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TOPOLOGICAL STABILITY AND SHADOWING PROPERTY FOR GROUP ACTIONS ON METRIC SPACES

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Abstract. In this paper, we introduce the notions of expansiveness, shadowing property and topological stability for group actions on metric spaces and give a version of Walters's stability theorem for group actions on locally compact metric spaces. Moreover, we show that if G is a finitely generated virtually nilpotent group and there exists $g \in G$ such that if T_g is expansive and has the shadowing property, then T is topologically stable.

1. Introduction

Expansiveness is a strong symbol of chaotic dynamics that has been studied extensively. Recall that a homeomorphism f of a compact metric space X is *expansive* if there is $\delta > 0$ such that, for any x and y in X, $d(f^n(x), f^n(y)) < \delta$ for all $n \in \mathbb{Z}$ implies $x = y$. In other words, expansiveness requests that any two different points x and y must get separated in a uniform distance δ at certain moment n. Such a constant δ is called an *expansive constant* of f. Note that this property is independent of the choice of metrics for X.

And a dynamical system has a shadowing property if any sufficiently precise approximate orbit is close to some exact orbit. More precisely, for a homeomorphism f of a compact metric space X, a sequence $\{x_n\}_{n\in\mathcal{Z}}$ of points of X is called a δ-*pseudo orbit* ($\delta > 0$) of f if $d(f(x_n), x_{n+1}) < \delta$ for all $n \in \mathcal{Z}$. We say that f has the *shadowing property* if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any δ-pseudo orbit $\{x_n\}_{n\in\mathcal{Z}}$ there is a point $x \in X$ satisfying $d(f^n(x), x_n) < \varepsilon$ for all $n \in \mathcal{Z}$. The shadowing property is also independent of the choice of metrics for X if X is compact.

These properties are very often appearing in several branches of the theory of dynamical systems, and in particular, they are usually playing an important role in the investigation of the stability theory.

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Walters [8] introduced the notion of topological stability, a kind of stability for homeomorphisms in which continuous perturbations are allowed, and [9] proved that every expansive homeomorphism with the shadowing property on a compact metric space is topologically stable. Afterwards, Lee and Morales [2] introduced the concepts of topological stability and shadowing property for Borel measures on compact metric spaces, and showed that any expansive measure with the shadowing property is topologically stable. Lee *et al.* [3] propose the notions of expansiveness, shadowing property and topological stability for homeomorphisms on noncompact metric spaces and extend the Walters's stability theorem to homeomorphisms on locally compact metric spaces.

Recently, Osipov [5] introduced a notion of shadowing property for actions of finitely generated groups and proved a shadowing lemma for actions of nilpotent groups. In [1], Chung and Lee extended the notion of topological stability from homeomorphisms to group actions on compact metric spaces, and proved that if an action of a finitely generated group action is expansive and has the shadowing property then it is topologically stable.

In this paper, we introduce the notions of expansiveness, shadowing property and topological stability for group actions on metric spaces and give a version of Walters's stability theorem for group actions on locally compact metric spaces. For the case of finitely generated nilpotent groups we prove that an action of the whole group is topologically stable if the action of at least one element is topologically stable. More precisely, we prove the following two theorems on a locally compact metric space X.

Theorem 1.1. *Let G be a finitely generated group. If an action $T \in Act(G,X)$ is expansive and has the shadowing property, then it is topologically stable.*

Theorem 1.2. *Let G be a finitely generated virtually nilpotent group and $T \in Act(G,X)$. If there exists an element $g \in G$ such that T_g is expansive and has the shadowing property, then T is topologically stable.*

2. Preliminaries

First of all, we recall some concepts and notations on compact metric spaces that we will use in the paper.

Let (X,d) be a compact metric space and G be a finitely generated group. We denote by $Act(G,X)$ the set of all continuous actions T of G on X.

We say that $T \in Act(G,X)$ is *expansive* if there exists a constant $c > 0$ called an expansive constant of T such that for every $x \neq y$, one has $\sup_{g\in G} d(T_g x, T_g y) > c$.

