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KILLING STRUCTURE JACOBI OPERATOR OF A REAL HYPERSURFACE IN A COMPLEX PROJECTIVE SPACE

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Abstract. We prove non-existence of real hypersurfaces with Killing structure Jacobi operator in complex projective spaces. We also classify real hypersurfaces in complex projective spaces whose structure Jacobi operator is Killing with respect to the k-th generalized Tanaka-Webster connection.

1. Introduction

Let $\mathbb{C}P^m$, $m \geq 2$, be a *complex projective space* endowed with the metric g of constant holomorphic sectional curvature 4. Let M be a *connected real hypersurface* of $\mathbb{C}P^m$ without boundary. Let ∇ be the Levi-Civita connection on M and J the complex structure of $\mathbb{C}P^m$. Take a locally defined unit normal vector field N on M and denote by $\xi = -JN$. This is a tangent vector field to M called the structure vector field on M. On M there exists an almost contact metric structure (ϕ, ξ, η, g) induced by the Kaehlerian structure of $\mathbb{C}P^m$, where ϕ is the tangent component of J and η is an one-form given by $\eta(X) = g(X, \xi)$ for any X tangent to M. The classification of homogeneous real hypersurfaces in $\mathbb{C}P^m$ was obtained by Takagi, see [6], [13], [14], [15]. His classification contains 6 types of real hypersurfaces. Among them we find type (A_1) real hypersurfaces that are geodesic hyperspheres of radius r, $0 < r < \frac{\pi}{2}$ and type (A_2) real hypersurfaces that are tubes of radius r, $0 < r < \frac{\pi}{2}$, over totally geodesic complex projective spaces $\mathbb{C}P^n$, $0 < n < m-1$. We will call both types of real hypersurfaces type (A) real hypersurfaces.

Ruled real hypersurfaces can be described as follows: Take a regular curve γ in $\mathbb{C}P^m$ with tangent vector field X. At each point of γ there is a unique $\mathbb{C}P^{m-1}$ cutting γ so as to be orthogonal not only to X but also to JX. The union of these hyperplanes is called a ruled real hypersurface. It will be an

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embedded hypersurface locally, although globally it will in general have self-intersections and singularities. Equivalently, a ruled real hypersurface satisfies that the maximal holomorphic distribution on M, $\mathbb{D}$, given at any point by the vectors orthogonal to ξ, is integrable with integral manifolds $\mathbb{C}P^{m-1}$, or $g(A\mathbb{D}, \mathbb{D}) = 0$. For examples of ruled real hypersurfaces see [7] or [9].

The Tanaka-Webster connection, [16], [18], is the canonical affine connection defined on a non-degenerate, pseudo-Hermitian CR-manifold. As a generalization of this connection, Tanno, [17], defined the generalized Tanaka-Webster connection for contact metric manifolds by

$$\hat{\nabla}_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi - \eta(X)\phi Y. \tag{1.1}$$

Using the naturally extended affine connection of Tanno's generalized Tanaka-Webster connection, Cho defined the g-Tanaka-Webster connection $\hat{\nabla}^{(k)}$ for a real hypersurface M in $\mathbb{C}P^m$ given, see [4], [5], by

$$\hat{\nabla}^{(k)}_X Y = \nabla_X Y + g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y \tag{1.2}$$

for any X,Y tangent to M where k is a non-zero real number. Then $\hat{\nabla}^{(k)}\eta = 0$, $\hat{\nabla}^{(k)}\xi = 0$, $\hat{\nabla}^{(k)}g = 0$, $\hat{\nabla}^{(k)}\phi = 0$. In particular, if the shape operator of a real hypersurface satisfies $\phi A + A\phi = 2k\phi$, the g-Tanaka-Webster connection coincides with the Tanaka-Webster connection.

Here we can consider the tensor field of type (1,2) given by the difference of both connections $F^{(k)}(X,Y) = g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y$, for any X, Y tangent to M, see [8] Proposition 7.10, pages 234–235. We will call this tensor the k-th Cho tensor on M. Associated to it, for any X tangent to M and any nonnull real number k we can consider the tensor field of type (1,1) $F^{(k)}_X$, given by $F^{(k)}_X Y = F^{(k)}(X,Y)$ for any $Y \in TM$. This operator will be called the k-th Cho operator corresponding to X. The torsion of the connection $\hat{\nabla}^{(k)}$ is given by $\hat{T}^{(k)}(X,Y) = F^{(k)}_X Y - F^{(k)}_Y X$ for any X, Y tangent to M.

The Jacobi operator R_X with respect to a unit vector field X is defined by $R_X = R(\cdot, X)X$, where R is the curvature tensor field on M. Then we see that R_X is a self-adjoint endomorphism of the tangent space. It is related to Jacobi vector fields, which are solutions of the second-order differential equation (the Jacobi equation) $\nabla_{\dot{\gamma}}(\nabla_{\dot{\gamma}}Y) + R(Y, \dot{\gamma})\dot{\gamma} = 0$ along a geodesic γ in M. The Jacobi operator with respect to the structure vector field ξ, R_ξ, is called the structure Jacobi operator on M.

Let $\mathcal{L}$ denote the Lie derivative of a Riemannian manifold $(\bar{M}, \bar{g})$ with Levi-Civita connection $\bar{\nabla}$. Generalizing the notion of Killing vector field (a vector field on $\bar{M}$ is Killing if $\mathcal{L}_X \bar{g} = 0$), Blair, [1], introduced the notion of Killing tensor along a geodesic γ of $\bar{M}$. A tensor of type (1,1) K on $\bar{M}$ is called Killing along γ if $(\nabla_{\dot{\gamma}}K)\dot{\gamma} = 0$. We will say K is Killing if it is Killing along any geodesic of $\bar{M}$. Therefore K is Killing if $(\bar{\nabla}_X K)X = 0$ for any vector field X tangent to $\bar{M}$. Equivalently

$$(\bar{\nabla}_X K)Y + (\bar{\nabla}_Y K)X = 0$$

for any vector fields X, Y tangent to $\bar{M}$.

