

A NATURAL TOPOLOGICAL MANIFOLD STRUCTURE OF PHASE TROPICAL HYPERSURFACES

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ABSTRACT. First, we define phase tropical hypersurfaces in terms of a degeneration data of smooth complex algebraic hypersurfaces in $(\mathbb{C}^*)^n$. Next, we prove that complex hyperplanes are homeomorphic to their degeneration called phase tropical hyperplanes. More generally, using Mikhalkin’s decomposition into pairs-of-pants of smooth algebraic hypersurfaces, we show that a phase tropical hypersurface with smooth tropicalization is naturally a topological manifold. Moreover, we prove that a phase tropical hypersurface is naturally homeomorphic to a symplectic manifold.

1. Introduction

In this paper we deal with smooth algebraic hypersurfaces in the complex projective space $\mathbb{C}\mathbb{P}^n$. Let V be a smooth hypersurface in $\mathbb{C}\mathbb{P}^n$ of degree d . Recall that for a fixed degree, generically a hypersurface in the projective space is smooth and transverse to all coordinate hyperplanes and all their intersections. Moreover, hypersurfaces in $\mathbb{C}\mathbb{P}^n$ with the same degree are all diffeomorphic, and if we equip these hypersurfaces with the Fubini-Study symplectic form on $\mathbb{C}\mathbb{P}^n$, then they are also symplectomorphic. We denote by \mathring{V} the intersection $V \cap (\mathbb{C}^*)^n$ where $(\mathbb{C}^*)^n$ is the complement of the coordinate hyperplanes in $\mathbb{C}\mathbb{P}^n$. In this case, \mathring{V} is given by some polynomial equation. One can degenerate the complex standard structure of the complex algebraic torus to a worst possible degeneration, called “maximal degeneration” by M. Kontsevich and Y. Soibelman (see [7] and [8]), and see what kind of geometry can have a degeneration of our variety \mathring{V} . After taking the logarithm, $(\mathbb{C}^*)^n$ degenerates or, in other words, collapse onto \mathbb{R}^n , and our hypersurface onto a balanced rational

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polyhedral complex Γ was called *tropical variety*. One can ask the following question: *What kind of topology and geometry one can have on a nice lifting in $(\mathbb{C}^*)^n$ of this balanced rational polyhedral complex?* This paper gives an answer to this question using tools from tropical and phase tropical geometry.

Tropical geometry is a recent area of mathematics that can be seen as a limiting aspect (or “degeneration”) of algebraic geometry. In tropical geometry complex curves viewed as Riemann surfaces turn to metric graphs which is one dimensional combinatorial object, and n -dimensional complex varieties turn to n -dimensional polyhedral complexes with some properties such as the balancing condition. In other words, tropical varieties are finite dimensional polyhedral complexes with some additional properties. As example, the tropical projective space \mathbb{TP}^1 is a smooth projective tropical variety homeomorphic to the segment. In general, the tropical projective space \mathbb{TP}^n is a smooth projective tropical variety homeomorphic to the n -dimensional simplex. Moreover, as in the classical algebraic geometry, a projective tropical n -variety V is a certain n -dimensional polyhedral complex in \mathbb{TP}^N . One of the most interesting projective tropical varieties are obtained by the tropical limit of a family of projective algebraic varieties V_t with $1 \leq t < \infty$ and t tends to ∞ . To be more precise, they are the limit of amoebas where amoebas of algebraic (or analytic) varieties are their image under the logarithm with base a real number t . For example, every tropical hypersurface is provided by such way. Tropical objects are somehow, the image of a classical objects under the logarithm with base infinity, they are also called non-Archimedean amoebas.

Phase tropical varieties are some lifting of tropical varieties in the complex algebraic torus. More precisely, for any strictly positive real number t we define the self diffeomorphism H_t of $(\mathbb{C}^*)^n$. This defines a new complex structure J_t on $(\mathbb{C}^*)^n$ denoted by J_t different from the standard complex structure if $t \neq e^{-1}$. One way to define phase tropical varieties, is to take the limit \mathring{V}_∞ with respect to the Hausdorff metric on compact sets in $(\mathbb{C}^*)^n$ of a family of J_t -holomorphic varieties $\{\mathring{V}_t\}_{t \in [e^{-1}, \infty)}$ when t goes to ∞ . First, in case of hypersurfaces, we prove that if the hypersurfaces \mathring{V}_t are smooth with the same degree, in other words their defining polynomials have the same Newton polytope Δ , then for a sufficiently large t the \mathring{V}_t 's are homeomorphic to their degeneration \mathring{V}_∞ , and the compactification M_∞ of \mathring{V}_∞ in the toric variety X_Δ associated to Δ (see Subsection 3.3 for the precise definition of X_Δ) have the same properties, and we have the following:

Theorem 1.1. *Let $\mathring{V}_t \subset (\mathbb{C}^*)^n$ be a family of smooth complex algebraic hypersurfaces with a fixed degree Δ , and denote by \mathring{V}_∞ the phase tropical hypersurface associated to the family $\{\mathring{V}_t\}_t$ in other words, the limit of $H_t(\mathring{V}_t)$ when t goes to ∞ . Then for a sufficiently large $t \gg 0$ the following statements hold:*

- (i) *the hypersurface \mathring{V}_t is homeomorphic to \mathring{V}_∞ ;*
- (ii) *the compactification M_∞ of \mathring{V}_∞ in the toric variety X_Δ associated to Δ is homeomorphic to V_t , where V_t is the closure of \mathring{V}_t in X_Δ .*

The first part of Theorem 1.1 were proved at the same time by G. Kerr and I. Zharkov (see [6]). Their method used a barycentric subdivision of a smooth tropical hypersurface Γ with a special orientation on faces, and they define a skeleton, in other words, a deformation retract of V_∞ , of the phase tropical hypersurface V_∞ . Using the description of the coamoeba of a hyperplane and some technical topological lemma they get the result. Our method opens up more geometric perspectives.

Moreover, using the fact that pairs-of-pants possess a natural symplectic structure which gives rise to the standard symplectic structure on the complex projective space $\mathbb{C}\mathbb{P}^n$ after compactification, in other words collapsing the pair-of-pants boundary, and the gluing of pairs-of-pants can be done in a natural way symplectically, to obtain a natural symplectic structure on a manifold homeomorphic to our phase tropical hypersurface.

Let $(\mathring{V}_t, \iota_t^*(\omega)) \subset ((\mathbb{C}^*)^n, \omega)$ be a family of smooth symplectic hypersurfaces where ι_t is the inclusion map $\iota_t : \mathring{V}_t \hookrightarrow (\mathbb{C}^*)^n$, and ω is the symplectic form on the complex algebraic torus $(\mathbb{C}^*)^n$ defined by:

$$(1) \quad \omega = \frac{1}{2\sqrt{-1}} \sum_{i=1}^n \frac{dz_i}{z_i} \wedge \frac{d\bar{z}_i}{\bar{z}_i}.$$

Moreover, assume that the phase tropical hypersurface \mathring{V}_∞ which is the limit with respect to the Hausdorff metric on compact sets in $(\mathbb{C}^*)^n$ exists. In Theorem 1.2 we construct a symplectic manifold (M_∞, ω_{nat}) homeomorphic to \mathring{V}_∞ and symplectomorphic to $(\mathring{V}_t, \iota_t^*(\omega))$ for sufficiently large t .

Theorem 1.2. *Let $\mathring{V}_t \subset (\mathbb{C}^*)^n$ be a family of smooth complex algebraic hypersurfaces with a fixed degree Δ , and denote by \mathring{V}_∞ the phase tropical hypersurface associated to the family $\{\mathring{V}_t\}_t$, in other words, the limit of $H_t(\mathring{V}_t)$ when t goes to ∞ . There exists a symplectic manifold (M_∞, ω_{nat}) such that for a sufficiently large $t \gg 0$ the following statements hold:*

- (i) *the hypersurface \mathring{V}_∞ is homeomorphic to the symplectic manifold (M_∞, ω_{nat}) ;*
- (ii) *the hypersurfaces $(\mathring{V}_t, \iota_t^*(\omega))$ and (M_∞, ω_{nat}) are symplectomorphic.*

We will use the natural logarithm, in other words, with base the Napier's constant e , so that the Archimedean amoeba of a subvariety of the complex torus $(\mathbb{C}^*)^n$ is its image under the coordinatewise logarithm map. Recall that amoebas were introduced by Gelfand, Kapranov, and Zelevinsky in 1994 [2]. The coamoeba of a subvariety of $(\mathbb{C}^*)^n$ is its image under the coordinatewise argument map to the real torus $(S^1)^n$. Coamoebas were introduced by Passare in a talk in 2004 (see [12] and [13] for more details about coamoebas).

Our goal is to open a new way to prove the following: *If under the dequantization of a non-singular complex varieties tends to a non-singular phase tropical variety, then the dequantization provides us an isotopy between the*

complex variety and its corresponding phase tropical variety. This is the original Oleg Viro's conjecture in one of his talk in 2010 at MSRI, Berkeley. As a consequence we can obtain e.g. Mikhalkin's correspondance theorem, and other applications in mirror symmetry which is the subject of a forthcoming work in progress.

