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RIGIDITY CHARACTERIZATIONS OF COMPLETE RIEMANNIAN MANIFOLDS WITH α-BACH-FLAT

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Abstract. For complete manifolds with α-Bach tensor (which is defined by (1.2)) flat, we provide some rigidity results characterized by some point-wise inequalities involving the Weyl curvature and the traceless Ricci curvature. Moveover, some Einstein metrics have also been characterized by some $L^{\frac{n}{2}}$-integral inequalities. Furthermore, we also give some rigidity characterizations for constant sectional curvature.

1. Introduction

Let (M^n, g) be an n-dimensional Riemannian manifold with $n \geq 3$, and denote the Riemannian curvature tensor, Ricci curvature and scalar curvature of the metric g by R_{ijkl}, R_{ij} and R, respectively. It is well-known that, in order to study the conformal relativity, Bach [1] introduced the following Bach tensor

$$B_{ij} = \frac{1}{n-3} W_{ikjl,lk} + \frac{1}{n-2} W_{ikjl} R_{kl}, \tag{1.1}$$

where the indices after the comma denotes the covariant derivatives. In particular, a Bach-flat (that is, $B_{ij} = 0$) metric g is a critical metric of the functional (see [1,2])

$$\mathcal{W}_g = \int_M |W_g|^2 dV_g$$

in the case of $n = 4$. Some manifolds with Bach-flat have been studied, for example, see [3,4,6,7,11,13,16] and the references therein. The authors in [14] introduced the following α-Bach tensor:

$$B^{\alpha}_{ij} = \frac{1}{n-3} W_{ikjl,lk} + \frac{\alpha}{n-2} W_{ikjl} R_{kl}, \tag{1.2}$$

where α is a real constant, which is a natural generalization to the above Bach tensor. In particular, B^1_{ij} is exactly the Bach tensor; $B^0_{ij} = \frac{1}{n-2} C_{kij,k}$ is the divergence of Cotton tensor (see the formula (2.4)).

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The purpose of this paper is to study rigidity results for complete Riemannian manifolds with α-Bach-flat. In order to state our results, we first introduce two constants.

$$C_n = \begin{cases} \frac{\sqrt{6}}{4}, & \text{if } n = 4; \\ \frac{4\sqrt{10}}{15}, & \text{if } n = 5; \\ \frac{n-2}{\sqrt{n(n-1)}} + \frac{n^2-n-4}{2\sqrt{(n-2)(n-1)n(n+1)}}, & \text{if } n \geq 6, \end{cases} \tag{1.3}$$

$$E_n = \begin{cases} \sqrt{6}, & \text{if } n = 4; \\ \frac{4(n-1)}{n(n-2)}\left(\frac{n-2}{\sqrt{n(n-1)}} + \frac{n^2-n-4}{2\sqrt{(n-2)(n-1)n(n+1)}}\right)^{-1}, & \text{if } n \geq 6. \end{cases} \tag{1.4}$$

We let $\owedge$ denote the Kulkarni-Nomizu product and $\mathring{\mathrm{Ric}}$ be the traceless Ricci tensor.

Now, we can state our main results as follows:

Theorem 1.1. *Let (M^n, g) be a complete α-Bach-flat manifold with positive constant scalar curvature and*

$$\int_M |\mathring{R}_{ij}|^2 < \infty. \tag{1.5}$$

Then M^n is Einstein provided that any case of the following occurs:

(1) $\alpha = 0$ *and*

$$\left|W + \frac{n}{\sqrt{2n}(n-2)}\mathring{\mathrm{Ric}} \owedge g\right| < \sqrt{\frac{2}{(n-1)(n-2)}}\, R; \tag{1.6}$$

(2) $\alpha > 0$ *or* $\alpha \leq -2$, *and*

$$\left|(1+\alpha)W + \frac{n}{\sqrt{2n}(n-2)}\mathring{\mathrm{Ric}} \owedge g\right| \leq \sqrt{\frac{2}{(n-1)(n-2)}}\, R; \tag{1.7}$$

(3) $-2 < \alpha < 0$ *and*

$$|\mathring{R}_{ij}| < \frac{1}{\sqrt{n(n-1)}} R. \tag{1.8}$$

In particular, if (1.7) *holds with* $\alpha \geq -1 + nC_n\sqrt{\frac{2}{(n-1)(n-2)}}$ *or* $\alpha \leq -1 - nC_n\sqrt{\frac{2}{(n-1)(n-2)}}$, *then it is of positive constant sectional curvature.*