The purpose of the present paper is to study real hypersurfaces M in $\mathbb{C}P^m$ whose structure Jacobi operator is Killing. We will prove the following:

Theorem 1. *There does not exist any real hypersurface in $\mathbb{C}P^m$, $m \geq 3$, with Killing structure Jacobi operator.*

We will say that the structure Jacobi operator of M is Killing with respect to the k-th generalized Tanaka-Webster connection if $(\hat{\nabla}^{(k)}_X R_\xi)X = 0$ for any vector field X tangent to M. Equivalently

$$(\hat{\nabla}^{(k)}_X R_\xi)Y + (\hat{\nabla}^{(k)}_Y R_\xi)X = 0$$

for any vector fields X, Y tangent to M. We prove the following:

Theorem 2. *Let M be a real hypersurface in $\mathbb{C}P^m$, $m \geq 2$. Let k be a nonnull constant. Then the structure Jacobi operator of M is Killing with respect to the k-th generalized Tanaka-Webster connection if and only if M is locally congruent to either a tube of radius $\frac{\pi}{4}$ over a complex submanifold of $\mathbb{C}P^m$ or to a type* (A) *real hypersurface with radius $r \neq \frac{\pi}{4}$.*

2. Preliminaries

Throughout this paper, all manifolds, vector fields, etc., will be considered of class C^∞ unless otherwise stated. Let M be a connected real hypersurface in $\mathbb{C}P^m$, $m \geq 2$, without boundary. Let N be a locally defined unit normal vector field on M. Let ∇ be the Levi-Civita connection on M and (J, g) the Kaehlerian structure of $\mathbb{C}P^m$.

For any vector field X tangent to M we write $JX = \phi X + \eta(X)N$, and $-JN = \xi$. Then (ϕ, ξ, η, g) is an almost contact metric structure on M, see [2]. That is, we have

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.1}$$

for any tangent vectors X, Y to M. From (2.1) we obtain

$$\phi\xi = 0, \quad \eta(X) = g(X, \xi). \tag{2.2}$$

From the parallelism of J we get

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi \tag{2.3}$$

and

$$\nabla_X \xi = \phi AX \tag{2.4}$$

for any X, Y tangent to M, where A denotes the shape operator of the immersion. As the ambient space has holomorphic sectional curvature 4, the equations of Gauss and Codazzi are given, respectively, by

$$\begin{aligned} R(X, Y)Z = {} & g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ & - 2g(\phi X, Y)\phi Z + g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{2.5}$$

and

$$(\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi \tag{2.6}$$

for any tangent vectors X, Y, Z to M, where R is the curvature tensor of M. We will call the maximal holomorphic distribution $\mathbb{D}$ on M to the following one: at any $p \in M$, $\mathbb{D}(p) = \{X \in T_pM \,|\, g(X, \xi) = 0\}$. We will say that M is Hopf if ξ is principal, that is, $A\xi = \alpha\xi$ for a certain function α on M.

From the above formulas we have that the structure Jacobi operator on M is given by

$$R_\xi(X) = X - \eta(X)\xi + g(A\xi, \xi)AX - g(AX, \xi)A\xi \tag{2.7}$$

for any X tangent to M. Therefore its covariant derivative is given by

$$\begin{aligned}(\nabla_X R_\xi)Y = & -g(Y, \phi AX)\xi - \eta(Y)\phi AX + g(\nabla_X A\xi, \xi)AY \\ & + g(A\xi, \phi AX)AY + g(A\xi, \xi)(\nabla_X A)Y - g(Y, \nabla_X A\xi)A\xi \\ & - g(AY, \xi)\nabla_X A\xi.\end{aligned} \tag{2.8}$$

In the sequel we need the following results:

Theorem 2.1 ([11]). *Let M be a real hypersurface of $\mathbb{C}P^m$, $m \geq 2$. Then the following are equivalent*:

(1) *M is locally congruent to either a geodesic hypersphere or a tube of radius r, $0 < r < \frac{\pi}{2}$, over a totally geodesic $\mathbb{C}P^n$, $0 < n < m-1$.*
(2) *$\phi A = A\phi$.*

Theorem 2.2 ([10]). *If ξ is a principal curvature vector with corresponding principal curvature α, it is constant and if $X \in \mathbb{D}$ is principal with principal curvature λ, then ϕX is principal with principal curvature $\frac{\alpha\lambda+2}{2\lambda-\alpha}$.*

3. Proof of Theorem 1

Suppose R_ξ is Killing and M is Hopf with $A\xi = \alpha\xi$. If $X \in \mathbb{D}$, then $R_\xi(X) = X + \alpha AX$. As $(\nabla_X R_\xi)\xi + (\nabla_\xi R_\xi)X = 0$ we get, bearing in mind that α is constant, $-R_\xi(\phi AX) + \nabla_\xi(X + \alpha AX) - R_\xi(\nabla_\xi X) = 0$. That is, $-R_\xi(\phi AX) + \alpha\nabla_\xi AX - \alpha A\nabla_\xi X = 0$. Thus we have

$$R_\xi(\phi AX) = \alpha(\nabla_\xi A)X$$

for any $X \in \mathbb{D}$.