This paper is organized as follows. In Section 2, we explain preliminary results in this area. In Section 3, we define phase tropical hypersurface and describe tropical localization. In Section 4, we describe examples of coamoebas and phase tropical hypersurfaces. In Section 5, we give the proof of Theorem 1.1. In Section 6, we construct in a natural way a symplectic structure on a manifold homeomorphic to a given phase tropical variety which proves Theorem 1.2.

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2. Preliminaries

In this section we recall basic concepts of tropical hypersurfaces relevant for our paper. For the general case we can see [9] with more details. We consider algebraic hypersurfaces \mathring{V} in the complex algebraic torus $(\mathbb{C}^*)^n$, where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and $n \geq 1$ an integer. This means that \mathring{V} is the zero locus of a polynomial:

$$(2) \quad f(z) = \sum_{\alpha \in \text{supp}(f)} a_\alpha z^\alpha, \quad z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n},$$

where each a_α is a non-zero complex number and $\text{supp}(f)$ is a finite subset of \mathbb{Z}^n , called the support of the polynomial f , and its convex hull in \mathbb{R}^n is called the Newton polytope of f that we denote by Δ_f . Moreover, we assume that $\text{supp}(f) \subset \mathbb{N}^n$ and f has no factor of the form z^α .

The *amoeba* \mathcal{A}_f of an algebraic variety $\mathring{V} \subset (\mathbb{C}^*)^n$ is by definition (see M. Gelfand, M. M. Kapranov and A. V. Zelevinsky [2]) the image of \mathring{V} under the map:

$$\begin{aligned} \text{Log} : (\mathbb{C}^*)^n &\longrightarrow \mathbb{R}^n \\ (z_1, \dots, z_n) &\longmapsto (\log |z_1|, \dots, \log |z_n|). \end{aligned}$$

Let \mathbb{K} be the field of Puiseux series with real powers, which is the field of series $a(t) = \sum_{j \in A_a} \xi_j t^j$ with $\xi_j \in \mathbb{C}^*$ and A_a is a well-ordered subset of \mathbb{R} meaning that any subsets of A_a has a smallest element. It is well known that

the field \mathbb{K} is algebraically closed of characteristic zero. Moreover, it has a non-Archimedean valuation $\text{val}(a) = -\min A_a$:

$$\begin{cases} \text{val}(ab) = \text{val}(a) + \text{val}(b), \\ \text{val}(a + b) \leq \max\{\text{val}(a), \text{val}(b)\}, \end{cases}$$

and we set $\text{val}(0) = -\infty$. Let $g \in \mathbb{K}[z_1, \dots, z_n]$ be a polynomial as in (2). If $\langle \cdot, \cdot \rangle$ denotes the scalar product in \mathbb{R}^n , then the following piecewise affine linear convex function $g_{trop}(x) = \max_{\alpha \in \text{supp}(g)} \{\text{val}(a_\alpha) + \langle \alpha, x \rangle\}$, which is in the same time the Legendre transform of the function $\nu : \text{supp}(g) \rightarrow \mathbb{R}$ defined by $\nu(\alpha) = \min A_{a_\alpha}$, is called the *tropical polynomial* associated to g .

Definition. The tropical hypersurface Γ_g is the set of points in \mathbb{R}^n where the tropical polynomial g_{trop} is not smooth (called the corner locus of g_{trop}).

We have the following Kapranov’s theorem (see [5]):

Theorem 2.1 ([5], Kapranov). *The tropical hypersurface Γ_g defined by the tropical polynomial g_{trop} is the subset of \mathbb{R}^n image under the valuation map of the algebraic hypersurface with defining polynomial g .*

Γ_g is also called the non-Archimedean amoeba of the zero locus of g in $(\mathbb{K}^*)^n$.

Let g be a polynomial as above, Δ its Newton polytope, and $\tilde{\Delta}$ its extending Newton polytope, i.e., $\tilde{\Delta} := \text{convexhull}\{(\alpha, r) \in \text{supp}(g) \times \mathbb{R} \mid r \geq \min A_{a_\alpha}\}$. Let us extend the above function ν defined on $\text{supp}(g)$ to all Δ as following:

$$\begin{aligned} \nu : \Delta &\longrightarrow \mathbb{R} \\ \alpha &\longmapsto \min\{r \mid (\alpha, r) \in \tilde{\Delta}\}. \end{aligned}$$

By taking the linear subsets of the lower boundary of $\tilde{\Delta}$, it is clear that the linearity domains of ν define a convex subdivision $\tau = \{\Delta_1, \dots, \Delta_l\}$ of Δ . Let $y = \langle x, v_i \rangle + r_i$ be the equation of the hyperplane $Q_i \subset \mathbb{R}^n \times \mathbb{R}$ containing points of coordinates $(\alpha, \nu(\alpha))$ with $\alpha \in \text{Vert}(\Delta_i)$.

There is a duality between the subdivision τ and the subdivision of \mathbb{R}^n induced by Γ_g , where each connected component of $\mathbb{R}^n \setminus \Gamma_g$ is dual to some vertex of τ_f and each k -cell of Γ_g is dual to some $(n - k)$ -cell of τ . In particular, each $(n - 1)$ -cell of Γ_g is dual to some edge of τ . If $x \in E_{\alpha\beta}^* \subset \Gamma_g$, then $\langle \alpha, x \rangle - \nu(\alpha) = \langle \beta, x \rangle - \nu(\beta)$, so $\langle \alpha - \beta, x - v_i \rangle = 0$. This means that v_i is a vertex of Γ_g dual to some Δ_i having $E_{\alpha\beta}$ as edge (see [4] for more details).

Definition. A tropical hypersurface $\Gamma \subset \mathbb{R}^n$ is smooth if and only if its dual subdivision is a triangulation where the Euclidean volume of every triangle is equal to $\frac{1}{n!}$.

Let $\hat{V} \subset (\mathbb{C}^*)^n$ be an algebraic hypersurface defined by a polynomial $f(z) = \sum_{\alpha_i \in A} a_{\alpha_i} z^{\alpha_i}$, with support $A = \{\alpha_1, \dots, \alpha_l, \alpha_{l+1}, \dots, \alpha_r\} \subset \mathbb{Z}^n$, and $A' = \{\alpha_{l+1}, \dots, \alpha_r\} = \text{Im}(\text{ord})$ where ord is the order mapping from the set of complement components of the amoeba \mathcal{A} of \hat{V} to $\Delta \cap \mathbb{Z}^n$ (see [1]). It was

shown by Mikael Passare and Hans Rullgård (see [14]) that the spine Γ of the amoeba \mathcal{A} is a non-Archimedean amoeba defined by the tropical polynomial

$$f_{trop}(x) = \max_{\alpha \in A'} \{c_\alpha + \langle \alpha, x \rangle\},$$

where c_α are a constants defined by:

$$(3) \quad c_\alpha = \mathbb{R} \left(\frac{1}{(2\pi i)^n} \int_{\text{Log}^{-1}(x)} \log \left| \frac{f(z)}{z^\alpha} \right| \frac{dz_1 \wedge \cdots \wedge dz_n}{z_1 \cdots z_n} \right)$$

where $x \in E_\alpha$, $z = (z_1, \dots, z_n) \in (\mathbb{C}^*)^n$. In other words, the spine of \mathcal{A} is defined as the set of points in \mathbb{R}^n where the piecewise affine linear function f_{trop} is not differentiable. Let us denote by τ the convex subdivision of Δ dual to the tropical variety Γ . Then the set of vertices of τ is precisely the image of the order mapping (i.e., A'). By duality, this means that the convex subdivision $\tau = \cup_{i=1}^r \Delta_{v_i}$ of Δ is determined by a piecewise affine linear map $\nu : \Delta \rightarrow \mathbb{R}$ so that:

- (i) $\nu|_{\Delta_{v_i}}$ is affine linear for each v_i .
- (ii) if $\nu|_U$ is affine linear for some open set $U \subset \Delta$, then there exists v_i such that $U \subset \Delta_{v_i}$.
- (iii) $\nu(\alpha) = -c_\alpha$ for any $\alpha \in \text{Im}(\text{ord})$.

We define the *generalized s-Passare-Rullgård function* by the following:

Definition. Let $s = (s_1, \dots, s_l) \in \mathbb{R}_+^l$ and $\nu_{PR}^s : A \rightarrow \mathbb{R}$ be the function, called the generalized s -Passare-Rullgård function, is defined by:

$$\nu_{PR}^s(\alpha) = \begin{cases} -c_\alpha & \text{if } \alpha \in \text{Im}(\text{ord}), \\ \langle \alpha_j, a_v \rangle + b_v + s_j & \text{if } \alpha = \alpha_j \text{ for } j = 1, \dots, l, \end{cases}$$

where $\alpha_j \in \Delta_v$, $\Delta_v \in \tau$ and $y = \langle x, a_v \rangle + b_v$ is the equation of the hyperplane in $\mathbb{R}^n \times \mathbb{R}$ containing the points of coordinates $(\beta; -c_\beta)$ with $\beta \in \text{Vert}(\Delta_v)$.