It is well-known that there is no complete noncompact Einstein manifold with positive constant scalar curvature. Hence, the following result follows directly:

Corollary 1.2. *Let M^n be a complete noncompact α-Bach-flat manifold with positive constant scalar curvature. Then we have*

$$\int_M |\mathring{R}_{ij}|^2 = \infty \tag{1.9}$$

provided that any case of (1.6) *with* $\alpha = 0$, (1.7) *with* $\alpha > 0$ *or* $\alpha \leq -2$, *or* (1.8) *with* $-2 < \alpha < 0$ *occurs.*

Recall that the Sobolev constant $Q_g(M)$ is defined by

$$Q_g(M) = \inf_{0 \neq v \in C_0^\infty(M)} \frac{\int_M \left(|\nabla v|^2 + \frac{n-2}{4(n-1)} R_g v^2\right)}{\left(\int_M |v|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}}. \tag{1.10}$$

It is noticed that there exist complete noncompact manifolds with negative scalar curvature which have positive Sobolev constant. For example, any simply connected complete manifold with $W_{ijkl} = 0$ has positive Sobolev constant (see [19]). It follows from (1.10) that, for any v,

$$Q_g(M)\Big(\int_M |v|^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}} \leq \int_M \Big(|\nabla v|^2 + \frac{n-2}{4(n-1)} R_g v^2\Big). \tag{1.11}$$

By virtue of Sobolev constant, we can prove the following.

Theorem 1.3. *Let* (M^n, g) *be a complete* α*-Bach-flat manifold with constant scalar curvature,* $Q_g(M) > 0$ *and satisfying* (1.5).

(1) *If* $3 \leq n \leq 6$, $R \geq 0$ *and*

$$\Big(\int_M \Big|(1+\alpha)W + \frac{n}{\sqrt{2n}(n-2)} \mathring{\mathrm{Ric}} \owedge g\Big|^{\frac{n}{2}}\Big)^{\frac{2}{n}} < \sqrt{\frac{2(n-1)}{n-2}} Q_g(M), \tag{1.12}$$

then M^n *is Einstein. In particular, if* α *satisfies*

$$\alpha \geq -1 + \frac{1}{F_n}\sqrt{\frac{2(n-1)}{n-2}}, \quad \text{or} \quad \alpha \leq -1 - \frac{1}{F_n}\sqrt{\frac{2(n-1)}{n-2}}, \tag{1.13}$$

where

$$F_n = \begin{cases} \frac{2\sqrt{15}-4}{\sqrt{10}}, & \text{if } n = 5; \\ E_n, & \text{if } n \neq 5, \end{cases} \tag{1.14}$$

then it must be of constant sectional curvature.

(2) *If* $n \geq 7$ *and*

$$\Big(\int_M \Big|(1+\alpha)W + \frac{n}{\sqrt{2n}(n-2)} \mathring{\mathrm{Ric}} \owedge g\Big|^{\frac{n}{2}}\Big)^{\frac{2}{n}} < \frac{4}{n-2}\sqrt{\frac{2(n-1)}{n-2}} Q_g(M), \tag{1.15}$$

then M^n *is Einstein. In particular, if* α *satisfies*

$$\alpha \geq -1 + \frac{4}{(n-2)F_n}\sqrt{\frac{2(n-1)}{n-2}}, \quad \text{or} \quad \alpha \leq -1 - \frac{4}{(n-2)F_n}\sqrt{\frac{2(n-1)}{n-2}}, \tag{1.16}$$

then it must be of constant sectional curvature.