The shadowing property of group action on compact metric spaces were studied in the recent literatures [5–7]. For given finitely generating set A of G and $\delta > 0$, a *δ-pseudo orbit* of $T \in Act(G,X)$ with respect to A is a sequence $\{x_g\}_{g\in G}$ in X such that $d(T_a x_g, x_{ag}) < \delta$ for all $a \in A$, $g \in G$. T is said to have *the shadowing property with respect to A* if for every $\varepsilon > 0$, there exists $\delta > 0$ such that any δ-pseudo orbit $\{x_g\}_{g\in G}$ for T with respect to A is ε-shadowed by some point x of X, that is, $d(T_g x, x_g) < \varepsilon$ for all $g \in G$. We say that an action $T \in Act(G, X)$ has *the shadowing property* if T has the shadowing property with respect to A for a finitely generating set A of G.

In [1], Chung and Lee extended the notion of topological stability from homeomorphisms to group actions on compact metric spaces as following: Let A be a finitely generating set of G. We say that T is A-topologically stable if for every $\varepsilon > 0$, there exists $\delta > 0$ such that if S is another continuous action of G on X with $d_A(T, S) := \sup_{x \in X, a \in A} d(T_a x, S_a x) < \delta$, then there exists a continuous map $f : X \to X$ with $T_g f = f S_g$, for every $g \in G$ and $d(f, Id_X) := \sup_{x \in X} d(f(x), x) \leq \varepsilon$. An action T is said to be *topologically stable* if it is A-topologically stable for a symmetric finitely generating set A of G.

Note that above concepts of expansive, shadowing property and topological stability of T does not depend on the choice of a compatible metric d of X. Moreover, they also proved that the shadowing and topological stability do not depend on the choice of symmetric finitely generating sets, see for example Lemma 2.2 in [6] and [1].

Now we introduce the notions of expansiveness, shadowing property and topological stability for group actions on metric spaces such that which are dynamical properties. We say that two actions T and S of finitely generating group G on metric spaces X and Y respectively are *conjugate* if there is a homeomorphism $h : X \to Y$ such that $hT_g(x) = S_g h(x)$ for all $x \in X$ and $g \in G$. Here, h is called a conjugacy. A property P is called a *dynamical property* if for any two conjugate actions T and S, T has the property P if and only if S does. Throughout this paper we denote $\mathcal{C}(X)$ the collection of continuous functions from X to $(0, \infty)$. For any $\varepsilon, \delta \in \mathcal{C}(X)$, we will write $\varepsilon < \delta$ whenever $\varepsilon(x) < \delta(x)$ for all $x \in X$.

Definition. Let A be a finitely generating set of G and let $T \in Act(G, X)$. We say that T is A-topologically stable if for every $\varepsilon \in \mathcal{C}(X)$, there is $\delta \in \mathcal{C}(X)$ such that if S is another continuous action of G on X with $d(T_a x, S_a x) < \delta(T_a x)$ for all $x \in X, a \in A$ then there exists a continuous map $f : X \to X$ with $T_g f = f S_g$ for all $g \in G$ and $d(f(x), Id(x)) \leq \varepsilon(f(x))$ for all $x \in X$.

It obviously see that the definition of topological stability of a homeomorphism introduced in [8] coincides with our definition when $G = \mathbb{Z}$, $A = \{1\}$ if X is compact. Furthermore, we have topological stability of T does not depend on the choice of a symmetric finitely generating set A of G. Recall that A is symmetric if for any $a \in A, a^{-1} \in A$. If A is a finitely generating set of G, then there always exists a symmetric finitely generating set containing A. Throughout the paper, a finitely generating set A of G implies a symmetric finitely generating set. From the following lemma, we can see that topological stability of T does not depend on the symmetric finitely generating set A of G.

Lemma 2.1. *Let A and B be symmetric finitely generating sets of G. For any $T \in Act(G,X)$, T is A-topologically stable if and only if it is B-topologically stable.*

To prove Lemma 2.1 we need the following two lemmas which are showed in [3].

Lemma 2.2. *Let (X,d) and (Y,d') be two metric spaces. Then a function f from X to Y is continuous if and only if for any $\varepsilon \in \mathcal{C}(Y)$, there exists $\delta \in \mathcal{C}(X)$ such that if $d(x,y) < \delta(x)$ $(x,y \in X)$, then $d'(f(x),f(y)) < \varepsilon(f(x))$.*

Lemma 2.3. *For any $\alpha \in \mathcal{C}(X)$, there is $\gamma \in \mathcal{C}(X)$ such that $\gamma(x) < \inf\{\alpha(y) \mid y \in B(x,\gamma(x))\}$ for all $x \in X$.*

For simplicity, we write $\delta \ll \varepsilon$ $(\delta,\varepsilon \in \mathcal{C}(X))$ if $\delta(x) < \inf\{\varepsilon(y) \mid y \in B(x,\delta(x))\}$ for any $x \in X$.