Assume that we have a hypersurface $\mathring{V} \subset (\mathbb{C}^*)^n$ defined by the polynomial $f(z) = \sum_{\alpha \in A} a_\alpha z^\alpha$ with $a_\alpha \in \mathbb{C}^*$, A a finite subset of \mathbb{Z}^n and $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}$. We denote by Δ the convex hull of A in \mathbb{R}^n which is the Newton polytope of f . We can consider the family of hypersurfaces $\mathring{V}_{f(t; s)} \subset (\mathbb{C}^*)^n$ defined by the following family of polynomials:

$$(4) \quad f_{(t; s)}(z) = \sum_{\alpha \in A} \xi_\alpha t^{\nu_{PR}^s(\alpha)} z^\alpha,$$

with $\xi_\alpha = a_\alpha e^{\nu_{PR}^s(\alpha)}$, and we view this family as a deformation of f .

Let us denote by $\mathcal{C}oh_A(\Delta)$ the set of coherent (i.e., convex) triangulations of Δ such that the set of vertices of all its elements is contained in A . For each $\tau \in \mathcal{C}oh_A(\Delta)$, assume $\nu : \Delta \rightarrow \mathbb{R}$ is a convex function defining τ . Let $f^{(\tau)}$ be the non-Archimedean polynomial defined by:

$$f^{(\tau)}(z) = \sum_{\alpha \in A} a_\alpha t^{\nu(\alpha)} z^\alpha.$$

We denote by $co\mathcal{A}_{\mathbb{C}}(f)$ (resp. $co\mathcal{A}_{\mathbb{K}}(f)$) the complex coamoeba (resp. non-Archimedean coamoeba) of the hypersurface with defining polynomial f .

3. Phase tropical hypersurfaces

3.1. Phase tropical hypersurfaces

For every strictly positive real number t we define the self diffeomorphism H_t of $(\mathbb{C}^*)^n$ by:

$$H_t : (\mathbb{C}^*)^n \longrightarrow (\mathbb{C}^*)^n$$

$$(z_1, \dots, z_n) \longmapsto \left(|z_1|^{-\frac{1}{\log t}} \frac{z_1}{|z_1|}, \dots, |z_n|^{-\frac{1}{\log t}} \frac{z_n}{|z_n|} \right).$$

This defines a new complex structure on $(\mathbb{C}^*)^n$ denoted by $J_t = (dH_t)^{-1} \circ J \circ (dH_t)$ where J is the standard complex structure.

A J_t -holomorphic hypersurface \mathring{V}_t is a holomorphic hypersurface with respect to the J_t complex structure on $(\mathbb{C}^*)^n$. It is equivalent to say that $\mathring{V}_t = H_t(\mathring{V})$ where $\mathring{V} \subset (\mathbb{C}^*)^n$ is an holomorphic hypersurface for the standard complex structure J on $(\mathbb{C}^*)^n$.

Recall that the Hausdorff distance between two closed subsets A, B of a metric space (E, d) is defined by:

$$d_H(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b)\}.$$

Here $E = \mathbb{R}^n \times (S^1)^n$ is equipped with the distance defined as the product of the Euclidean metric on \mathbb{R}^n and the flat metric on $(S^1)^n$.

Definition. A phase tropical hypersurface $\mathring{V}_\infty \subset (\mathbb{C}^*)^n$ is the limit (with respect to the Hausdorff metric on compact sets in $(\mathbb{C}^*)^n$) of a sequence of a J_t -holomorphic hypersurfaces $\mathring{V}_t \subset (\mathbb{C}^*)^n$ when t tends to ∞ .

We have an algebraic definition of phase tropical hypersurfaces in case of curves which is called complex tropical curves (see [10]) as follows:

Let $a \in \mathbb{K}^*$ be the Puiseux series $a = \sum_{j \in A_a} \xi_j t^j$ with $\xi \in \mathbb{C}^*$ and $A_a \subset \mathbb{R}$ is a well-ordered set with smallest element. Then we have a non-Archimedean valuation on \mathbb{K} defined by $\text{val}(a) = -\min A_a$. We complexify the valuation map as follows:

$$w : \mathbb{K}^* \longrightarrow \mathbb{C}^*$$

$$a \longmapsto w(a) = e^{\text{val}(a) + i \arg(\xi_{-\text{val}(a)})}.$$

Let Arg be the argument map $\mathbb{K}^* \rightarrow S^1$ defined by $\text{Arg}(a) = e^{i \arg(\xi_{-\text{val}(a)})}$ for any Puiseux series $a = \sum_{j \in A_a} \xi_j t^j$. Note that this map extends the map $\mathbb{C}^* \rightarrow S^1$ defined by $\rho e^{i\theta} \mapsto e^{i\theta}$ which we denote by Arg .

Applying this map coordinatewise we obtain a map:

$$W : (\mathbb{K}^*)^n \longrightarrow (\mathbb{C}^*)^n$$

Theorem 3.1 (Mikhalkin, 2002). *The set $\mathring{V}_\infty \subset (\mathbb{C}^*)^n$ is a phase tropical hypersurface if and only if there exists an algebraic hypersurface $\mathring{V}_\mathbb{K} \subset (\mathbb{K}^*)^n$ over \mathbb{K} such that $\overline{W(\mathring{V}_\mathbb{K})} = \mathring{V}_\infty$, where $\overline{W(\mathring{V}_\mathbb{K})}$ is the closure of $W(\mathring{V}_\mathbb{K})$ in $(\mathbb{C}^*)^n \approx \mathbb{R}^n \times (S^1)^n$ as a Riemannian manifold with metric defined by the standard Euclidean metric of \mathbb{R}^n and the standard flat metric of the real torus.*

Let $f_t(z) = \sum_\alpha a_\alpha t^{-v(\alpha)} z^\alpha$ be a polynomial with a parameter t , and $\mathring{V}_t = \{f_t = 0\} \subset (\mathbb{C}^*)^n$. The family of f_t can be viewed as a single polynomial in $\mathbb{K}[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$. We have the following theorems (see [10], [11], and [15]):

Theorem 3.2 (Mikhalkin, Rullgård (2001)). *The amoebas \mathcal{A}_t of \mathring{V}_t converge in the Hausdorff metric to the non-archimedean amoeba $\mathcal{A}_\mathbb{K}$ when $t \rightarrow \infty$.*

Theorem 3.3 (Mikhalkin). *The sets $H_t(\mathring{V}_t)$ converge in the Hausdorff metric to $W(\mathring{V}_\mathbb{K})$ when $t \rightarrow \infty$.*

3.2. Tropical localization

Let ν be the piecewise affine linear map defined in Section 2, and $\tilde{\Delta}$ be the extended polyhedron of Δ associated to ν , that is the convex hull of the set $\{(\alpha, u) \in \Delta \times \mathbb{R} \mid u \geq \nu(\alpha)\}$. For any $\Delta_{v_i} \in \tau$, let $\lambda(x) = \langle x, a_{v_i} \rangle + b_{v_i}$ be the affine linear map defined on Δ such that $\lambda|_{\Delta_{v_i}} = \nu|_{\Delta_{v_i}}$ where $\langle \cdot, \cdot \rangle$ is the scalar product in \mathbb{R}^n , $a_{v_i} = (a_{v_i, 1}, \dots, a_{v_i, n}) \in \mathbb{R}^n$ (which is the coordinates of the vertex of the spine Γ , dual to Δ_{v_i}), and b_{v_i} is a real number. Let $s \in \mathbb{R}_+^l$ as above and put $\nu' = \nu_{PR}^{(s)} - \lambda$ and we define the family of polynomials $\{f'_{(t,s)}\}_{t \in (0, \frac{1}{e}]}$ by:

$$f'_{(t,s)}(z) = \sum_{\alpha \in A} \xi_\alpha t^{\nu'(\alpha)} z^\alpha,$$

where $\xi_\alpha \in \mathbb{C}$. Then we have:

$$\begin{aligned} f'_{(t,s)}(z) &= t^{-b_v} \sum_{\alpha \in A} \xi_\alpha t^{\nu_{PR}^{(s)}(\alpha)} (z_1 t^{-a_{v_i, 1}})^{\alpha_1} \dots (z_n t^{-a_{v_i, n}})^{\alpha_n} \\ &= t^{-b_v} f_{(t,s)} \circ \Phi_{\Delta_{v_i}, t}^{-1}(z), \end{aligned}$$

where $f_{(t,s)}$ is the polynomial defined in (4), and $\Phi_{\Delta_{v_i}, t}$ is the self diffeomorphism of $(\mathbb{C}^*)^n$ defined by:

$$\begin{aligned} \Phi_{\Delta_{v_i}, t} &: (\mathbb{C}^*)^n &\longrightarrow (\mathbb{C}^*)^n \\ (z_1, \dots, z_n) &\longmapsto (z_1 t^{a_{v_i, 1}}, \dots, z_n t^{a_{v_i, n}}). \end{aligned}$$