Corollary 1.4. *Let* M^n *be a complete noncompact* α*-Bach-flat manifold with positive constant scalar curvature and* $Q_g(M) > 0$. *Then we have* (1.9) *holds provided that any case of* (1.12) *with* $3 \leq n \leq 6$ *or* (1.15) *with* $n \geq 7$ *occurs.*

Furthermore, we can obtain the following rigidity results for constant sectional curvature:

Theorem 1.5. *Let (M^n, g) be a complete α-Bach-flat manifold with constant scalar curvature, $Q_g(M) > 0$ and*

$$\int_M |\mathring{R}_{ijkl}|^2 < \infty. \tag{1.17}$$

(1) *If $3 \le n \le 6$, $R > 0$ and*

$$\left(\int_M |\mathring{R}_{ijkl}|^{\frac{n}{2}}\right)^{\frac{2}{n}} < \left[(n-2)|\alpha|\sqrt{\frac{n-2}{2(n-1)}} + B_n\right]^{-1} Q_g(M), \tag{1.18}$$

then it must be of constant sectional curvature, where

$$B_n = \frac{2(n-2)}{\sqrt{n(n-1)}} + \frac{n^2-n-4}{\sqrt{(n-2)(n-1)n(n+1)}} + \sqrt{\frac{(n-1)(n-2)}{n}}. \tag{1.19}$$

(2) *If $n \ge 7$, $R > 0$ and*

$$\left(\int_M |\mathring{R}_{ijkl}|^{\frac{n}{2}}\right)^{\frac{2}{n}} < \frac{4}{n-2}\left[(n-2)|\alpha|\sqrt{\frac{n-2}{2(n-1)}} + B_n\right]^{-1} Q_g(M), \tag{1.20}$$

then it must be of constant sectional curvature.

(3) *If $n \ge 10$, $R < 0$ and*

$$\left(\int_M |\mathring{R}_{ijkl}|^{\frac{n}{2}}\right)^{\frac{2}{n}} < \frac{8(n-1)}{n(n-2)}\left[(n-2)|\alpha|\sqrt{\frac{n-2}{2(n-1)}} + B_n\right]^{-1} Q_g(M), \tag{1.21}$$

then it must be of constant sectional curvature.

(4) *If $R = 0$ and (1.18) holds, then it must be of constant sectional curvature.*

Corollary 1.6. *Let M^n be a complete noncompact α-Bach-flat manifold with positive constant scalar curvature and $Q_g(M) > 0$. Then we have*

$$\int_M |\mathring{R}_{ijkl}|^2 = \infty \tag{1.22}$$

provided that any case of (1.18) with $3 \le n \le 6$ or (1.20) with $n \ge 7$ occurs.

Remark 1.7. For Bach-flat manifolds, our estimate (1.12) improves the (1.8) in [13].

Remark 1.8. The authors in [6] proved that, for complete Bach-flat manifolds, if $\left(\int_M |\mathring{R}_{ijkl}|^{\frac{n}{2}}\right)^{\frac{2}{n}}$ is small enough, then it must be of constant sectional curvature. Our Theorem 1.5 gives upper bounds of such constant for complete α-Bach-flat manifolds, in some sense.

Remark 1.9. Taking $\alpha = 1$ and $\alpha = 0$ in our Theorems, we can deduce corresponding conclusions for manifolds with Bach-flat and divergence-free Cotton tensor, respectively.

2. Preliminaries

It is well-known that the Weyl curvature tensor and the Cotton tensor are defined by

$$\begin{aligned} W_{ijkl} &= R_{ijkl} - \frac{1}{n-2}(R_{ik}g_{jl} - R_{il}g_{jk} + R_{jl}g_{ik} - R_{jk}g_{il}) \\ &\quad + \frac{R}{(n-1)(n-2)}(g_{ik}g_{jl} - g_{il}g_{jk}) \\ &= R_{ijkl} - \frac{1}{n-2}(\mathring{R}_{ik}g_{jl} - \mathring{R}_{il}g_{jk} + \mathring{R}_{jl}g_{ik} - \mathring{R}_{jk}g_{il}) \\ &\quad - \frac{R}{n(n-1)}(g_{ik}g_{jl} - g_{il}g_{jk}), \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} C_{ijk} &= R_{kj,i} - R_{ki,j} - \frac{1}{2(n-1)}(R_{,i}g_{jk} - R_{,j}g_{ik}) \\ &= \mathring{R}_{kj,i} - \mathring{R}_{ki,j} + \frac{n-2}{2n(n-1)}(R_{,i}g_{jk} - R_{,j}g_{ik}), \end{aligned} \tag{2.2}$$

respectively, where $\mathring{R}_{ij} = R_{ij} - \frac{1}{n}Rg_{ij}$ denotes the traceless Ricci tensor. From the definition of the Cotton tensor, it is easy to check that

$$C_{ijk,k} = 0, \qquad C_{iij} = C_{iji} = C_{ijj} = 0.$$

Moreover, the divergence of the Weyl curvature tensor is related to the Cotton tensor by

$$-\frac{n-3}{n-2}C_{ijk} = W_{ijkl,l}. \tag{2.3}$$

In particular, applying (2.3) in (1.2), the α-Bach tensor can be written as

$$B^{\alpha}_{ij} = \frac{1}{n-2}(C_{kij,k} + \alpha W_{ikjl}R_{kl}). \tag{2.4}$$