Suppose that $X \in \mathbb{D}$ satisfies $AX = \lambda X$. From Theorem 2.2 $A\phi X = \mu\phi X$, where $\mu = \frac{\alpha\lambda+2}{2\lambda-\alpha}$. Therefore $\lambda\phi X + \lambda\alpha A\phi X = \alpha(\nabla_\xi A)X$. That is,

$$(\lambda + \lambda\alpha\mu)\phi X = \alpha(\nabla_\xi A)X. \tag{3.1}$$

As also $R_\xi(\phi A\phi X) = \alpha(\nabla_\xi A)\phi X$, we obtain

$$-(\mu + \lambda\alpha\mu)X = \alpha(\nabla_\xi A)\phi X. \tag{3.2}$$

Taking the scalar product of (3.1) and ϕX we have

$$\lambda + \lambda\alpha\mu = \alpha g((\nabla_\xi A)X, \phi X) = \alpha g((\nabla_\xi A)\phi X, X) = -\mu - \lambda\alpha\mu.$$

Thus $\lambda+\mu=-2\alpha\lambda\mu$. Bearing in mind the value of μ it follows $\lambda+\frac{\alpha\lambda+2}{2\lambda-\alpha}=-2\alpha\lambda\frac{\alpha\lambda+2}{2\lambda-\alpha}$. Therefore

$$(1+\alpha^2)\lambda^2+2\alpha\lambda+1=0. \tag{3.3}$$

But this equation with unknown λ does not admit any real root, which is impossible.

That means our real hypersurface is non Hopf. Then, we write $A\xi=\alpha\xi+\beta U$, where U is a unit vector field in $\mathbb{D}$ and β a function on M non-vanishing on a certain neighborhood of a point p. From now on the calculations will be made on such a neighborhood. We will call $\mathbb{D}_U=\{X\in\mathbb{D}/g(X,U)=g(X,\phi U)=0\}$. As $(\nabla_\xi R_\xi)\xi=0$, we have $-R_\xi(\phi A\xi)=-\beta R_\xi(\phi U)=0$. As $\beta\neq 0$ we get $R_\xi(\phi U)=0$ and this yields

$$\alpha\neq 0 \tag{3.4}$$

and

$$A\phi U=-\frac{1}{\alpha}\phi U. \tag{3.5}$$

From (2.8) our condition $(\nabla_X R_\xi)Y+(\nabla_Y R_\xi)X=0$ becomes

$$\begin{aligned}
&-g(Y,\phi AX)\xi-\eta(Y)\phi AX+g(\nabla_X A\xi,\xi)AY+g(A\xi,\phi AX)AY\\
&+\alpha(\nabla_X A)Y-g(Y,\nabla_X A\xi)A\xi-g(AY,\xi)\nabla_X A\xi-g(X,\phi AY)\xi\\
&-\eta(X)\phi AY+g(\nabla_Y A\xi,\xi)AX+g(A\xi,\phi AY)AX+\alpha(\nabla_Y A)X\\
&-g(X,\nabla_Y A\xi)A\xi-g(AX,\xi)\nabla_Y A\xi=0
\end{aligned} \tag{3.6}$$

for any X,Y tangent to M. From (3.6) and Codazzi equation we obtain

$$\begin{aligned}
&2\alpha(\nabla_X A)Y\\
&=\alpha\{\eta(X)\phi Y-\eta(Y)\phi X-2g(\phi X,Y)\xi\}\\
&\quad+g(Y,\phi AX)\xi+\eta(Y)\phi AX-g(\nabla_X A\xi,\xi)AY-g(A\xi,\phi AX)AY\\
&\quad+g(Y,\nabla_X A\xi)A\xi+g(AY,\xi)\nabla_X A\xi+g(X,\phi AY)\xi+\eta(X)\phi AY\\
&\quad-g(\nabla_Y A\xi,\xi)AX-g(A\xi,\phi AY)AX+g(X,\nabla_Y A\xi)A\xi\\
&\quad+g(AX,\xi)\nabla_Y A\xi.
\end{aligned} \tag{3.7}$$

In particular, if $Y=\xi$ we get

$$\begin{aligned}
2\alpha(\nabla_X A)\xi= &-\alpha\phi X+\phi AX-g(A\xi,\phi AX)A\xi+\alpha\nabla_X A\xi\\
&+g(X,\phi A\xi)\xi+\eta(X)\phi A\xi-g(\nabla_\xi A\xi,\xi)AX\\
&+g(X,\nabla_\xi A\xi)A\xi+g(AX,\xi)\nabla_\xi A\xi
\end{aligned} \tag{3.8}$$

for any X tangent to M. Bearing in mind (3.7) and (3.8) and that

$$g((\nabla_X A)\xi,Y)=g((\nabla_X A)Y,\xi)$$

we have

$$
\begin{aligned}
& -\alpha g(\phi X,Y)+g(X,\phi AY)-g(\nabla_Y A\xi,\xi)g(AX,\xi)\\
& -g(A\xi,\phi AY)g(AX,\xi)+\alpha g(X,\nabla_Y A\xi)+g(AX,\xi)g(\nabla_Y A\xi,\xi)\\
& = g(X,\phi A\xi)\eta(Y)\\
& +\eta(X)g(\phi A\xi,Y)-g(\nabla_\xi A\xi,\xi)g(AX,Y)+g(X,\nabla_\xi A\xi)g(A\xi,Y)\\
& +g(AX,\xi)g(\nabla_\xi A\xi,Y)
\end{aligned}
\tag{3.9}
$$

for any X,Y tangent to M.

From (3.8), bearing in mind (3.5) it follows

$$
\begin{aligned}
\alpha\nabla_U A\xi = {} & 2\alpha A\phi AU-\alpha\phi U+\phi AU\\
& -g(\nabla_\xi A\xi,\xi)AU+g(U,\nabla_\xi A\xi)A\xi+\beta\nabla_\xi A\xi
\end{aligned}
\tag{3.10}
$$

and taking $X\in\mathbb{D}_U$, $Y=U$ in (3.9) we obtain

$$g(X,\phi AU)+\alpha g(X,\nabla_U A\xi)=-g(\nabla_\xi A\xi,\xi)g(AX,U)+\beta g(X,\nabla_\xi A\xi) \tag{3.11}$$

for any $X\in\mathbb{D}_U$.