This means that the polynomials $f'_{(t,s)}$ and $f_{(t,s)} \circ \Phi_{\Delta_{v_i}, t}^{-1}$ define the same hypersurface. Whence we have:

$$\mathring{V}_{f'_{(t,s)}} = \mathring{V}_{f_{(t,s)} \circ \Phi_{\Delta_{v_i}, t}^{-1}} = \Phi_{\Delta_{v_i}, t}(\mathring{V}_{f_{(t,s)}}),$$

where \mathring{V}_g denotes algebraic hypersurface in $(\mathbb{C}^*)^n$ with defining polynomial g . Let U_{v_i} be a small ball in \mathbb{R}^n with center the vertex of $\Gamma_{(t,s)}$ dual to

Δ_{v_i} where $\Gamma_{(t;s)}$ is the spine of the amoeba $\mathcal{A}_{H_t(\check{V}_{f(t;s)})}$ where H_t denotes the self diffeomorphism of $(\mathbb{C}^*)^n$ defined as in Subsection 3.1, and $\text{Log}_t = \text{Log} \circ H_t$. Let $f_{(t;s)}^{\Delta_{v_i}}$ be the truncation of $f_{(t;s)}$ to Δ_{v_i} , and $\check{V}_{\infty, \Delta_{v_i}}$ be the complex tropical hypersurface with tropical coefficients of index $\alpha \in \Delta_{v_i}$, in other words, $\check{V}_{\infty, \Delta_{v_i}} = \lim_{t \rightarrow 0} H_t(\check{V}_{f_{(t;s)}^{\Delta_{v_i}}})$. Using Kapranov's theorem (see [5]), we obtain the following Proposition called a tropical localization by Mikhalkin (see [10]):

Proposition 3.4. *Let s be in \mathbb{R}_+^l . For any $\varepsilon > 0$ there exists t_0 such that if $t \geq t_0$ then the image under $\Phi_{\Delta_{v_i}, t} \circ H_t^{-1}$ of $H_t(\check{V}_{f(t;s)}) \cap \text{Log}^{-1}(U_{v_i})$ is contained in the ε -neighborhood of the image under $\Phi_{\Delta_{v_i}, t} \circ H_t^{-1}$ of the phase tropical hypersurface $\check{V}_{\infty, \Delta_{v_i}}$ corresponding to the family $\{\check{V}_{f(t;s)}\}_t$, with respect to the product metric in $(\mathbb{C}^*)^n \approx \mathbb{R}^n \times (S^1)^n$.*

Proof. By decomposition of $f'_{(t,s)}$, we obtain:

$$(5) \quad f'_{(t,s)}(z) = t^{-b_v} \sum_{\alpha \in \Delta_v \cap A} \xi_\alpha t^{\nu(\alpha) - \langle \alpha, a_v \rangle} z^\alpha + \sum_{\alpha \in A \setminus \Delta_v} \xi_\alpha t^{\nu(\alpha) - \langle \alpha, a_v \rangle - b_v} z^\alpha.$$

On the other hand, we have the following commutative diagram:

$$(6) \quad \begin{array}{ccc} (\mathbb{C}^*)^n & \xrightarrow{\Phi_{\Delta_v, t}} & (\mathbb{C}^*)^n \\ \text{Log}_t \downarrow & & \downarrow \text{Log}_t \\ \mathbb{R}^n & \xrightarrow{\phi_{\Delta_v}} & \mathbb{R}^n, \end{array}$$

such that if $v = (a_{v,1}, \dots, a_{v,n}) \in \mathbb{R}^n$ is the vertex of the tropical hypersurface Γ dual to the element Δ_v of the subdivision τ , then $\phi_{\Delta_v}(x_1, \dots, x_n) = (x_1 - a_{v,1}, \dots, x_n - a_{v,n})$. Let U_v be a small open ball in \mathbb{R}^n centered at v .

Assume that $\text{Log}_t(z) \in \phi_{\Delta_v}(U_v)$ and z is not singular in \check{V}_t . Then the second sum in (5) converges to zero when t goes to infinity, because by the choice of z and U_v , the tropical monomials in $f'_{trop, (t,s)}$, corresponding to lattice points of Δ_v , dominates the monomials corresponding to lattice points of $A \setminus \Delta_v$. But the first sum in (5) is just a polynomial defining the hypersurface $\Phi_{\Delta_v, t}(V_{f_{(t,s)}^{\Delta_v}})$.

By the commutativity of diagram (6), if $z \in \check{V}_{f_{(t,s)'}}$ is such that $\text{Log}_t(z) \in \phi_{\Delta_v}(U_v)$ then $\text{Log}_t \circ \Phi_{\Delta_v, t}^{-1}(z) \in U_v$, and hence $H_t(\Phi_{\Delta_v, t}^{-1}(z)) \in \text{Log}^{-1}(U_v)$. Hence the image under $\Phi_{\Delta_v, t} \circ H_t^{-1}$ of $H_t(\check{V}_{f(t;s)}) \cap \text{Log}^{-1}(U_v)$ is contained in an ε -neighborhood of the image under $\Phi_{\Delta_v, t} \circ H_t^{-1}$ of $H_t(\check{V}_{f_{(t,s)}^{\Delta_v}})$ for sufficiently large t and the proposition is proved because $\check{V}_{\infty, \Delta_v}$ is the limit when t tends to ∞ of the sequence of J_t -holomorphic hypersurfaces $H_t(V_{f_{(t,s)}^{\Delta_v}})$ by taking a discrete sequence t_k converging to ∞ if necessary. In particular the set of arguments of $\check{V}_{\infty, f} \cap \text{Log}^{-1}(v)$ is contained in the set of arguments of

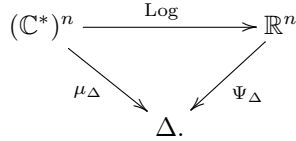
V_{∞, Δ_v} , in other words, $\text{Arg}(\mathring{V}_{\infty, f} \cap \text{Log}^{-1}(v)) \subseteq \text{Arg}(\mathring{V}_{\infty, \Delta_v})$. If it is not the case, we can get away too after applying $\Phi_{\Delta_v, t} \circ H_t^{-1}$ for sufficiently large t . \square

3.3. Toric varieties

To every convex polyhedron $\Delta \subset \mathbb{R}^n$ with integer vertices, there is a complex toric variety X_{Δ} containing $(\mathbb{C}^*)^n$. Indeed, we can consider the Veronese embedding $\rho : (\mathbb{C}^*)^n \rightarrow \mathbb{C}\mathbb{P}^{\#(\Delta \cap \mathbb{Z}^n) - 1}$ defined by the monomial map associated to $\Delta \cap \mathbb{Z}^n$: $(z_1, \dots, z_n) \mapsto z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}$ for each $\alpha := (\alpha_1, \dots, \alpha_n) \in \Delta \cap \mathbb{Z}^n$; and X_{Δ} is defined as the closure of the image of $(\mathbb{C}^*)^n$. Then the Fubini-Study symplectic form on the projective spaces $\mathbb{C}\mathbb{P}^{\#(\Delta \cap \mathbb{Z}^n) - 1}$ defines a natural symplectic form on X_{Δ} . In particular we obtain a symplectic form ω_{Δ} on $(\mathbb{C}^*)^n$ invariant under the Hamiltonian action of the real torus $(S^1)^n$. This gives a moment map μ_{Δ} with respect to ω_{Δ} :

$$\begin{aligned} \mu_{\Delta} : (\mathbb{C}^*)^n &\longrightarrow \Delta \\ z &\longmapsto \frac{\sum_{\alpha \in \Delta \cap \mathbb{Z}^n} \sum_{i=1}^n \alpha_i |z_i^{2\alpha_i}|}{\sum_{\alpha \in \Delta \cap \mathbb{Z}^n} \sum_{i=1}^n |z_i^{2\alpha_i}|}, \end{aligned}$$

which is an embedding with image the interior of Δ .



The maps Log and μ_{Δ} both have orbits $(S^1)^n$ as fibers, and we obtain a reparametrization of \mathbb{R}^n which we denote by Ψ_{Δ} (see [2]).

Definition. Let $\Gamma \subset \mathbb{R}^n$ be an n -dimensional balanced polyhedral complex, and Δ its dual convex lattice polyhedron. $\bar{\Gamma} \subset \Delta$ is the compactification of Γ by taking $\Psi_{\Delta}(\Gamma)$ in Δ . $\bar{\Gamma} \setminus \Psi_{\Delta}(\Gamma)$ is called the boundary of $\bar{\Gamma}$.

Let f be a Laurent polynomial in $\mathbb{C}[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$, and Δ be its Newton polytope. Let $\mathring{V} := \{z \in (\mathbb{C}^*)^n \mid f(z) = 0\}$ be the hypersurface in $(\mathbb{C}^*)^n$ with defining polynomial f . Let X_{Δ} be the complex toric variety as defined before. We denote by V the closure of the hypersurface \mathring{V} in X_{Δ} .

Let Δ be a compact convex lattice polyhedron such that the singularity of its corresponding toric variety X_{Δ} are on the vertices of Δ . Let $(\mathbb{C}^*)^{\#(\Delta \cap \mathbb{Z}^n)}$ be the set of all polynomial $f(z) = \sum_{\alpha \in \Delta \cap \mathbb{Z}^n} a_{\alpha} z^{\alpha}$ such that $a_{\alpha} \neq 0$. Then for a generic choice of a polynomial, the closure V in X_{Δ} of the zero set of f is a smooth hypersurface transverse to all toric subvarieties $X_{\Delta'}$, corresponding to the faces $\Delta' \subset \Delta$. In particular, all such hypersurfaces V are diffeomorphic,

even symplectomorphic if they are equipped with the symplectic form coming from the one of X_Δ .