Proof of Lemma 2.1. Suppose T is A-topologically stable. Then for any $\varepsilon \in \mathcal{C}(X)$, there exists $\delta' \in \mathcal{C}(X)$ corresponding to ε such that if for any continuous action S of G on X with $d(T_a x, S_a x) < \delta'(T_a x)$ for all $x \in X, a \in A$, then there exists a continuous map $f : X \to X$ with $T_g f(x) = f S_g(x)$ for every $g \in G$, $x \in X$ and $d(f(x), Id(x)) \leq \varepsilon(f(x))$ for all $x \in X$. It suffices to show that there exists $\delta \in \mathcal{C}(X)$ such that for any $S \in Act(G,X)$, if

$$d(T_b x, S_b x) < \delta(T_b x)$$

for any $b \in B$, then

$$d(T_a x, S_a x) < \delta^{'}(T_a x)$$

for any $a \in A$. Put $m := \max_{a \in A} l_B(a)$, where l_B is the word length metric on G induced by B. Choose $\delta_1, \delta_2, \ldots, \delta_m \in \mathcal{C}(X)$ such that

$$\delta_m \ll \delta_{m-1} \ll \cdots \ll \delta_1 \ll \frac{\delta'}{m}.$$

Since B is finite and the action T is continuous, by Lemma 2.2 there exists $\delta \in \mathcal{C}(X)$ such that $d(T_h x, T_h y) < \delta_1(T_h x)$ for $x, y \in X$ with $d(x,y) < \delta(x)$ and for $h \in G$ with $l_B(h) \leq m$.

For any $a \in A$, we write a as $b_1 \cdots b_{l(a)}$, where $l(a) = l_B(a) \leq m$, $b_i \in B$, $i = 1, \ldots, l(a)$. Then for any $S \in Act(G,X)$ with $d(T_b x, S_b x) < \delta(T_b x)$ for any $b \in B, x \in X$, we have

$$\begin{aligned} d(T_a x, S_a x) &= d(T_{b_1 \cdots b_{l(a)}} x, S_{b_1 \cdots b_{l(a)}} x) \\ &\leq d(T_{b_1 \cdots b_{l(a)-1}} T_{b_l(a)} x, T_{b_1 \cdots b_{l(a)-1}} S_{b_l(a)} x) \\ &\quad + d(T_{b_1 b_2 \cdots b_{l(a)-2}} T_{b_{l(a)-1}} S_{b_{l(a)}} x, T_{b_1 b_2 \cdots b_{l(a)-2}} S_{b_{l(a)-1}} S_{b_{l(a)}} x) \\ &\quad + \cdots + d(T_{b_1} T_{b_2} S_{b_3 \cdots b_{l(a)-1} b_{l(a)}} x, T_{b_1} S_{b_2} S_{b_3 \cdots b_{l(a)-1} b_{l(a)}} x) \\ &\quad + d(T_{b_1} S_{b_1 \cdots b_{l(a)-1} b_{l(a)}} x, S_{b_1 \cdots b_{l(a)}} x) \\ &< \delta_m(T_{b_1 \cdots b_{l(a)-1}} T_{b_l(a)} x) + \delta_m(T_{b_1 b_2 \cdots b_{l(a)-2}} T_{b_{l(a)-1}} S_{b_{l(a)}} x) \\ &\quad + \cdots + \delta_m(T_{b_1} S_{b_1 \cdots b_{l(a)-1} b_{l(a)}} x). \end{aligned}$$

For $0 \le i \le l(a)$, by the choice of δ_m, we observe that

$$\delta_m(T_{b_1 \cdots b_{l(a)-i}} S_{b_{l(a)-i+1} \cdots b_{l(a)}} x) < \delta_{m-1}(T_{b_1 \cdots b_{l(a)-i-1}} S_{b_{l(a)-i} \cdots b_{l(a)}} x)$$

and

$$\delta_m(T_{b_1 \cdots b_{l(a)-i}} S_{b_{l(a)-i+1} \cdots b_{l(a)}} x) < \delta_{m-1}(T_{b_1 \cdots b_{l(a)-i}} S_{b_{l(a)-i+1} \cdots b_{l(a)}} x).$$