Introducing (3.11) in (3.10) we have

$$g(\phi AU,X)+\alpha g(A\phi AU,X)=0 \tag{3.12}$$

for any $X\in\mathbb{D}_U$. Therefore $\phi AU+\alpha A\phi AU\in Span\{\xi,U,\phi U\}$. As $g(\phi AU+\alpha A\phi AU,\xi)=0$ from (3.5), $g(\phi AU+\alpha A\phi AU,U)=\alpha g(AU,\phi AU)=0$ and $g(\phi AU+\alpha A\phi AU,\phi U)=g(AU,U)+\alpha g(A\phi U,\phi AU)=g(AU,U)-g(\phi U,\phi AU)$ $=0$, we get

$$\phi AU+\alpha A\phi AU=0. \tag{3.13}$$

Taking $X=U$, $Y\in\mathbb{D}_U$ in (3.9) we obtain

$$\alpha g(U,\nabla_Y A\xi)=-g(\nabla_\xi A\xi,\xi)g(AU,Y)+\beta g(\nabla_\xi A\xi,Y) \tag{3.14}$$

for any $Y\in\mathbb{D}_U$. From (3.8) it follows

$$\alpha\nabla_Y A\xi=2\alpha A\phi AY-\alpha\phi Y+\phi AY-g(\nabla_\xi A\xi,\xi)AY+g(Y,\nabla_\xi A\xi)A\xi. \tag{3.15}$$

Introducing (3.15) in (3.14) we have $2\alpha g(A\phi AY,U)-g(\nabla_\xi A\xi,\xi)g(AY,U)+\beta g(Y,\nabla_\xi A\xi)=-g(\nabla_\xi A\xi,\xi)g(AU,Y)+\beta g(\nabla_\xi A\xi,Y)$. Therefore, as $\alpha\neq 0$, for any $Y\in\mathbb{D}_U$, $g(A\phi AY,U)=0$. That is, $A\phi AU\in Span\{\xi,U,\phi U\}$. As $g(A\phi AU,\xi)=\beta g(\phi AU,U)=0$, $g(A\phi AU,U)=g(\phi AU,AU)=0$ and $g(A\phi AU,\phi U)=g(\phi AU,A\phi U)=-\frac{1}{\alpha}g(\phi AU\ ,\phi U)=-\frac{1}{\alpha}g(AU,U)$ we conclude

$$A\phi AU=-\frac{1}{\alpha}g(AU,U)\phi U. \tag{3.16}$$

From (3.13) and (3.16) we get $\phi AU=g(U,AU)\phi U$. This yields

$$AU=\beta\xi+g(AU,U)U \tag{3.17}$$

and this means that $\mathbb{D}_U$ is A-invariant.

Take $X,Y\in\mathbb{D}_U$ in (3.9). We have

$$-\alpha g(\phi X,Y)+g(X,\phi AY)+\alpha g(X,\nabla_Y A\xi)=-g(\nabla_\xi A\xi,\xi)g(AX,Y). \tag{3.18}$$

Moreover, taking $Y \in \mathbb{D}_U$ in (3.8) we obtain

$$\alpha\nabla_Y A\xi = 2\alpha A\phi AY - \alpha\phi Y + \phi AY - g(\nabla_\xi A\xi, \xi)AY + g(Y, \nabla_\xi A\xi)A\xi. \tag{3.19}$$

From (3.18) and (3.19) we get

$$2g(X, \phi AY) + 2\alpha g(A\phi AY, X) = 0 \tag{3.20}$$

for any $X, Y \in \mathbb{D}_U$. Then $\phi AY + \alpha A\phi AY = 0$ for any $Y \in \mathbb{D}_U$. Suppose that $Y \in \mathbb{D}_U$ is unit and satisfies $AY = \lambda Y$. Then $\lambda\phi Y + \lambda\alpha A\phi Y = 0$. Thus, either $\lambda = 0$ or $A\phi Y = -\frac{1}{\alpha}\phi Y$. Interchanging X and Y in (3.20) we have $A\phi Y + \alpha A\phi AY = 0$. Therefore $A\phi Y = \phi AY$. If $AY = 0$, $A\phi Y = 0$ and if $A\phi Y = -\frac{1}{\alpha}\phi Y = \phi AY$, then $AY = -\frac{1}{\alpha}Y$. Therefore the unique principal curvatures appearing on $\mathbb{D}_U$ are 0 and $-\frac{1}{\alpha}$.

Taking $X = \phi U$ in (3.8) we obtain

$$\begin{aligned} \alpha\nabla_{\phi U} A\xi = {} & 2\alpha A\phi A\phi U + \alpha U + \phi A\phi U + \beta g(A\phi U, \phi U)A\xi + \beta\xi \\ & + \frac{1}{\alpha} g(\nabla_\xi A\xi, \xi)\phi U + g(\phi U, \nabla_\xi A\xi)A\xi. \end{aligned} \tag{3.21}$$

And taking $X = U$, $Y = \phi U$ in (3.9) it follows

$$-\alpha + \frac{1-\beta^2}{\alpha} + \alpha g(U, \nabla_{\phi U} A\xi) = \beta g(\nabla_\xi A\xi, \phi U). \tag{3.22}$$

From (3.21) and (3.22) we have

$$g(AU, U) = \frac{\beta^2 - 1}{\alpha}. \tag{3.23}$$

From Proposition 3.3 in [12], there must be a principal curvature on $\mathbb{D}_U$ not equal to $-\frac{1}{\alpha}$.

Codazzi equation applied to $X \in \mathbb{D}_U$ and ϕX such that $AX = A\phi X = 0$ yields $\beta^2 = 1$. Our Theorem follows from Propositions 3.1 and 3.2 in [12].