4. Examples of coamoebas and phase tropical hypersurfaces

(a) Let \mathring{V} be the line in $(\mathbb{C}^*)^2$ defined by the polynomial $f(z, w) = r_1 e^{i\alpha_1} z + r_2 e^{i\alpha_2} w + r_3 e^{i\alpha_3}$ where r_i are real positive numbers and $\alpha_1 > \alpha_3 > \alpha_2 > 0$. Then its coamoeba is as displayed in Figure 1. The equations of the external hyperplanes are given by (1) $y = x + \alpha_1 - \alpha_2 + (2k + 1)\pi$, (2) $x = \alpha_3 - \alpha_1 + (2l + 1)\pi$, and (3) $y = \alpha_3 - \alpha_2 + (2m + 1)\pi$ with k, l and m in \mathbb{Z} (the external hyperplanes are seen in \mathbb{R}^2 the universal covering of the torus).

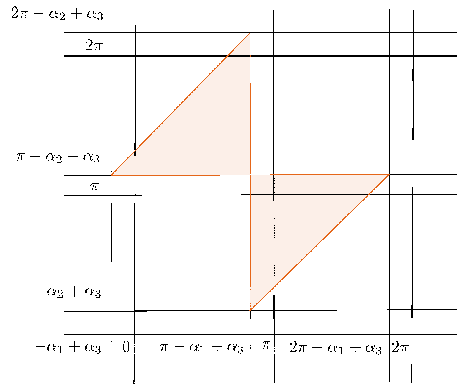


FIGURE 1. The coamoeba of the line in $(\mathbb{C}^*)^2$ defined by the polynomial $f(z, w) = r_1 e^{i\alpha_1} z + r_2 e^{i\alpha_2} w + r_3 e^{i\alpha_3}$ where r_i are real positive numbers and $\alpha_1 > \alpha_3 > \alpha_2 > 0$.

We can remark that in this case there are no extra-pieces, and all the boundary of the closure of this coamoeba in the torus is contained in three external hyperplanes.

(b) Consider now the example of a parabola. Let $\mathring{V}_f \subset (\mathbb{C}^*)^2$ the curve defined by the polynomial $f(z, w) = w - z^2 + 2z - \lambda$ with $\lambda > 1$. Consider the parametrization defined by:

$$\begin{cases} z(r, \alpha) = r e^{i\alpha}, \\ w(r, \alpha) = r^2 e^{2i\alpha} - 2r e^{i\alpha} + \lambda, \end{cases}$$

with $r > 0$ and $\alpha \in [0, 2\pi]$. We have to compute the argument of $r^2 e^{2i\alpha} - 2r e^{i\alpha} + \lambda$, with $r \in \mathbb{R}_+^*$. Let $a = \lambda - 1$, so we have $w(r, \alpha) = (r e^{i\alpha} - 1)^2 + a$ and then $\beta = \arg(w(r, \alpha)) = \arg\left[\frac{(r e^{i\alpha} - 1) - i\sqrt{a}}{(r e^{-i\alpha} - 1) - i\sqrt{a}}\right]$.

- (i) Let $0 \leq \alpha \leq \arctan \sqrt{a}$ then $0 \leq \beta \leq 2\alpha$ if $1 + \tan^2 \alpha \leq r^2 < \infty$ and $g_\alpha(r) \leq \beta \leq 2\pi$ if $0 < r^2 < 1 + \tan^2 \alpha$ where for each α , g_α is

- a differentiable function with one maximum in the interval $0 < r^2 < 1 + \tan^2 \alpha$ (see Figure 2);
- (ii) If $\pi \geq \alpha \geq \arctan \sqrt{a}$ then $2\alpha \leq \beta \leq 2\pi$;
- (iii) For $\alpha > \pi$ we have the conjugate of the sets in (i) and (ii).

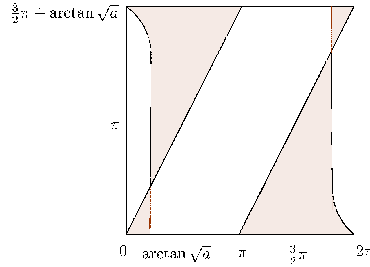


FIGURE 2. Coamoeba of a parabola.

We can view a parabola as an algebraic curve $\mathring{V}_{f_{\mathbb{K}}}$ over the field of the Puiseux series with real powers \mathbb{K} , defined by the polynomial $f_{\mathbb{K}}(z, w) = f_t(z, w) = t^0 w - t^0 z^2 + 2t^0 z - t^{-\text{Log } \lambda}$ with $z, w \in \mathbb{K}^*$ and $t \in \mathbb{R}_+^*$. It is clear that the limit of the coamoebas of the curves \mathring{V}_{f_t} converge to the coamoeba of the phase tropical curve with tropical coefficients $a_{01} = 1, a_{00} = -\lambda$ and $a_{20} = -1$, which are the coefficients with index in $\text{Vert}(\tau)$ where τ is the triangulation of the Newton polygon of f dual to Γ , with Γ the tropical curve that is the spine of the amoeba of \mathring{V}_f (see Figure 3, the coamoeba of a phase tropical parabola).

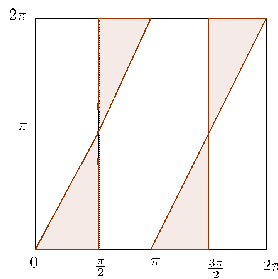


FIGURE 3. Coamoeba of a parabola with coefficients only in the vertices of the Newton polygon of its defining polynomial.

We can see in Figure 2 extra-pieces in the coamoeba of our parabola.

- (c) Let V_λ be the complex curve defined by the polynomial $f(z, w) = \lambda + z + w + zw$ with $\lambda \in \mathbb{R}^*$ with Newton polygon the standard square of vertices $(0, 0), (1, 0), (0, 1)$ and $(1, 1)$.

1st case. Assume $0 < \lambda < 1$, and we parametrize $z = re^{i\alpha}$ with $\alpha \in [0, 2\pi]$ and $r \in \mathbb{R}_+^*$. So $\arg(w(r, \alpha)) = \theta(r, \alpha)$ with:

$$\theta(r, \alpha) = \arcsin \left(\frac{-r(1 - \lambda) \sin \alpha}{((\lambda + r(1 + \lambda) \cos \alpha + r^2)^2 + r^2(1 - \lambda)^2 \sin^2 \alpha)^{\frac{1}{2}}} \right)$$

and we have $\frac{\partial \theta}{\partial r}(r, \alpha) = 0$ if and only if $r = \pm\sqrt{\lambda}$, so $r = \sqrt{\lambda}$ and the maximum of the argument of w is attained at $r = \sqrt{\lambda}$, this means that we have

$$\theta_{\max}(\alpha) = \arcsin \left(\frac{-\sqrt{\lambda}(1 - \lambda) \sin \alpha}{((2\lambda + \sqrt{\lambda}(1 + \lambda) \cos \alpha)^2 + \lambda(1 - \lambda)^2 \sin^2 \alpha)^{\frac{1}{2}}} \right).$$

If $0 < \lambda < 1$ it can be viewed as a parameter, and hence as an element of \mathbb{K}^* , which means that the curve V_λ is viewed as an algebraic curve over \mathbb{K} , i.e. $V_\lambda^{\mathbb{K}} = \{(z, w) \in (\mathbb{K}^*)^2 \mid \lambda + z + w + zw = 0\}$ and $\text{Log}_{\mathbb{K}}(V_\lambda^{\mathbb{K}})$ is the tropical curve with tropical polynomial $f_{\text{trop}}(x, y) = \max\{x, y, x + y, -1\}$. We have $\text{Log}^{-1}(v_1) \cap W(V_\lambda^{\mathbb{K}})$ is the union of the two sets of $S^1 \times S^1$ with boundary the two half of the cycles $\delta_1 = \{\alpha = \pi\}$ and $\delta_2 = \{\beta = \pi\}$ and the half of the cycle defined by the graph of the function θ_{\max} , which is homotopic to the product of δ_1 and δ_2 . We have the same result for the vertex v_2 .

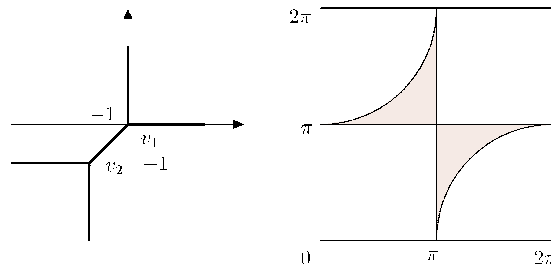


FIGURE 4. The spine of the amoeba of the hyperbola defined by the polynomial f_λ with $0 < \lambda < 1$ and its coamoeba.