Then we get

$$d(T_a x, S_a x) < m\delta_1(S_{b_1 \cdots b_{l(a)}} x) < \delta^{'}(T_a x).$$

This means that T is B-topologically stable, and so we completes the proof. □

Definition. An action $T \in Act(G, X)$ is said to be topologically stable if it is A-topologically stable for a symmetric finitely generating set A of G.

Now we will introduce the concepts of expansiveness and shadowing property of group actions on metric spaces.

Definition. We say that an action $T \in Act(G, X)$ is expansive if there exists a function $e \in \mathcal{C}(X)$ called an expansive function of T such that if

$$d(T_g x, T_g y) < e(T_g x)$$

for all $g \in G$, then $x = y$.

Remark 2.4. If for some $g \in G$, T_g is expansive, then the whole action T is expansive. Then it is easy to see that if $G_1 \le G$ is a subgroup of G and T_{G_1} is expansive, so T is expansive too.

Proof. We suppose T_g is expansive, by the definition of expansiveness of homeomorphisms on metric spaces in [3], there exists an expansive function $e \in \mathcal{C}(X)$ such that for two different points $x, y \in X$,

$$d(T_g^n x, T_g^n y) \ge e(T_g^n x)$$

for some $n \in \mathbb{Z}$. Since $T_g^n = T_{g^n}$ and $g^n \in G$, we know T is expansive. □

Definition. Let A be a finitely generating set of G.

(1) For given $\delta \in \mathcal{C}(X)$, a δ-pseudo orbit of $T \in Act(G, X)$ with respect to A is a sequence $\{x_g\}_{g \in G}$ in X such that

$$d(T_a x_g, x_{ag}) < \delta(T_a x_g)$$

for all $a \in A, g \in G$.

(2) An action $T \in Act(G, X)$ is said to have the shadowing property with respect to A if for every $\varepsilon > 0$, there exists $\delta > 0$ such that any δ-pseudo orbit $\{x_g\}_{g \in G}$ for T with respect to A is ε-shadowed by some point x of X, that is, $d(T_g x, x_g) < \varepsilon(T_g x)$ for all $g \in G$.

Similarly, we can show that the shadowing property of T does not depend on the choice of symmetric finitely generating sets of G.

Definition. We say that an action $T \in Act(G, X)$ has the shadowing property if T has the shadowing property with respect to a symmetric finitely generating set A of G.

Note that the shadowing property of T_g and T are independent. More detail examples could be found in [4].

Proposition 2.5. *Expansiveness, shadowing property and topological stability for group actions on metric spaces are dynamical properties; that is, properties are conjugacy invariant.*

Proof. Let G be a finitely generated group. Take $T \in Act(X, G)$ and $S \in Act(Y, G)$ where (X, d) and (Y, d') are two metric spaces. Suppose T and S are conjugate with a conjugacy h.

Firstly, we assume that T is expansive with an expansive function $e \in \mathcal{C}(X)$. Since h is continuous, by Lemma 2.2, take $\varepsilon \in \mathcal{C}(Y)$ such that if $d'(x, y) < \varepsilon(x)$, then $d(h^{-1}(x), h^{-1}(y)) < e(h^{-1}(x))$. Suppose that $d'(S_g(x), S_g(y)) < \varepsilon(S_g(x))$ for all $g \in G$ for some $x, y \in Y$. Then

$$\begin{aligned} d(T_g h^{-1}(x), T_g h^{-1}(y)) &= d(h^{-1} S_g(x), h^{-1} S_g(y)) \\ &< e(h^{-1} S_g(x)) \\ &= e(T_g h^{-1}(x)) \end{aligned}$$

for all $g \in G$. By expansiveness of T, we have $h^{-1}(x) = h^{-1}(y)$, so $x = y$. This means that S is expansive.

Let A be a symmetric finitely generating set of G.