We let

$$\mathring{R}_{ijkl} = R_{ijkl} - \frac{R}{n(n-1)}(g_{ik}g_{jl} - g_{il}g_{jk}).$$

Then we have

$$|\mathring{R}_{ijkl}|^2 = |W_{ijkl}|^2 + \frac{4}{n-2}|\mathring{R}_{ij}|^2, \tag{2.5}$$

which shows that

$$|\mathring{R}_{ij}|^2 \le \frac{n-2}{4}|\mathring{R}_{ijkl}|^2 \tag{2.6}$$

with equality occurring if and only if $W_{ijkl} = 0$. In particular, if the scalar curvature is constant, then (2.2) shows

$$W_{ijkl,l} = -\frac{n-3}{n-2}C_{ijk} = -\frac{n-3}{n-2}(\mathring{R}_{kj,i} - \mathring{R}_{ki,j}) = \frac{n-3}{n-2}\mathring{R}_{ijkl,l}. \tag{2.7}$$

Lemma 2.1 ([7,12])**.** *On every Riemannian manifold (M^n, g), for any $\rho \in \mathbb{R}$, the following estimate holds*

$$\begin{aligned}
&\Big| - W_{ijkl}\mathring{R}_{jl}\mathring{R}_{ik} + \rho\mathring{R}_{ij}\mathring{R}_{jk}\mathring{R}_{ki}\Big| \\
&\leq \sqrt{\frac{n-2}{2(n-1)}}\Big(|W|^2 + \frac{2(n-2)\rho^2}{n}|\mathring{R}_{ij}|^2\Big)^{\frac{1}{2}}|\mathring{R}_{ij}|^2 \qquad (2.8)\\
&= \sqrt{\frac{n-2}{2(n-1)}}\Big|W + \frac{\rho}{\sqrt{2n}}\mathring{\mathrm{Ric}} \owedge g\Big||\mathring{R}_{ij}|^2.
\end{aligned}$$

The next lemma comes from the Lemma 2.3 and Remark 2.4 in [9] (or see [8, 10, 18]):

Lemma 2.2. *On every Einstein manifold (M^n, g), we have*

$$\frac{1}{2}\Delta|W|^2 \geq \frac{n+1}{n-1}|\nabla|W||^2 + \frac{2}{n}R|W|^2 - 2C_n|W|^3, \tag{2.9}$$

where C_n is defined by (1.3)*. In particular, if the scalar curvature of Einstein metric g is positive, then it is of constant positive sectional curvature, provided either*

$$C_n|W| < \frac{1}{n}R, \tag{2.10}$$

or

(1) *for $n \neq 5$,*

$$\Big(\int_M |W|^{\frac{n}{2}}\Big)^{\frac{2}{n}} < E_n\, Q_g(M), \tag{2.11}$$

where E_n is given by (1.4);

(2) *for $n = 5$,*

$$\Big(\int_M |W|^{\frac{5}{2}}\Big)^{\frac{2}{5}} \leq \frac{2\sqrt{15}-4}{\sqrt{10}}\, Q_g(M). \tag{2.12}$$

3. Proof of main results

3.1. Proof of Theorem 1.1

Denote by $p \in M^n$ and B_r a fixed point and the geodesic ball of M^n of radius r centered at p, respectively. Let ϕ_r be the nonnegative cut-off function defined on M^n satisfying

$$\phi_r = \begin{cases} 1, & \text{on } B_r \\ 0, & \text{on } M^n \backslash B_{r+1} \end{cases} \tag{3.1}$$

with $|\nabla\phi_r| \leq 2$ on $B_{r+1}\backslash B_r$. Using the above cut-off function, we have the following result (see Lemma 2.1 in [13]):