2nd case. Suppose $\lambda > 1$, and let $\tau = \frac{1}{\lambda}$. So, $\lambda = \tau^{-1}$ and then $V_\tau^{\mathbb{K}} = \{(z, w) \in (\mathbb{K}^*)^2 \mid \tau^{-1} + z + w + zw = 0\}$. Hence $\text{Log}_{\mathbb{K}}(V_\tau^{\mathbb{K}})$ is the tropical curve with tropical polynomial $f_{\text{trop}}(x, y) = \max\{x, y, x + y, +1\}$. Hence, we have:

$$\begin{aligned} \text{Log}^{-1}(v_1) \cap W(V_\tau^{\mathbb{K}}) &= \{(\alpha, \beta) \in S^1 \times S^1 \mid 0 \leq \alpha \leq \pi, \theta_{\max}(\alpha) \leq \beta \leq \pi\} \\ &\cup \{(\alpha, \beta) \in S^1 \times S^1 \mid \pi \leq \alpha \leq 2\pi, \pi \leq \beta \leq \theta_{\max}(\alpha)\}. \end{aligned}$$

3rd case. Assume $\lambda = 1 = t^0$, so we have $f_1(z, w) = (1 + z)(1 + w)$, and the corresponding tropical curve is the union of two axes, and $\text{Log}^{-1}(v_1) \cap W(V_1^{\mathbb{K}})$ is the union of two circles (the valuation of the constant coefficient is zero in this case).

4th case. Suppose $\lambda < 0$ and $\lambda \neq -1$. If $|\lambda| < 1$, then consider λ as a parameter and we have the tropical curve of the first case (it means that the

valuation of the constant coefficient is negative). So, if we put $z(t) = t^{-x}e^{i\alpha}$, then $w(t)_\alpha = -\left(\frac{t-t^{-x}e^{i\alpha}}{-1-t^{-x}e^{i\alpha}}\right)$ and then $\text{Log}^{-1}(v_1) \cap W(V_\lambda^{\mathbb{K}})$ is the closure in $S^1 \times S^1$ of the set

$$\left(\alpha, \lim_{t \rightarrow 0, x \rightarrow 0} \arg(w(t)_\alpha)\right)$$

with $0 \leq \alpha \leq 2\pi$. We then obtain the union of two triangles. For the second vertex we have $\text{Log}^{-1}(v_2) \cap W(V_\lambda^{\mathbb{K}})$ is the closure in $S^1 \times S^1$ of the set

$$\left(\alpha, \lim_{t \rightarrow 0, x \rightarrow 1} \arg(w(t)_\alpha)\right)$$

with $0 \leq \alpha \leq 2\pi$, and we obtain the union of two triangles.

5th case. Suppose $\lambda < -1$ and write $\lambda = \tau^{-1}$ with $-1 < \tau < 0$. So we have the tropical curve of the second case (this means that the valuation of the constant coefficient is positive).

5. A topological manifold structure on phase tropical hypersurfaces

5.1. Manifold structure on phase tropical hyperplanes

In [10], Mikhalkin gives the following definition of a generalized pair-of-pants:

Definition. Let $\mathcal{H} \subset \mathbb{C}\mathbb{P}^n$ be an arrangement of $n + 2$ generic hyperplanes in $\mathbb{C}\mathbb{P}^n$. Let $\mathcal{U} \subset \mathbb{C}\mathbb{P}^n$ be the union of their tubular ε -neighborhood for a small $0 < \varepsilon \ll 1$. The complement $\mathcal{P}_n = \mathbb{C}\mathbb{P}^n \setminus \mathcal{U}$ is called the n -dimensional pair-of-pants, and $\mathcal{P}_n = \mathbb{C}\mathbb{P}^n \setminus \mathcal{H}$ is called the n -dimensional open pair-of-pants.

As $\mathcal{H} \subset \mathbb{C}\mathbb{P}^n$ is unique up to the action of the projective special linear group $PSL_{n+1}(\mathbb{C})$, then \mathcal{P}_n can be given a canonical complex structure. The one dimensional pair-of pants \mathcal{P}_1 is diffeomorphic to the Riemann sphere punctured at 3 points. Moreover, Mikhalkin constructs a foliation \mathcal{F} of the complement in \mathbb{R}^n of the complex defined by the standard tropical hyperplane Γ_n . As before, if $v \in \Gamma$ is a vertex, then there exists a neighborhood U_v of v in Γ and an affine linear transformation F with linear part A_v in $SL_n(\mathbb{Z})$ such that up to a translation in \mathbb{R}^n , ${}^tA_v^{-1}(U_v)$ is a neighborhood of the origin in Γ_n . Let W_v be a neighborhood of $F(U_v)$. According to Mikhalkin, a partition of unity gives a foliation \mathcal{F}_Γ of a neighborhood W of Γ .

Let $\pi_{\mathcal{F}_\Gamma} : W(\Gamma) \rightarrow \Gamma$ the projection along \mathcal{F}_Γ . By Theorem 5.4 of Mikhalkin and Rullgard, $\text{Log}_t(V_t) \subset W(\Gamma)$ for $t \gg 0$. Let

$$\lambda_t := \pi_{\mathcal{F}_\Gamma} \circ \text{Log}_t : V_t \rightarrow \Gamma.$$

The example of hyperplanes in the projective space is fundamental for our Theorem 1.1. So, let $H = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid z_1 + \dots + z_n + 1 = 0\} \subset \mathbb{C}\mathbb{P}^n$ be a hyperplane. Consider its toric part $\mathring{H} = H \cap (\mathbb{C}^*)^n$. Let us denote by

$\mathcal{A}_n \subset \mathbb{R}^n$ the amoeba of \mathring{H} and by $\Gamma_n \subset \mathbb{R}^n$ the tropical hyperplane defined by the tropical polynomial:

$$f_{trop}(x_1, \dots, x_n) = \max\{0, x_1, \dots, x_n\}.$$

It is well known that $\Gamma_n \subset \mathcal{A}_n$ and it is called the spine of the amoeba \mathcal{A}_n . Moreover, Γ_n is a strong deformation retract of \mathcal{A}_n (see [14]). The number of connected components of the complement of the amoeba \mathcal{A}_n in \mathbb{R}^n is equal to $n + 1$. Each component \mathcal{C}_i of $\mathbb{R}^n \setminus \mathcal{A}_n$ is equal to the subset of \mathbb{R}^n where one the functions $\{0, x_1, \dots, x_n\}$ is maximal.

Let us recall Mikhalkin’s construction of the foliation mentioned above ([10], Section 4.3) to obtain a singular foliation of the amoeba \mathcal{A}_n . More precisely, let \mathcal{L}_i be the foliation of the complement component of Γ corresponding to x_i (i.e., the set of \mathbb{R}^n where the tropical polynomial f_{trop} achieved its maximum) into straight lines parallel to the gradient $v_i := \frac{\partial}{\partial x_i}$ of x_i for $i = 1, \dots, n$ and in the component corresponding to the constant function equal to 0 we consider the foliation into straight lines parallel to the vector with coordinates $v_0 = (1, \dots, 1)$. Consider $\pi_i : \mathcal{C}_i \rightarrow \Gamma_n$ the linear projection onto Γ_n and parallel to the vector v_i . Let π the following map:

$$\pi : \mathcal{A}_n \setminus \Gamma_n \rightarrow \Gamma_n,$$

where $\pi|_{\mathcal{C}_i \cap \mathcal{A}_n} = \pi_i|_{\mathcal{A}_n}$ for each $i = 0, 1, \dots, n$. The foliations of the \mathcal{C}_i ’s glue to a global foliation \mathcal{L} of \mathcal{A}_n which has singularities at Γ_n and the leaves passing through a point p in an open $(n - 1 - k)$ -cell of Γ_n is homeomorphic to the union of $k + 2$ segments having a common boundary point p (in other word a cone over $k + 2$ points). We can smooth the foliation \mathcal{L} over all open $(n - 1)$ -cells of Γ_n , but not at the lower dimensional cells because their leaves are not even a topological manifolds. The only leaves diffeomorphic to a manifold are those passing through open $(n - 1)$ -cells which are diffeomorphic to the closed interval $[-1, +1]$. Let us denote the foliation obtained by this smoothing by \mathcal{F} .