4. Proof of Theorem 2

If we suppose that $(\hat{\nabla}^{(k)}_X R_\xi)Y + (\hat{\nabla}^{(k)}_Y R_\xi)X = 0$ for any X, Y tangent to M, developing it we get

$$\begin{aligned} & (\nabla_X R_\xi)Y + (\nabla_Y R_\xi)X \\ = {} & -g(\phi AX, R_\xi(Y))\xi + k\eta(X)\phi R_\xi(Y) - \eta(Y)R_\xi(\phi AX) \\ & - k\eta(X)R_\xi(\phi Y) - g(\phi AY, R_\xi(X))\xi + k\eta(Y)\phi R_\xi(X) \\ & - \eta(X)R_\xi(\phi AY) - k\eta(Y)R_\xi(\phi X). \end{aligned} \tag{4.1}$$

From (2.8) it follows

$$
\begin{aligned}
&- g(\phi AX,Y)\xi - \eta(Y)\phi AX + g(\nabla_X A\xi,\xi)AY + g(A\xi,\phi AX)AY \\
&+ \alpha(\nabla_X A)Y - g(Y,\nabla_X A\xi)A\xi - g(AY,\xi)\nabla_X A\xi - g(X,\phi AY)\xi \\
&- \eta(X)\phi AY + g(\nabla_Y A\xi,\xi)AX + g(A\xi,\phi AY)AX + \alpha(\nabla_Y A)X \\
&- g(X,\nabla_Y A\xi)A\xi - g(AX,\xi)\nabla_Y A\xi \\
= &- g(\phi AX,R_\xi(Y))\xi + k\eta(Y)\phi R_\xi(Y) - \eta(Y)R_\xi(\phi AX) \\
&- k\eta(X)R_\xi(\phi Y) - g(\phi AY,R_\xi(X))\xi + k\eta(Y)\phi R_\xi(X) \\
&- \eta(X)R_\xi(\phi AY) - k\eta(Y)R_\xi(\phi X)
\end{aligned}
\tag{4.2}
$$

for any X, Y tangent to M.

Let us suppose M is Hopf with $A\xi = \alpha\xi$. If at a point $p \in M$ $\alpha \neq 0$, there exists an open neighborhood of p on which α does not vanish. All the following computations are made on this neighborhood.

From (4.2) it follows

$$
\begin{aligned}
&\alpha(\nabla_X A)Y + \alpha(\nabla_Y A)X \\
= &\, g(Y,\phi AX)\xi + \eta(Y)\phi AX + g(Y,\nabla_X A\xi)A\xi \\
&+ g(AY,\xi)\nabla_X A\xi + g(X,\phi AY)\xi + \eta(X)\phi AY \\
&+ g(X,\nabla_Y A\xi)A\xi + g(AX,\xi)\nabla_Y A\xi \\
&- g(\phi AX,R_\xi(Y))\xi + k\eta(X)\phi R_\xi(Y) - \eta(Y)R_\xi(\phi AX) \\
&- k\eta(X)R_\xi(\phi Y) - g(\phi AY,R_\xi(X))\xi + k\eta(Y)\phi R_\xi(X) \\
&- \eta(X)R_\xi(\phi AY) - k\eta(Y)R_\xi(\phi X)
\end{aligned}
\tag{4.3}
$$

for any X, Y tangent to M. From Codazzi equation we have

$$
\begin{aligned}
&2\alpha(\nabla_X A)Y \\
= &\, \alpha\eta(X)\phi Y - \alpha\eta(Y)\phi X - 2\alpha g(\phi X,Y)\xi + g(Y,\phi AX)\xi \\
&+ \eta(Y)\phi AX + g(Y,\nabla_X A\xi)A\xi + g(AY,\xi)\nabla_X A\xi \\
&+ g(X,\phi AY)\xi + \eta(X)\phi AY + g(X,\nabla_Y A\xi)A\xi \\
&+ g(AX,\xi)\nabla_Y A\xi - g(\phi AX,R_\xi(Y))\xi + k\eta(X)\phi R_\xi(Y) \\
&- \eta(Y)R_\xi(\phi AX) - k\eta(X)R_\xi(\phi Y) - g(\phi AY,R_\xi(X))\xi \\
&+ k\eta(Y)\phi R_\xi(X) - \eta(X)R_\xi(\phi AY) - k\eta(Y)R_\xi(\phi X)
\end{aligned}
\tag{4.4}
$$

for any X, Y tangent to M

In particular, if $Y = \xi$ we get

$$
\begin{aligned}
2\alpha(\nabla_X A)\xi = &- \alpha\phi X + \phi AX + \alpha\nabla_X A\xi + g(X,\nabla_\xi A\xi)A\xi \\
&- R_\xi(\phi AX) + k\phi R_\xi(X) - kR_\xi(\phi X)
\end{aligned}
\tag{4.5}
$$

for any X tangent to M. From (4.4), (4.5) and the fact that $g((\nabla_X A)Y, \xi) = g(Y, (\nabla_X A)\xi)$ we have

$$\begin{aligned} & -\alpha g(\phi X, Y) + g(AY, \xi)g(\nabla_X A\xi, \xi) + g(X, \phi AY) \\ & + \alpha g(X, \nabla_Y A\xi) - g(\phi AY, R_\xi(X)) \\ = \; & g(X, \nabla_\xi A\xi)g(A\xi, Y) + \alpha g(\nabla_X A\xi, Y) + kg(\phi R_\xi(X), Y) \\ & - kg(R_\xi(\phi X), Y) \end{aligned} \tag{4.6}$$

for any X, Y tangent to M. Taking $X, Y \in \mathbb{D}$ in (4.6) we obtain

$$\begin{aligned} & -\alpha g(\phi X, Y) + g(X, \phi AY) + \alpha^2 g(X, \phi AY) - g(\phi AY, R_\xi(X)) \\ = \; & \alpha^2 g(\phi AX, Y) + kg(\phi R_\xi(X), Y) - kg(R_\xi(\phi X), Y), \end{aligned} \tag{4.7}$$

where we have applied that α is constant.