Lemma 3.1. *Let (M^n, g) be a complete Riemannian manifold with constant scalar curvature. Then for any $\theta \in \mathbb{R}$, we have*

$$\begin{aligned}
&\int_M |\nabla \mathring{R}_{ij}|^2 \phi_r^2 \\
&\geq \frac{2\theta}{\theta^2+1+\epsilon_1} \int_M \Big(W_{ijkl} \mathring{R}_{ik} \mathring{R}_{jl} - \frac{n}{n-2} \mathring{R}_{ij} \mathring{R}_{jk} \mathring{R}_{ki} - \frac{1}{n-1} R |\mathring{R}_{ij}|^2 \Big) \phi_r^2 \\
&\quad - \frac{4\theta^2}{\epsilon_1(\theta^2+1+\epsilon_1)} \int_M |\mathring{R}_{ij}|^2 |\nabla \phi_r|^2,
\end{aligned} \tag{3.2}$$

where ϵ_1 is a positive constant.

On the other hand, for complete manifolds with α-Bach-flat, we can prove the following:

Lemma 3.2. *Let (M^n, g) be a complete Riemannian manifold with constant scalar curvature. If the α-Bach tensor is flat, we have*

$$\begin{aligned}
\int_M |\nabla \mathring{R}_{ij}|^2 \phi_r^2 \leq \frac{1}{1-\epsilon_2} \int_M \Big(&(1+\alpha) W_{ijkl} \mathring{R}_{jl} \mathring{R}_{ik} - \frac{n}{n-2} \mathring{R}_{ij} \mathring{R}_{jk} \mathring{R}_{ki} \\
&- \frac{1}{n-1} R |\mathring{R}_{ij}|^2 \Big) \phi_r^2 + \frac{1}{\epsilon_2(1-\epsilon_2)} \int_M |\mathring{R}_{ij}|^2 |\nabla \phi_r|^2,
\end{aligned} \tag{3.3}$$

where $\epsilon_2 \in (0,1)$ is a constant.

Proof. For α-Bach-flat manifolds, Lemma 2.1 in [14] gives

$$\begin{aligned}
\frac{1}{2} \Delta |\mathring{R}_{ij}|^2 = |\nabla \mathring{R}_{ij}|^2 - (1+\alpha) W_{ikjl} \mathring{R}_{ij} \mathring{R}_{kl} + \frac{n}{n-2} \mathring{R}_{ij} \mathring{R}_{jk} \mathring{R}_{ki} \\
+ \frac{1}{n-1} R |\mathring{R}_{ij}|^2,
\end{aligned} \tag{3.4}$$

which shows

$$\begin{aligned}
\mathring{R}_{ij} \Delta \mathring{R}_{ij} = &- (1+\alpha) W_{ikjl} \mathring{R}_{ij} \mathring{R}_{kl} + \frac{n}{n-2} \mathring{R}_{ij} \mathring{R}_{jk} \mathring{R}_{ki} \\
&+ \frac{1}{n-1} R |\mathring{R}_{ij}|^2
\end{aligned} \tag{3.5}$$

coming from that the scalar curvature is constant. Therefore,

$$\begin{aligned}
\int_M |\nabla \mathring{R}_{ij}|^2 \phi_r^2 = &- \int_M \mathring{R}_{ij} \Delta \mathring{R}_{ij} \phi_r^2 - \int_M \mathring{R}_{ij} \mathring{R}_{ij,k} (\phi_r^2)_k \\
= &\int_M \Big((1+\alpha) W_{ijkl} \mathring{R}_{jl} \mathring{R}_{ik} - \frac{n}{n-2} \mathring{R}_{ij} \mathring{R}_{jk} \mathring{R}_{ki} \\
&- \frac{1}{n-1} R |\mathring{R}_{ij}|^2 \Big) \phi_r^2 - \int_M \mathring{R}_{ij} \mathring{R}_{ij,k} (\phi_r^2)_k \\
\leq &\int_M \Big((1+\alpha) W_{ijkl} \mathring{R}_{jl} \mathring{R}_{ik} - \frac{n}{n-2} \mathring{R}_{ij} \mathring{R}_{jk} \mathring{R}_{ki} \\
&- \frac{1}{n-1} R |\mathring{R}_{ij}|^2 \Big) \phi_r^2 + \epsilon_2 \int_M |\nabla \mathring{R}_{ij}|^2 \phi_r^2 + \frac{1}{\epsilon_2} \int_M |\mathring{R}_{ij}|^2 |\nabla \phi_r|^2,
\end{aligned} \tag{3.6}$$

finishing the proof of Lemma 3.2.