Proposition 5.1. *A phase tropical hyperplane $H_\infty \subset (\mathbb{C}^*)^n$ is homeomorphic to a hyperplane in the projective space $\mathbb{C}\mathbb{P}^n$ minus $n + 1$ generic hyperplanes.*

Proof. Since each phase tropical hyperplane is a translated in $(\mathbb{C}^*)^n$ of the following phase tropical hyperplane $H_\infty = W(\{(z_1, \dots, z_n) \in (\mathbb{K}^*)^n \mid z_1 + \dots + z_n + 1 = 0\})$, then it suffices to consider this case. Let us start by the case of phase plane tropical line in $(\mathbb{C}^*)^2$. In the case of lines the inverse image by the logarithmic map of the vertex of the tropical line $\Gamma := \text{Log}(\mathcal{H})$ is a union of two triangles whose vertices pairwise identified, and the inverse image by the logarithmic map of any point in the interior of its rays is a circle (see Example (a)). This means that the inverse image of each ray is a holomorphic annulus \mathcal{R}_j for $j = 1, 2, 3$. It is clear now that a phase tropical line in $(\mathbb{C}^*)^2$ is diffeomorphic to a sphere punctured in three points. In fact, if we denote v_0 the vertex of Γ and \mathcal{R}_j for $j = 1, 2, 3$ are the three rays going to the infinity,

then the phase tropical line in $(\mathbb{C}^*)^2$ is homeomorphic the gluing of the closure $\overline{\text{Log}^{-1}(v_0)}$ in the real torus $(S^1)^2$ and the three semi-open holomorphic annulus $\mathcal{R}_j = \text{Log}^{-1}(\mathcal{R}_j)$ for $j = 1, 2, 3$. A complete description of $\overline{\text{Log}^{-1}(v_0)}$ is given in [13]. For any dimension, it is the same as the complement in the real torus $(S^1)^n$ of an open zonotope (i.e., the coamoeba of a hyperplane). In case where $n > 2$, using the description of the coamoeba of a hyperplane given in Theorem 3.3 [13] and the description of the $(n-1)$ -dimensional pair-of-pants given in Proposition 2.24 [10], one can check the phase tropical hyperplane $(\mathbb{C}^*)^n$ is homeomorphic to the complex projective space $\mathbb{C}\mathbb{P}^{n-1}$ minus a tubular neighborhood of the union \mathcal{H} of $n + 1$ hyperplanes in $\mathbb{C}\mathbb{P}^{n-1}$. Let us be more explicit.

The hyperplane $H_{\mathbb{K}} := \{(z_1, \dots, z_n) \in (\mathbb{K}^*)^n \mid z_1 + \dots + z_n + 1 = 0\}$ can be parametrized as follows:

$$\begin{cases} z_1(t) &= t^{-x_1} e^{i\alpha_1}, \\ z_2(t) &= t^{-x_2} e^{i\alpha_2}, \\ \vdots &\quad \quad \quad \vdots \\ z_{n-1}(t) &= t^{-x_{n-1}} e^{i\alpha_{n-1}}, \\ z_n(t) &= -1 - \sum_{j=1}^{j=n-1} t^{-x_j} e^{i\alpha_j} \end{cases}$$

with $x_j \in \mathbb{R}$ and $0 \leq \alpha_j \leq 2\pi$ for $j = 1, \dots, n-1$. If we denote $\mathring{H}_t \subset (\mathbb{C}^*)^n$ the hyperplane given by the parametrization for a fixed t . Then all the family of hyperplanes $\{\mathring{H}_t\}_{0 < t \leq 1}$ is viewed as a single hyperplane in $(\mathbb{K}^*)^n$ and we have $\mathring{H}_\infty = W(\mathring{H}_{\mathbb{K}})$ where W is the map from $(\mathbb{K}^*)^n$ to $(\mathbb{C}^*)^n$ defined in Section 3. Also, the tropical hyperplane Γ_n is the image by the logarithmic map of \mathring{H}_∞ . The following lemma gives a complete topological description of \mathring{H}_∞ . \square

Lemma 5.2. *Let $\mathring{H}_\infty \subset (\mathbb{C}^*)^n$ be a phase tropical hyperplane and Γ_n its image by the logarithmic map. Then the inverse image of a point in the interior of an l -cell $\sigma \subset \Gamma_n$ is the product of a real l -torus with the coamoeba of a hyperplane in $(\mathbb{C}^*)^{n-l}$, i.e., if $x = (x_1, \dots, x_n) \in \mathring{\sigma}$, then we have:*

$$\text{Log}^{-1}(x) = (S^1)^l \times \text{co}\mathcal{A}(n-1-l),$$

where $\text{co}\mathcal{A}(n-1-l)$ is the coamoeba of a $(n-1-l)$ -plane in $(\mathbb{C}^*)^{n-l}$.

Proof. Let x be a point in the interior of an l -cell, then there exist x_{j_1}, \dots, x_{j_l} strictly negative and all the other x_j are equal to zero. As \mathring{H}_∞ is the limit when t tends to zero (if we want t goes to infinity then we can make the change of variable in the parametrization, t by $\frac{1}{t}$), then for any fixed $\alpha_{j_1}, \dots, \alpha_{j_l}$ we obtain the coamoeba of a hyperplane in $(\mathbb{C}^*)^{n-l}$ (recall that $\lim_{t \rightarrow 0} t^{-x_{j_u}} = 0$, because $x_{j_u} < 0$ for any $u = 1, \dots, l$). But $0 \leq \alpha_{j_u} \leq 2\pi$, which means that the fiber over x is the product of the torus $(S^1)^l$ with the coamoeba of a hyperplane in $(\mathbb{C}^*)^{n-l}$. In particular, the inverse image of a 0-cell is the coamoeba of a hyperplane in $(\mathbb{C}^*)^n$ which is equal to its phase limit set, and its topological description is given in [13]. \square

Lemma 5.2 gives a complete description of the phase tropical hyperplane \mathring{H}_∞ , which coincide with the description of a hyperplane in the projective space $\mathbb{C}\mathbb{P}^n$ minus $n + 1$ generic hyperplanes.

5.2. A topological manifold structure on phase tropical hypersurfaces

In the general case, let us denote by Γ the tropical variety limit of the family of amoebas $\{\mathcal{A}_t\}$, where \mathcal{A}_t is the amoeba of the variety \mathring{V}_t . Moreover, suppose that the tropical hypersurface Γ is smooth in the sense that every vertex of Γ is dual to a simplex of Euclidean volume equal to $\frac{1}{n!}$. Therefore, locally for any vertex v of Γ there exists an open neighborhood U_v homeomorphic to the standard tropical hyperplane, in other words, tropical pair-of-pants. More precisely, there exists an affine linear transformation of \mathbb{R}^n whose linear part A_v belongs to $SL_n(\mathbb{Z})$ such that U_v is the image of the standard tropical hyperplane by ${}^tA_v^{-1}$. Namely, \bar{U}_v has $n + 1$ boundary components isomorphic to an $(n - 2)$ -dimensional tropical hyperplane in \mathbb{R}^{n-1} where \mathbb{R}^{n-1} can be viewed as a boundary component of the tropical projective space $\mathbb{P}\mathbb{T}^n$ represented by the standard simplex.

Let v_1 and v_2 be two adjacent vertices of Γ , in other words, there exists a compact edge e with boundary v_1 and v_2 . Then U_{v_i} has a boundary component $\mathcal{B}_{ij} := \partial_j U_{v_i}$ that can be viewed as a component of the boundary of a tubular neighborhood of a boundary component $\mathcal{B}_{ji} := \partial_i U_{v_j}$. In other words, there exists an open neighborhood $U_{v_1 v_2}$ of v_1 and v_2 containing U_{v_1} and U_{v_2} such that $U_{v_1 v_2}$ is the interior of the gluing of \bar{U}_{v_1} and \bar{U}_{v_2} along their boundaries \mathcal{B}_{ij} and \mathcal{B}_{ji} are joined by a vertical edge and all the other edges adjacent to v_i ($i = 1, 2$) are horizontal, in other words they are mutually parallel, such that the orientation reversing diffeomorphism is given by $(z_1, \dots, z_{n-1}, z_n) \mapsto (z_1, \dots, z_{n-1}, \bar{z}_n)$.

After gluing all pieces, we obtain a manifold $W_\infty(\Gamma)$ with boundary coming from unbounded 1-cells of Γ_f where each unbounded 1-cell will corresponds to \mathcal{B}_{ij} for some vertex v_i . Each \mathcal{B}_{ij} is a circle fibration over a union of lower dimensional pair-of-pants \mathcal{P}_{n-2} (see Figure 5). We can remark that $W_\infty(\Gamma)$ is a topological description of the decomposition of $\mathring{H}_\infty = W(\mathring{V}_{\mathbb{K}})$, where $\mathring{V}_{\mathbb{K}}$ is the hypersurface of $(\mathbb{K}^*)^n$ representing the family $\{\mathring{V}_t\}$. In other words, the family $\{\mathring{V}_t\}$ is viewed as a single hypersurface in the algebraic torus $(\mathbb{K}^*)^n$.

Let us denote by $M_\infty(\Gamma)$ the result of collapsing all fibers of these fibrations on the boundary $\partial W_\infty(\Gamma)$ of $W_\infty(\Gamma)$. Then $M_\infty(\Gamma)$ is a smooth manifold. Indeed, this construction coincide locally with collapsing the boundary on $\bar{\mathcal{P}}_{n-1}$ which results in the projection space $\mathbb{C}\mathbb{P}^{n-1}$ which is smooth.