As for any $X \in \mathbb{D}$, $R_\xi(X) = X + \alpha AX$, (4.7) yields

$$-\phi X + (k - \alpha)A\phi X - (k + \alpha)\phi AX + A\phi AX = 0 \tag{4.8}$$

for any $X \in \mathbb{D}$. But interchanging X and Y we obtain

$$\phi X - (k - \alpha)\phi AX + (k + \alpha)A\phi X - A\phi AX = 0. \tag{4.9}$$

From (4.8) and (4.9) we get $2kA\phi X - 2k\phi AX = 0$ for any $X \in \mathbb{D}$. This yields $A\phi = \phi A$ and from Theorem 2.1, M is locally congruent to a real hypersurface of type (A). The converse is immediate.

Suppose now that $\alpha = 0$ on M. From (4.2) we get

$$\begin{aligned} & -g(Y, \phi AX)\xi - \eta(Y)\phi AX - g(X, \phi AY)\xi - \eta(X)\phi AY \\ = \; & -g(\phi AX, R_\xi(Y))\xi + k\eta(X)\phi R_\xi(Y) - \eta(Y)R_\xi(\phi AX) \\ & - k\eta(X)R_\xi(\phi Y) - g(\phi AY, R_\xi(X))\xi \\ & + k\eta(Y)\phi R_\xi(X) - \eta(X)R_\xi(\phi AY) - k\eta(Y)R_\xi(\phi X). \end{aligned} \tag{4.10}$$

In this case $R_\xi(X) = X - \eta(X)\xi$ and, for any $X \in \mathbb{D}$, $R_\xi(X) = X$. If $X, Y \in \mathbb{D}$, (4.10) gives $-g(Y, \phi AX)\xi - g(X, \phi AY)\xi = -g(\phi AX, Y)\xi - g(\phi AY, X)\xi$. If $X = \xi$ and Y is tangent to M we obtain $-\phi AY = -\phi AY$, proving that any real hypersurface such that $A\xi = 0$ satisfies our condition. The results follows from [3].

In the following we suppose M is non Hopf and as in the previous section we write $A\xi = \alpha\xi + \beta U$ for a unit $U \in \mathbb{D}$. Let p be a point of M such that $\alpha \neq 0$. This is true on a neighborhood of such a point and we will make the calculations on that neighborhood.

Bearing in mind (4.2) and Codazzi equation $\alpha(\nabla_X A)Y - \alpha(\nabla_Y A)X = \alpha\eta(X)\phi Y - \alpha\eta(Y)\phi X - 2\alpha g(\phi X, Y)\xi$ we obtain

$$\begin{aligned} &2\alpha(\nabla_X A)Y \\ &= g(Y,\phi AX)\xi + \eta(Y)\phi AX - g(\nabla_X A\xi,\xi)AY - g(A\xi,\phi AX)AY \\ &\quad + g(Y,\nabla_X A\xi)A\xi + g(AY,\xi)\nabla_X A\xi + g(X,\phi AY)\xi + \eta(X)\phi AY \\ &\quad - g(\nabla_Y A\xi,\xi)AX - g(A\xi,\phi AY)AX + g(X,\nabla_Y A\xi)A\xi \\ &\quad - g(AX,\xi)\nabla_Y A\xi - g(\phi AX, R_\xi(Y))\xi + k\eta(X)\phi R_\xi(Y) \\ &\quad - \eta(Y)R_\xi(\phi AX) - k\eta(X)R_\xi(\phi Y) - g(\phi AY, R_\xi(X))\xi \\ &\quad + k\eta(Y)\phi R_\xi(X) - \eta(X)R_\xi(\phi AY) - k\eta(Y)R_\xi(\phi X) \\ &\quad + \alpha\eta(X)\phi Y - \alpha\eta(Y)\phi X - 2\alpha g(\phi X, Y)\xi \end{aligned} \tag{4.11}$$

for any X, Y tangent to M. In particular, for $Y = \xi$, (4.11) yields

$$\begin{aligned} &2\alpha(\nabla_X A)\xi \\ &= \phi AX - g(A\xi,\phi AX)A\xi + \alpha\nabla_X A\xi + g(X,\phi A\xi)\xi \\ &\quad + \eta(X)\phi A\xi - g(\nabla_\xi A\xi,\xi)AX + g(X,\nabla_\xi A\xi)A\xi \\ &\quad - g(AX,\xi)\nabla_\xi A\xi - R_\xi(\phi AX) - g(\phi A\xi, R_\xi(X))\xi + k\phi R_\xi(X) \\ &\quad - \eta(X)R_\xi(\phi A\xi) - kR_\xi(\phi X) - \alpha\phi X \end{aligned} \tag{4.12}$$

for any X tangent to M. From (4.11) and (4.12) we have

$$\begin{aligned} &- g(\nabla_X A\xi,\xi)g(AY,\xi) + g(X,\phi AY) - g(A\xi,\phi AY)g(AX,\xi) \\ &\quad + \alpha g(X,\nabla_Y A\xi) - g(AX,\xi)g(\nabla_Y A\xi,\xi) - g(\phi AY, R_\xi(X)) \\ &= \eta(X)g(\phi A\xi, Y) - g(\nabla_\xi A\xi,\xi)g(AX,Y) - g(\phi A\xi, R_\xi(X))\eta(Y) \\ &\quad + kg(\phi R_\xi(X), Y) - \eta(X)g(R_\xi(\phi A\xi), Y) - kg(R_\xi(\phi X), Y) \end{aligned} \tag{4.13}$$

for any X, Y tangent to M. For $X = \xi$, (4.13) gives $-\alpha g(A\xi, \phi AY) = g(\phi A\xi, Y) - g(R_\xi(\phi A\xi), Y)$ for any Y tangent to M. That is,

$$\begin{aligned} -\alpha\beta g(U,\phi AY) &= \beta g(\phi U, Y) - \beta g(R_\xi(\phi U), Y) \\ &= \beta g(\phi U, Y) - \beta g(\phi U, Y) - \alpha\beta g(A\phi U, Y) \\ &= \alpha\beta g(U, \phi AY). \end{aligned}$$