Now, with the help of Lemma 3.1 and Lemma 3.2, we will complete the proof of Theorem 1.1. Combing (3.2) and (3.3), we obtain

$$\begin{aligned}
&\frac{[4\epsilon_2(1-\epsilon_2)+\epsilon_1]\theta^2+\epsilon_1(1+\epsilon_1)}{\epsilon_1\epsilon_2(1-\epsilon_2)(\theta^2+1+\epsilon_1)}\int_M |\mathring{R}_{ij}|^2|\nabla\phi_r|^2\\
\geq\ &-\frac{(1+\alpha)\theta^2-2(1-\epsilon_2)\theta+(1+\alpha)(1+\epsilon_1)}{(1-\epsilon_2)(\theta^2+1+\epsilon_1)}\int_M W_{ijkl}\mathring{R}_{jl}\mathring{R}_{ik}\phi_r^2\\
&+\frac{\theta^2-2(1-\epsilon_2)\theta+(1+\epsilon_1)}{(1-\epsilon_2)(\theta^2+1+\epsilon_1)}\frac{n}{n-2}\int_M \mathring{R}_{ij}\mathring{R}_{jk}\mathring{R}_{ki}\phi_r^2\\
&+\frac{\theta^2-2(1-\epsilon_2)\theta+(1+\epsilon_1)}{(1-\epsilon_2)(\theta^2+1+\epsilon_1)}\frac{1}{n-1}\int_M R|\mathring{R}_{ij}|^2\phi_r^2. \qquad (3.7)
\end{aligned}$$

For all $\epsilon_2\in(0,1)$, we have

$$\theta^2-2(1-\epsilon_2)\theta+(1+\epsilon_1)=[\theta-(1-\epsilon_2)]^2+[(1+\epsilon_1)-(1-\epsilon_2)^2]>0, \qquad (3.8)$$

and hence (3.7) is equivalent to

$$\begin{aligned}
&\frac{[4\epsilon_2(1-\epsilon_2)+\epsilon_1]\theta^2+\epsilon_1(1+\epsilon_1)}{\epsilon_1\epsilon_2[\theta^2-2(1-\epsilon_2)\theta+(1+\epsilon_1)]}\int_M |\mathring{R}_{ij}|^2|\nabla\phi_r|^2\\
\geq\ &-\frac{(1+\alpha)\theta^2-2(1-\epsilon_2)\theta+(1+\alpha)(1+\epsilon_1)}{\theta^2-2(1-\epsilon_2)\theta+(1+\epsilon_1)}\int_M W_{ijkl}\mathring{R}_{jl}\mathring{R}_{ik}\phi_r^2\\
&+\frac{n}{n-2}\int_M \mathring{R}_{ij}\mathring{R}_{jk}\mathring{R}_{ki}\phi_r^2+\frac{1}{n-1}\int_M R|\mathring{R}_{ij}|^2\phi_r^2, \qquad (3.9)
\end{aligned}$$

which combining with (2.8) gives

$$\begin{aligned}
&\frac{[4\epsilon_2(1-\epsilon_2)+\epsilon_1]\theta^2+\epsilon_1(1+\epsilon_1)}{\epsilon_1\epsilon_2[\theta^2-2(1-\epsilon_2)\theta+(1+\epsilon_1)]}\int_M |\mathring{R}_{ij}|^2|\nabla\phi_r|^2\\
\geq\ &\int_M\Big[-\sqrt{\frac{n-2}{2(n-1)}}\Big|f_\alpha(\theta,\epsilon_1,\epsilon_2)W\\
&+\frac{n}{\sqrt{2n}(n-2)}\mathring{\mathrm{Ric}}\owedge g\Big|+\frac{1}{n-1}R\Big]|\mathring{R}_{ij}|^2\phi_r^2, \qquad (3.10)
\end{aligned}$$

where

$$f_\alpha(\theta,\epsilon_1,\epsilon_2)=\frac{(1+\alpha)\theta^2-2(1-\epsilon_2)\theta+(1+\alpha)(1+\epsilon_1)}{\theta^2-2(1-\epsilon_2)\theta+(1+\epsilon_1)}. \qquad (3.11)$$

