estimates for oscillatory singular integrals with Hölder class kernels*, J. Funct. Spaces **2019** (2019), Art. ID 8561402, 7 pp. https://doi.org/10.1155/2019/8561402

[3] L. Carleson and P. Sjölin, *Oscillatory integrals and a multiplier problem for the disc*, Studia Math. **44** (1972), 287–299. (errata insert). https://doi.org/10.4064/sm-44-3-287-299

[4] S. Chanillo and M. Christ, *Weak* $(1,1)$ *bounds for oscillatory singular integrals*, Duke Math. J. **55** (1987), no. 1, 141–155. https://doi.org/10.1215/S0012-7094-87-05508-6

[5] R. R. Coifman and C. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals*, Studia Math. **51** (1974), 241–250. https://doi.org/10.4064/sm-51-3-241-250

[6] E. T. Copson, *Asymptotic expansions*, reprint of the 1965 original, Cambridge Tracts in Mathematics, 55, Cambridge University Press, Cambridge, 2004.

[7] J. G. van der Corput, *Zahlentheoretische Abschätzungen*, Math. Ann. **84** (1921), no. 1-2, 53–79. https://doi.org/10.1007/BF01458693

[8] O. Costin, *Asymptotics and Borel Summability*, Monographs and Surveys in Pure & Appl. Math. #141, Chapman and Hall/CRC, 2008.

[9] A. Erdélyi, *Asymptotic Expansions*, Dover Publications, Inc., New York, 1956.

[10] L. Grafakos, *Classical and Modern Fourier Analysis*, Pearson Education, Inc., Upper Saddle River, NJ, 2004.

[11] E. Hlawka, *Über Integrale auf konvexen Körpern. I*, Monatsh. Math. **54** (1950), 1–36. https://doi.org/10.1007/BF01304101

[12] L. Hörmander, *Oscillatory integrals and multipliers on* FL^p, Ark. Mat. **11** (1973), 1–11. https://doi.org/10.1007/BF02388505

[13] ———, *The Analysis of Linear Partial Differential Operators I.*, Springer-Verlag, 1983.

[14] W. Littman, *Fourier transforms of surface-carried measures and differentiability of surface averages*, Bull. Amer. Math. Soc. **69** (1963), 766–770. https://doi.org/10.1090/S0002-9904-1963-11025-3

[15] Y. Pan, *Hardy spaces and oscillatory singular integrals*, Rev. Mat. Iberoamericana **7** (1991), no. 1, 55–64. https://doi.org/10.4171/RMI/105

[16] ———, *Boundedness of oscillatory singular integrals on Hardy spaces: II*, Indiana Univ. Math. J. **41** (1992), no. 1, 279–293. https://doi.org/10.1512/iumj.1992.41.41016

[17] D. H. Phong and E. M. Stein, *Hilbert integrals, singular integrals, and Radon transforms. I*, Acta Math. **157** (1986), no. 1-2, 99–157. `https://doi.org/10.1007/BF02392592`

[18] ———, *Operator versions of the van der Corput lemma and Fourier integral operators*, Math. Res. Lett. **1** (1994), no. 1, 27–33. `https://doi.org/10.4310/MRL.1994.v1.n1.a3`

[19] F. Ricci and E. M. Stein, *Harmonic analysis on nilpotent groups and singular integrals. I. Oscillatory integrals*, J. Funct. Anal. **73** (1987), no. 1, 179–194. `https://doi.org/10.1016/0022-1236(87)90064-4`

[20] C. D. Sogge and E. M. Stein, *Averages of functions over hypersurfaces in* $\mathbf{R}^n$, Invent. Math. **82** (1985), no. 3, 543–556. `https://doi.org/10.1007/BF01388869`

[21] E. M. Stein, *Interpolation of linear operators*, Trans. Amer. Math. Soc. **83** (1956), 482–492. `https://doi.org/10.2307/1992885`

[22] ———, *Beijing lectures in harmonic analysis*, Ann. Math. Studies # 112, Princeton Univ. Press, Princeton, NJ, 1986.

[23] ———, *Harmonic Analysis*: *Real Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Univ. Press, Princeton, NJ, 1993.

[24] A. N. Varchenko, *Newton polyhedra and estimation of oscillating integrals*, Func. Anal. Appl. **10** (1976), 175–196.

[25] V. V. Yurinskii, *On application of the van der Corput lemma to the estimation of the characteristic functions of certain singular distributions*, Theory of Prob. & Appl. **16** (1971), 387–388.

Hussain Al-Qassem
Department of Mathematics and Physics
Qatar University
Doha-Qatar
Email address: `husseink@qu.edu.qa`

Leslie Cheng
Department of Mathematics
Bryn Mawr College
Bryn Mawr, PA 19010, U.S.A.
Email address: `lcheng@brynmawr.edu`

Yibiao Pan
Department of Mathematics
University of Pittsburgh
Pittsburgh, PA 15260, U.S.A.
Email address: `yibiao@pitt.edu`