Therefore, as $\alpha\beta \neq 0$, $g(A\phi U, Y) = 0$ for any Y tangent to M. That is,

$$A\phi U = 0. \tag{4.14}$$

From (4.14) we obtain $R_\xi(\phi U) = \phi U$ and (4.12) becomes

$$\begin{aligned} \alpha\nabla_X A\xi = 2\alpha A\phi AX &+ \phi AX - g(\nabla_\xi A\xi,\xi)AX \\ &+ g(X,\nabla_\xi A\xi)A\xi - g(AX,\xi)\nabla_\xi A\xi - R_\xi(\phi AX) \\ &+ k\phi R_\xi(X) - kR_\xi(\phi X) - \alpha\phi X \end{aligned} \tag{4.15}$$

for any X tangent to M. If $X = \xi$ it follows $\alpha\nabla_\xi A\xi = -\alpha\nabla_\xi A\xi$. As $\alpha \neq 0$ we have

$$\nabla_\xi A\xi = 0 \tag{4.16}$$

and (4.15) becomes

$$\alpha\nabla_X A\xi = 2\alpha A\phi AX + \phi AX - R_\xi(\phi AX) + k\phi R_\xi(X) - kR_\xi(\phi X) - \alpha\phi X \tag{4.17}$$

for any X tangent to M.

From (4.13) we obtain

$$\begin{aligned} &\alpha g(\nabla_X A\xi, \xi)g(AY, \xi) - \alpha g(X, \phi AY) - \alpha^2 g(X, \nabla_Y A\xi) \\ &+ \alpha g(AX, \xi)g(\nabla_Y A\xi, \xi) + \alpha g(\phi AY, R_\xi(X)) \\ &= \alpha g(\phi A\xi, R_\xi(X))\eta(Y) - k\alpha g(\phi R_\xi(X), Y) + k\alpha g(R_\xi(\phi X), Y). \end{aligned} \tag{4.18}$$

Taking $X = \phi U$, $Y = \xi$ in (4.18) we get $\alpha g(\nabla_{\phi U} A\xi, \xi) = 2\beta$, but from (4.17) $\alpha\nabla_{\phi U} A\xi = k\phi R_\xi(\phi U) + kR_\xi(U) + U$. Therefore $\alpha g(\nabla_{\phi U} A\xi, \xi) = 0$, which yields $\beta = 0$, giving a contradiction.

Thus we must suppose $\alpha = 0$. Let $X \in \mathbb{D}$. Then $(\hat{\nabla}^{(k)}_X R_\xi)\xi + (\hat{\nabla}^{(k)}_\xi R_\xi)X = 0$ gives $(\hat{\nabla}^{(k)}_\xi R_\xi)X = 0$. That is, $\hat{\nabla}^{(k)}_\xi R_\xi(X) - R_\xi(\hat{\nabla}^{(k)}_\xi X) = 0$. This yields

$$\begin{aligned} (\nabla_\xi R_\xi)X &= -\beta g(\phi U, R_\xi(X))\xi + k\phi R_\xi(X) - kR_\xi(\phi X) \\ &= -\beta g(\phi U, X)\xi - k\beta g(A\xi, X)\phi U + kg(A\xi, \phi X)A\xi \\ &= -\beta g(\phi U, X)\xi - k\beta^2 g(U, X)\phi U + k\beta^2 g(U, \phi X)U \end{aligned} \tag{4.19}$$

for any $X \in \mathbb{D}$.

In particular,

$$\begin{aligned} (\nabla_\xi R_\xi)\phi U &= -\beta\xi - k\beta^2 U \\ &= \nabla_\xi \phi U - R_\xi(\nabla_\xi \phi U) \\ &= -g(\phi U, \phi A\xi)\xi + \beta^2 g(U, \nabla_\xi \phi U)U \\ &= -\beta\xi + \beta^2 g(U, \nabla_\xi \phi U)U, \end{aligned}$$

that implies

$$g(\nabla_\xi U, \phi U) = k. \tag{4.20}$$

Taking $X = U$ in (4.19) we have $(\nabla_\xi R_\xi)U = -k\beta^2\phi U = \nabla_\xi(1 - \beta^2)U - R_\xi(\nabla_\xi U) = \nabla_\xi(-\beta^2 U) = -\xi(\beta^2)U - \beta^2\nabla_\xi U$. Thus we obtain

$$\xi(\beta) = 0. \tag{4.21}$$

If $m \geq 3$ we define $\mathbb{D}_U$ as in the previous section. Taking $X \in \mathbb{D}_U$ in (4.19) we get $(\nabla_\xi R_\xi)(X) = 0 = \nabla_\xi X - R_\xi(\nabla_\xi X) = g(A\xi, \nabla_\xi X)A\xi = \beta^2 g(U, \nabla_\xi X)U$. This yields

$$g(\nabla_\xi U, X) = 0 \tag{4.22}$$

for any $X \in \mathbb{D}_U$. As $g(\nabla_\xi U, U) = g(\nabla_\xi U, \xi) = 0$, from (4.20) and (4.22) we have

$$\nabla_\xi U = k\phi U. \tag{4.23}$$

Take $X, Y \in \mathbb{D}$ in (4.2). In our case we get

$$\begin{aligned} &- g(Y, \phi AX)\xi - \beta g(Y, \nabla_X(\beta U))U - \beta g(Y, U)\nabla_X(\beta U) \\ &- g(X, \phi AY)\xi - \beta g(X, \nabla_Y(\beta U))U - \beta g(X, U)\nabla_Y(\beta U) \\ = &- g(\phi AX, R_\xi(Y))\xi - g(\phi AY, R_\xi(X))\xi \end{aligned} \tag{4.24}$$

for any $X, Y \in \mathbb{D}$.

The scalar product of (4.24) and U, bearing in mind $\beta \neq 0$, yields

$$2X(\beta)g(Y, U) + \beta g(Y, \nabla_X U) + 2Y(\beta)g(X, U) + \beta g(X, \nabla_Y U) = 0. \tag{4.25}$$

Taking $X = Y = U$ in (4.25) we obtain

$$U(\beta) = 0. \tag{4.26}$$

Taking $X \in \mathbb{D}$ and orthogonal to U, $Y = U$ in (4.25) we have

$$2X(\beta) + \beta g(X, \nabla_U U) = 0 \tag{4.27}$$

for any $X \in \mathbb{D}$ and orthogonal to U.