It suffices to suppose that T has the shadowing property with respect to A. For any $\varepsilon' \in \mathcal{C}(Y)$, choose $\varepsilon \in \mathcal{C}(X)$ such that if $d(x, y) < \varepsilon(x)$, then $d'(h(x), h(y)) < \varepsilon'(h(x))$. Take $\delta \in \mathcal{C}(X)$ corresponding to ε by the shadowing property of T, and let $\delta' \in \mathcal{C}(Y)$ be such that if $d'(x, y) < \delta'(x)$, then $d(h^{-1}(x), h^{-1}(y)) < \delta(h^{-1}(x))$ (by Lemma 2.2). Let $\{x_g\}_{g \in G}$ be a δ'-pseudo orbit of S with respect to A. Then $\{h^{-1}(x_g)\}$ is a δ-pseudo orbit of T with respect to A. Indeed, $d'(S_a(x_g), x_{ag}) < \delta'(S_a(x_g))$ implies

$$d(h^{-1} S_a(x_g), h^{-1}(x_{ag})) = d(T_a h^{-1}(x_g), h^{-1}(x_{ag})) < \delta(T_a h^{-1}(x_g))$$

for all $g \in G$ and $a \in A$. By the shadowing property of T, there is $x \in X$ such that $d(T_g(x), h^{-1}(x_g))) < \varepsilon(T_g(x))$ for all $g \in G$. Then we get

$$d'(hT_g(x), h(h^{-1}(x_g))) = d'(S_g h(x), x_g) < \varepsilon'(hT_g(x))$$

for all $g \in G$. This means that S has the shadowing property with respect to A.

Finally, without of loss generality, we can suppose that T is A-topologically stable. Claim that S is A-topologically stable. For any $\varepsilon' \in \mathcal{C}(Y)$, choose $\varepsilon \in \mathcal{C}(X)$ such that if $d(x, y) < \varepsilon(x)$, then $d'(h(x), h(y)) < \varepsilon'(h(x))$. Take $\delta \in \mathcal{C}(X)$ corresponding to ε by the topological stability of T. Select $\delta' \in \mathcal{C}(Y)$ such that if $d'(x, y) < \delta'(x)$, then $d(h^{-1}(x), h^{-1}(y)) < \delta(h^{-1}(x))$ (by Lemma

2.2). For any $K \in Act(Y, G)$ with $d'(S_a(x), K_a(x)) < \delta'(S_g(x))$ for all $x \in Y$ and $a \in A$, we have

$$d(h^{-1}S_a(x), h^{-1}K_a(x)) = d(T_a h^{-1}(x), h^{-1}K_a h(h^{-1}(x))) < \delta(T_a h^{-1}(x)).$$

Thus there is a continuous map $c : X \to X$ such that $ch^{-1}K_g h = T_g c$ and $d(c(x), x) < \varepsilon(c(x))$ for all $x \in X$. Then we get

$$(hch^{-1})K_g = S_g(hch^{-1}) \text{ and } d'(hch^{-1}(x), x) < \varepsilon'(hch^{-1}(x)).$$

This implies that S is A-topologically stable and so we complete the proof. □

3. Proof of Theorem 1 and Theorem 2

The following lemmas are using to prove Theorem 1.1.

Lemma 3.1. *Let $T \in Act(G, X)$ be an expansive action with the shadowing property with respect to a symmetric finitely generating set A of G. For any $\varepsilon \in \mathcal{C}(X)$, there is $\delta \in \mathcal{C}(X)$ such that any δ-pseudo orbit of T is ε-shadowed by a unique point in X.*

Proof. Let $\eta_1 \in \mathcal{C}(X)$ be such that $\varepsilon \ll \eta_1 \ll \eta/2$ and $\delta \in \mathcal{C}(X)$ corresponds to ε as in definition of shadowing, where η is an expansive function of T. Consider a δ-pseudo orbit $\{x_g\}_{g\in G}$ of T with respect to A, and let x, y be two points that both ε-shadow $\{x_g\}_{g\in G}$. Then one has

$$\begin{aligned} d(T_g x, T_g y) &\le d(T_g x, x_g) + d(x_g, T_g y) \\ &< \varepsilon(T_g x) + \varepsilon(T_g y) \\ &< \eta_1(x_g) + \eta_1(x_g) \\ &< \eta(T_g) \end{aligned}$$

for every $g \in G$. By the expansiveness of T, we get $x = y$. □

Note that on locally compact metric spaces, there is a fact as following.