5.3. Proof of Theorem 1.1

Since all smooth hypersurface with a fixed Newton polytope are isotopic, then we can choose any of them. More precisely, we will use for our subject

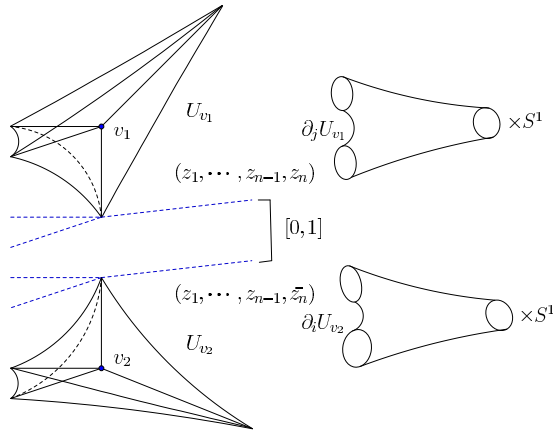


FIGURE 5. Gluing of two 2-dimensional pairs-of-pants in $(\mathbb{C}^*)^3$ along one component of their boundaries.

the convenient one. Let $f_t(x) = \sum_{j \in \Delta \cap \mathbb{Z}^n} a_j t^{-v(j)} z^j$ be a polynomial with a parameter t , and $\mathring{V}_t = \{f_t = 0\} \subset (\mathbb{C}^*)^n$. The family of f_t can be viewed as a single polynomial in $\mathbb{K}[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$. Therefore this family defines a hypersurface $\mathring{V}_{\mathbb{K}} \subset (\mathbb{K}^*)^n$. Let $\mathcal{A}_t := \text{Log}_t(\mathring{V}_t)$ and $\mathcal{A}_{\mathbb{K}} := \text{Log}_{\mathbb{K}}(\mathring{V}_{\mathbb{K}})$.

Let Γ be a maximally dual Δ -complex, in other words, all the element of its dual the subdivision are simplex of Euclidean volume $\frac{1}{n!}$ and $\nu : \Delta \cap \mathbb{Z}^n \rightarrow \mathbb{R}$ be the function such that $\Gamma = \Gamma_{\nu}$, i.e., Γ_{ν} is the tropical hypersurface defined by the tropical polynomial $\max_{\alpha \in \Delta \cap \mathbb{Z}^n} \{\nu(\alpha) + \langle \alpha, x \rangle\}$. Then we obtain a family of polynomial called a Viro-patchworking polynomial [16]

$$f_t(z) = \sum_{v \in \Delta \cap \mathbb{Z}^n} t^{-v(j)} z^j.$$

Let us denote $\mathring{V}_t \subset (\mathbb{C}^*)^n$ the zero locus of the polynomial f_t . Using a foliation of the amoeba of \mathring{V}_t Mikhalkin obtains a map $\lambda_t = \pi_{\mathcal{F}_{\Gamma}} \circ \text{Log}_t : \mathring{V}_t \rightarrow \Gamma$, and proves in Lemma 6.5, [10] that \mathring{V}_t is smooth for a sufficiently large $t \gg 0$.

First of all, Γ looks locally as a tropical hyperplane after a linear transformation with linear part $SL_n(\mathbb{Z})$. It means that Γ can be locally identified to a tropical hyperplane in \mathbb{R}^n by a linear transformation F of \mathbb{R}^n with a linear part in $SL_n(\mathbb{Z})$.

It was shown in Lemma 6.5 [10] that V_t is also smooth, and λ_t satisfies a nice properties. Indeed, for $t \gg 0$, \mathring{V}_t is smooth, and \mathring{V}_t is an union of finite number of open sets, where each set is the image of a small perturbation of a hyperplane. Hence, its compactification $V_t \subset X_{\Delta}$ is smooth and transverse to the coordinate hyperplanes. Also, for a large $t \gg 0$, V_t is isotopic to the variety $M_{\infty}(\Gamma)$ constructed above (which is a compactification of the phase tropical

variety $\mathring{V}_\infty = W_\infty(\Gamma)$ the lifting of Γ in $(\mathbb{C}^*)^n$, this comes from Theorem 4 of Mikhalkin [10], which proves the second statement of Theorem 1.1. This shows that \mathring{V}_∞ is also homeomorphic to \mathring{V}_t for sufficiently large $t \gg 0$ and the first statement of Theorem 1.1 is proved.

6. Construction of a natural symplectic structure on $M_\infty(\Gamma)$

Note that every pair-of-pants inherit a natural symplectic structure coming from the one of the projective space $\mathbb{C}\mathbb{P}^n$. Namely, the projective space $\mathbb{C}\mathbb{P}^n$ is obtained from a closed pair-of-pants after collapsing its boundary. Indeed, each component of the boundary of a pair-of pants \mathcal{P}^n is a S^1 -fibration over a lower dimensional pair-of-pants \mathcal{P}^{n-1} , and the result of collapsing all fibers of these S^1 -fibrations is precisely the projective space $\mathbb{C}\mathbb{P}^n$.

6.1. Proof of Theorem 1.2

Let $M_\infty(\Gamma)$ be the variety constructed in Section 5, which is a compactification of \mathring{V}_∞ in the toric variety X_Δ where Δ is the degree of our original hypersurface V . The variety $M_\infty(\Gamma)$ is obtained by gluing pairs-of-pants along a part of their boundary \mathcal{B}_j that is a product of a holomorphic cylinder (i.e., an annulus) in \mathbb{C}^* with a lower dimensional pair-of-pants \mathcal{P}^{n-2} (i.e., along $[0, 1] \times \mathcal{B}_j$). Moreover, each \mathcal{B}_j is a circle fibration over \mathcal{P}^{n-2} , where the fibers are precisely the fibers of the annulus over the interval $[0, 1]$:

$$\mathcal{B}_j \longrightarrow \mathcal{P}^{n-2}$$

is an S^1 -fibration, and

$$\mathcal{A} \times \mathcal{P}^{n-2} = [0, 1] \times \mathcal{B}_j \longrightarrow \mathcal{P}^{n-2}$$

is an annulus fibration, where \mathcal{A} is the annulus $[0, 1] \times S^1$.

Let us denote by $\omega_j^{(n-2)}$ the symplectic form on the pair-of-pants \mathcal{P}^{n-2} coming from the projective space $\mathbb{C}\mathbb{P}^{n-2}$ and $ds \wedge dt$ the symplectic form on $S^1 \times \mathbb{R}$. Hence, we obtain a symplectic form $\omega_j := ds \wedge dt + \omega_j^{(n-2)}$ on $[0, 1] \times \mathcal{B}_j$. It means that we have a symplectic form on parts where the gluing was done. Recall that $[0, 1] \times \mathcal{B}_j$ can be seen as a neighborhood of a boundary component of the pair-of-pants \mathcal{P}^{n-1} . On the other part of \mathcal{P}^{n-1} , i.e., $\mathcal{P}^{n-1} \setminus \cup_j([0, 1] \times \mathcal{B}_j)$, we already have the symplectic form of a pair-of-pants $\omega^{(n)}$ and the pull back of $\omega^{(n)}$ on the factor \mathcal{P}^{n-2} of any boundary component is precisely $\omega_j^{(n-2)}$.

However, when we glue $[0, 1] \times \mathcal{B}_j$ and $[0, 1] \times \mathcal{B}_i$ where the first part is equipped with the form $ds \wedge dt + \omega_j^{(n-2)}$ then the second should be equipped with the form $-ds \wedge dt + \omega_i^{(n-2)}$ because the gluing was done with a reversing orientation. Recall that the forms $\omega_j^{(n-2)}$ and $\omega_i^{(n-2)}$ are the same.

On the other hand, the symplectic forms outside of the gluing parts are well defined since each component is symplectically an open pair-of-pants which is a hyperplane in the complex algebraic torus $(\mathbb{C}^*)^n$. After taking the compactification of such hyperplanes in the projective space $\mathbb{C}\mathbb{P}^{n-1}$, the restriction of

these forms on the infinite parts $\mathbb{C}\mathbb{P}^{n-2}$'s are precisely the forms ω_j^{n-2} 's. This gives rise to a global symplectic form $\mathring{\omega}_{nat}$ on a manifold \mathring{M}_∞ homeomorphic to $\mathring{M}_\infty(\Gamma)$.

Let us denote by $\mathring{\omega}_t = \iota_t^*(\omega)$ the symplectic form on \mathring{V}_t where ι_t is the inclusion of \mathring{V}_t in the complex algebraic torus $(\mathbb{C}^*)^n$, ω is the symplectic form on $(\mathbb{C}^*)^n$ defined by (1). Using Moser's trick, Mikhalkin showed that $M_\infty(\Gamma)$ is symplectomorphic to V_t for a sufficiently large $t \gg 0$. Let us denote this symplectomorphism by ϕ . Hence we have the following commutative diagram:

$$(7) \quad \begin{array}{ccc} (\mathring{M}_\infty, \mathring{\omega}_{nat}) & \xrightarrow{\psi = \phi|_{\mathring{M}_\infty(\Gamma)}} & (\mathring{V}_t, \mathring{\omega}_t) \\ \downarrow j & & \downarrow i \\ (M_\infty, \omega_{nat}) & \xrightarrow{\phi} & (V_t, \omega_t). \end{array}$$

This means that for a sufficiently large $t \gg 0$, \mathring{M}_∞ is also symplectomorphic to \mathring{V}_t , and the second statement Theorem 1.2 is proved. Recall that we can prove Theorem 1.2 using a generalization of Moser's trick for non compact manifolds proved by R. E. Greene and K. Shiohama on 1979 in [3].

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