As $R_\xi(U) = (1-\beta^2)U$, if we develop $(\hat{\nabla}_U^{(k)} R_\xi)U = 0$, bearing in mind (4.26) we obtain $\beta^2 \nabla_U U + \beta^2 g(\phi AU, U)\xi = 0$. Therefore, $g(\nabla_U U, X) = 0$ for any $X \in \mathbb{D}$ and orthogonal to U. From (4.27) it follows

$$X(\beta) = 0 \tag{4.28}$$

for any $X \in \mathbb{D}$ and orthogonal to U. From (4.21), (4.26) and (4.28), β is a constant. By Codazzi equation $(\nabla_\xi A)U - (\nabla_U A)\xi = \phi U$. This yields

$$\nabla_\xi AU - kA\phi U - \beta \nabla_U U + A\phi AU = \phi U. \tag{4.29}$$

Its scalar product with ξ gives

$$g(\nabla_\xi AU, \xi) + 2\beta g(U, \phi AU) = 0. \tag{4.30}$$

As $g(AU, \xi) = \beta$, a constant, $g(\nabla_\xi AU, \xi) = -g(AU, \phi A\xi) = \beta g(\phi AU, U)$. From (4.30) we obtain

$$g(AU, \phi U) = 0. \tag{4.31}$$

This and above results yield

$$\nabla_U U = 0. \tag{4.32}$$

Let $X \in \mathbb{D}_U$ (if $m \geq 3$). Codazzi equation gives $(\nabla_X A)\xi - (\nabla_\xi A)X = -\phi X$. This yields $\beta \nabla_X U - A\phi AX - \nabla_\xi AX + A\nabla_\xi X = -\phi X$. Its scalar product with ξ implies $3\beta g(A\phi U, X) - \beta g(X, \nabla_\xi U) = 0$. From (4.23) we have $g(A\phi U, X) = 0$ for any $X \in \mathbb{D}_U$. Now, from (4.30) we conclude that $A\phi U = \gamma \phi U$ for a certain function γ.

Once again, from Codazzi equation $(\nabla_{\phi U} A)\xi - (\nabla_\xi A)\phi U = U$. This yields $\beta \nabla_{\phi U} U - A\phi A\phi U - \nabla_\xi(\gamma \phi U) + A\nabla_\xi \phi U = U$. Its scalar product with ξ gives

$-2\beta g(U,\phi A\phi U)+\gamma g(\phi U,\phi A\xi)+\beta g(\nabla_\xi \phi U,U)=0$. From (4.23) we have $3\gamma - k=0$. That is,

$$A\phi U=\frac{k}{3}\phi U. \tag{4.33}$$

Moreover, $(\nabla_{\phi U}A)U-(\nabla_U A)\phi U=2\xi$. That is,

$$\nabla_{\phi U}AU-A\nabla_{\phi U}U-\frac{k}{3}\nabla_U\phi U+A\nabla_U\phi U=2\xi. \tag{4.34}$$

Its scalar product with ξ implies

$$g(\nabla_{\phi U}AU,\xi)+\frac{k}{3}g(\phi U,\phi AU)+\beta g(\nabla_U\phi U,U)=2.$$

This gives $g(\nabla_{\phi U}AU,\xi)+\frac{k}{3}g(U,AU)=2$. As $g(AU,\xi)=\beta$ is constant, $g(\nabla_{\phi U}AU,\xi)=-g(AU,\phi A\phi U)$. Therefore $g(\phi AU,A\phi U)+\frac{k}{3}g(U,AU)=2$, which yields

$$g(AU,U)=\frac{3}{k}. \tag{4.35}$$

We have $(\hat{\nabla}^{(k)}_U R_\xi)\phi U+(\hat{\nabla}^{(k)}_{\phi U}R_\xi)U=0$. As $R_\xi(\phi U)=\phi U$ and $R_\xi(U)=(1-\beta^2)U$, this yields $\hat{\nabla}^{(k)}_U\phi U-R_\xi(\hat{\nabla}^{(k)}_U\phi U)+(1-\beta^2)\hat{\nabla}^{(k)}_{\phi U}U-R_\xi(\hat{\nabla}^{(k)}_{\phi U}U)=0$. That is, $\beta^2 g(U,\hat{\nabla}^{(k)}_U\phi U)U-\beta^2\hat{\nabla}^{(k)}_{\phi U}U=0$. As $g(U,\hat{\nabla}^{(k)}_U\phi U)=g(U,\nabla_U\phi U+g(\phi AU\phi U)\xi)=0$, we get $\hat{\nabla}^{(k)}_{\phi U}U=0=\nabla_{\phi U}U+g(\phi A\phi U,U)\xi=\nabla_{\phi U}U-\frac{k}{3}\xi$. Therefore

$$\nabla_{\phi U}U=\frac{k}{3}\xi. \tag{4.36}$$

The scalar product of (4.34) and U yields

$$-g(AU,\nabla_{\phi U}U)-\frac{k}{3}\beta+g(\nabla_U\phi U,AU)=0,$$

where we have applied that $g(AU,U)=\frac{3}{k}$ is constant. Therefore

$$g(\nabla_{\phi U}AU,U)=-g(AU,\nabla_{\phi U}U).$$

As $(\nabla_U\phi)U=-g(AU,U)\xi=-\frac{3}{k}\xi$ and $\nabla_U\phi U=(\nabla_U\phi)U+\phi\nabla_U U=(\nabla_U\phi)U$, from (4.32) and (4.35) we get $\frac{2k}{3}\beta+\frac{3}{k}\beta=0$. This implies $2k^2+9=0$, which is impossible, finishing the proof.

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