Lemma 3.2 ([3])**.** *Let X be a locally compact metric space. Then there exists $\alpha \in \mathcal{C}(X)$ such that $\overline{B(x, \alpha(x))}$ is compact in X for $x \in X$, where $B(x, \delta)$ denotes the δ-ball centered at x.*

Throughout the paper, we will use the notation $\alpha_X \in \mathcal{C}(X)$ such that $\overline{B(x, \alpha_X(x))}$ is compact for all $x \in X$.

Lemma 3.3. *Let $T \in Act(G, X)$ be an expansive action with an expansive function $e \in \mathcal{C}(X)$ with $e < \alpha_X$. Then for any $x_0 \in X$ and $\lambda \in \mathcal{C}(X)$, there exists a non-empty finite subset F of G satisfying that if $d(x_0, y) \ge \lambda(x_0)$, then there is some $g \in F$ such that $d(T_g x_0, T_g y) \ge \lambda(T_g x_0)$.*

Proof. Assume that there exists $\lambda \in \mathcal{C}(X)$ such that for any non-empty finite subset F of G, there exists $y_F \in X$ such that $d(x_0, y_F) \ge \lambda(x_0)$ and $d(T_g x_0, T_g y_F) < e(T_g x_0)$ for all $g \in G$. Choose a sequence of finite subsets F_n of G satisfying $e_G \in F_1 \subset F_2 \subset \cdots$ and $G = \cup_{n\in\mathbb{N}} F_n$. Then for every $n \in \mathbb{N}$, there exists $y_n \in F_n$ such that $d(x_0, y_n) \geq \lambda(x_0)$ and $d(T_g x_0, T_g y_n) < e(T_g x_0)$ for all $g \in G$.

Then we have

$$\{y_n\}_{n\in\mathbb{N}} \subset B(x_0, e(x_0)) \subset B(x_0, \alpha_X(x_0)).$$

Since $\overline{B(x_0, \alpha_X(x_0))}$ is compact in X, we may assume that there is a subsequence $\{y_{n_k}\}_{k\in\mathbb{N}}$ of $\{y_n\}_{n\in\mathbb{N}}$ which converges to y for some $y \in \overline{B(x_0, \alpha_X(x_0))}$. Then we have

$$d(T_g x_0, T_g y) = \lim_{k\to\infty} d(T_g x_0, T_g y_{n_k}) \leq e(T_g x_0)$$

for all $g \in G$ and

$$d(x_0, y) = \lim_{k\to\infty} d(x_0, y_{n_k}) \geq \lambda(x_0),$$

which contradicts with the expansiveness of T. □

Proof of Theorem 1.1. Let $e \in \mathcal{C}(X)$ be an expansive function of T with $e < \alpha_X$. Let $\gamma \in \mathcal{C}(X)$ be such that $\gamma \ll e$. For any $\varepsilon \in \mathcal{C}(X)$ with $\varepsilon < \gamma/4$. Let A be a finitely generating set of G. Choose $\delta \in \mathcal{C}(X)$ corresponding to ε by the shadowing property with respect to A of T. Let S be a continuous action of G on X with $d(T_g x, S_g x) < \delta(T_g x)$ for all $x \in X$, $g \in G$. Then we have that the S-orbit $\{S_g x\}_{g\in G}$ of each $x \in X$ is a δ-pseudo orbit of T. By Lemma 3.1, there is a unique point denoted by $f(x)$ whose T-orbit ε-shadows $\{S_g x\}_{g\in G}$. Hence we can define a map $f : X \to X$ by $f(x)$= the unique shadowing point of the δ-pseudo orbit $\{S_g(x)\}_{g\in G}$. Then we have

$$d(T_g f(x), S_g(x)) < \varepsilon(T_g f(x))$$

for all $x \in X$ and $g \in G$. In particular, we get $d(f(x), x) < \varepsilon(f(x))$ for every $x \in X$. Now we claim that $T_g f(x) = fS_g(x)$ for every $x \in X, g \in G$. In fact, we have

$$d(T_h f(S_g x), S_{hg} x) = d(T_h f(S_g x), S_h S_g x) < \varepsilon(T_h f(S_g x))$$

for all $x \in X$ and $g, h \in G$. On the other hand, we obtain

$$d(T_h T_g f(x), S_{hg} x) = d(T_{hg} f(x), S_{hg} x) < \varepsilon(T_{hg} f(x)).$$

Then we get $T_g f(x) = fS_g(x)$ for all $x \in X$, $g \in G$.

Next we will show that f is continuous at each $x_0 \in X$. Let $\lambda > 0$ be a constant. By Lemma 3.3, there exists a non-empty finite subset F of G such that whenever $d(x_0, y) < \lambda(x_0)$, then $d(T_g x_0, T_g y) < \lambda(T_g x_0)$ for all $g \in F$.

Let $\eta > 0$ be such that if $d(x_0, y) < \eta$ one has $d(S_g x_0, S_g(y)) < \frac{\gamma(fS_g x_0)}{4}$ for all $g \in F$. Then for any $y \in X$ with $d(x_0, y) < \eta$, we have

$$\begin{aligned} d(T_g f(x_0), T_g f(y)) &= d(fS_g(x_0), fS_g(y)) \\ &\leq d(fS_g(x_0), S_g(x_0)) + d(S_g(x_0), S_g(y)) + d(S_g(y), fS_g(y)) \\ &\leq \varepsilon(fS_g(x_0)) + \frac{\gamma(fS_g(x_0))}{4} + \varepsilon(fS_g(y)) \end{aligned}$$

$$< \frac{\gamma(fS_g(x_0)}{4} + \frac{\gamma(fS_g(x_0))}{4} + \frac{\gamma(fS_g(y))}{4}$$
$$< e(fS_g(x_0))$$

for every $g \in F$, thus $d(f(x_0), f(y)) < \lambda$. Therefore f is continuous.

Moreover, we have that map f is unique. Assume that there is another map f_1 such that $T_g f_1 = f_1 S_g$ for all $g \in G$ and $d(f_1(x), x) < \varepsilon(f_1(x))$. Then, for any $g \in G$, we have

$$\begin{aligned} d(T_g f(x), T_g f_1(x)) &\leq d(T_g f(x), S_g(x)) + d(S_g(x), T_g f_1(x)) \\ &= d(fS_g(x), S_g(x)) + d(S_g(x), f_1 S_g(x)) \\ &< \varepsilon(fS_g(x)) + \varepsilon(S_g x) \\ &< e(T_g f(x)). \end{aligned}$$

By the expansiveness of T, we get $f(x) = f_1(x)$ for all $x \in X$. □

Next we will provide a class of nilpotent group actions which are topologically stable.

First recall the definition of nilpotent group. Let G be a countable group. The lower central series of G is the sequence $\{G_i\}_{i\geq 0}$ of subgroups of G defined by $G_0 = G$ and $G_{i+1} = [G_i, G]$, where $[G_i, G]$ is the subgroup of G generated by all commutators

$$[a, b] := aba^{-1}b^{-1},\ a \in G_i,\ b \in G.$$

The group G is said to be *nilpotent* if there exists $n \geq 0$ such that $G_n = \{e_G\}$. The such smallest n is called the *nilpotent degree* of G. We say G is a finitely generated virtually nilpotent group if there exists a nilpotent subgroup H of G with finite index.

The following two lemmas were used to prove Theorem 1.2.

Lemma 3.4. *Let G be a finitely generated group and H be a finitely generated normal subgroup of G. Let T be a continuous action of G on X. If the restriction action T_H of T to H is expansive and has the shadowing property, then T has the shadowing property.*

Proof. Let A be a symmetric finitely generating set of H. Let B be the symmetric finitely generating set of G by adding more elements to A. Let $c \in \mathcal{C}(X)$ be an expansive function of T_H. Since B is finite, there exist $\eta \in \mathcal{C}(X)$ with $\eta < c_2/3$ such that if $d(x, y) < \eta(x)$ for every $x, y \in X$, then $d(T_b x, T_b y) < c_2(T_b x)/3$ for every $b \in B$, where $c_1, c_2 \in \mathcal{C}(X)$ with $c_2 \ll c_1 \ll c$. Let $\varepsilon \in \mathcal{C}(X)$ be a function with $\varepsilon < \eta$. We can choose $\delta \in \mathcal{C}(X)$ with $\delta < \varepsilon$ such that every δ-pseudo orbit for T_H with respect to A is ε-shadowed by some point of X. Let $\{x_g\}_{g\in G}$ be a δ-pseudo orbit of T with respect to B. For every $g \in G$, the sequence $\{x_{hg}\}_{h\in H}$ is a δ-pseudo orbit of T_H with respect to A. Since T_H is expansive, by Lemma 3.1, there exists a unique point $y_g \in X$ such that $d(T_h y_g, x_{hg}) < \varepsilon(T_h y_g)$ for every $h \in H$.

Now we claim that $y_g = T_g y_e$ for $g \in G$. Fix $g \in G$ and $b \in B$. For any $h \in H$, there exists $h' \in H$ such that $hb = bh'$. Then we have

$$d(T_h y_{bg}, x_{bh'g}) = d(T_h y_{bg}, x_{hbg}) < \varepsilon(T_h y_{bg})$$

and

$$d(T_b T_{h'} y_g, T_b x_{h'g}) < c_2(T_b T_{h'} y_g)/3.$$

Hence we get

$$\begin{aligned} d(T_h T_b y_g, T_h y_{bg}) &= d(T_b T_{h'} y_g, T_h y_{bg}) \\ &\le d(T_{bh'} y_g, T_b x_{h'g}) + d(T_b x_{h'g}, x_{bh'g}) + d(T_h y_{bg}, x_{bh'g}) \\ &< \frac{c_2(T_h(T_b y_g))}{3} + \delta(T_b x_{h'g}) + \varepsilon(T_h y_{bg}) \\ &< \frac{c_1(T_b x_{h'g})}{3} + \frac{c_1(x_{bh'g})}{3} + \frac{c_1(x_{bh'g})}{3} \\ &< c(T_h T_b y_g). \end{aligned}$$

Since T_H is expansive, we have $T_b y_g = y_{bg}$ for every $b \in B$. As B is a symmetric generating set of G, we get $T_g y_e = y_g$ for every $g \in G$. On the other hand, set $h = e_G$ we have $d(x_g, y_g) < \varepsilon$ for every $g \in G$ and hence $d(T_g y_e, x_g) < \varepsilon(T_g y_e)$ for $g \in G$. ☐

We cite the following proposition for the proof of Lemma 3.6.

Proposition 3.5 ([5]). *Let G be a finitely generated nilpotent group with nilpotent degree n. Denote $Q = [G, G]$ and $P = \langle Q, g\rangle$ (the minimal subgroup of G that contains Q and g).*

(1) *The group P is a normal subgroup of G.*
(2) *The group P is nilpotent of class at most $n - 1$.*

Lemma 3.6. *Let G be a finitely generated nilpotent group and T be a continuous action of G on X. If there exists $g \in G$ such that T_g is expansive and has the shadowing property, then T has the shadowing property.*

Proof. We prove by induction on the nilpotent degree n of G.

If $n = 1$, then the group G is abelian and hence $H = \langle g\rangle$ is a normal subgroup of G generated by g. Thus by applying Lemma 3.4, we have T has the shadowing property.

Let $n > 1$ and assume that the statement of the lemma is true for all nilpotent groups with nilpotent degree is less than or equal to $n - 1$. Put $G_1 := [G, G]$ and $K := \langle G_1, g\rangle$. Then by Proposition 3.5, K has the nilpotent degree at most $n - 1$. It is known that G_1 is finitely generated and hence K is finitely generated. Thus, from the induction assumption, we know T_K has the shadowing property. Since T_g is expansive and $g \in K$, we have T_K is expansive. As K is a normal subgroup of G, then T has the shadowing property by Lemma 3.4. Therefore we complete the proof. ☐

Proof of Theorem 2. Let H be an nilpotent normal subgroup of G with finite index. Then H is finitely generated. Since H has finite index in G, there exists $n \in N$ such that $g_n \in H$. Because T_g is expansive and has the shadowing property, $T_{g^n} = T_g^n$ is also expansive and has the shadowing property. Thus, from Lemma 3.6, the action T_H has the shadowing property. Since $g^n \in H$ and T_{g^n} is expansive, one has T_H is expansive (by Remark 2.4). Applying Theorem 1.1 and Lemma 3.4, we complete the proof. □

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