Case one: When $\alpha=0$, then $f_0(\theta,\epsilon_1,\epsilon_2)=1$ and (3.10) becomes

$$\begin{aligned}
&\frac{[4\epsilon_2(1-\epsilon_2)+\epsilon_1]\theta^2+\epsilon_1(1+\epsilon_1)}{\epsilon_1\epsilon_2[\theta^2-2(1-\epsilon_2)\theta+(1+\epsilon_1)]}\int_M |\mathring{R}_{ij}|^2|\nabla\phi_r|^2\\
\geq\ &\int_M\Big[-\sqrt{\frac{n-2}{2(n-1)}}\Big|W+\frac{n}{\sqrt{2n}(n-2)}\mathring{\mathrm{Ric}}\owedge g\Big|+\frac{1}{n-1}R\Big]|\mathring{R}_{ij}|^2\phi_r^2. \qquad (3.12)
\end{aligned}$$

Therefore, under the condition (1.6), the estimate (3.12) gives

$$
\begin{aligned}
0 &\leq \int_M \Big[-\sqrt{\frac{n-2}{2(n-1)}}\Big|W + \frac{n}{\sqrt{2n}(n-2)}\mathring{\mathrm{Ric}} \owedge g\Big| + \frac{1}{n-1}R\Big]|\mathring{R}_{ij}|^2\phi_r^2 \\
&\leq \frac{[4\epsilon_2(1-\epsilon_2)+\epsilon_1]\theta^2+\epsilon_1(1+\epsilon_1)}{\epsilon_1\epsilon_2[\theta^2-2(1-\epsilon_2)\theta+(1+\epsilon_1)]}\int_M |\mathring{R}_{ij}|^2|\nabla\phi_r|^2 \to 0 \qquad (3.13)
\end{aligned}
$$

as $r\to\infty$ coming from (1.5), which shows that M^n is Einstein.

Case two: When $\alpha > 0$ or $\alpha \leq -2$. We fix ϵ_1 and ϵ_2 and take θ satisfying

$$
\theta = -\sqrt{1+\epsilon_1}, \qquad (3.14)
$$

then (3.10) becomes

$$
\begin{aligned}
&\frac{[2\epsilon_2(1-\epsilon_2)+\epsilon_1]\sqrt{1+\epsilon_1}}{\epsilon_1\epsilon_2[\sqrt{1+\epsilon_1}+(1-\epsilon_2)]}\int_M |\mathring{R}_{ij}|^2|\nabla\phi_r|^2 \\
&\geq \int_M \Bigg[-\sqrt{\frac{n-2}{2(n-1)}}\Bigg|\Big(1+\frac{\alpha}{1+\frac{1-\epsilon_2}{\sqrt{1+\epsilon_1}}}\Big)W \\
&\quad + \frac{n}{\sqrt{2n}(n-2)}\mathring{\mathrm{Ric}} \owedge g\Bigg| + \frac{1}{n-1}R\Bigg]|\mathring{R}_{ij}|^2\phi_r^2. \qquad (3.15)
\end{aligned}
$$

Since W is perpendicular to $\mathring{\mathrm{Ric}} \owedge g$, we have that

$$
\begin{aligned}
&\Big|\Big(1+\frac{\alpha}{1+\frac{1-\epsilon_2}{\sqrt{1+\epsilon_1}}}\Big)W + \frac{n}{\sqrt{2n}(n-2)}\mathring{\mathrm{Ric}} \owedge g\Big|^2 \\
&= \Big(1+\frac{\alpha}{1+\frac{1-\epsilon_2}{\sqrt{1+\epsilon_1}}}\Big)^2|W|^2 + \frac{n}{2(n-2)^2}|\mathring{\mathrm{Ric}} \owedge g|^2 \\
&< (1+\alpha)^2|W|^2 + \frac{n}{2(n-2)^2}|\mathring{\mathrm{Ric}} \owedge g|^2 \\
&= \Big|(1+\alpha)W + \frac{n}{\sqrt{2n}(n-2)}\mathring{\mathrm{Ric}} \owedge g\Big|^2 \qquad (3.16)
\end{aligned}
$$

holds for any given ϵ_1, ϵ_2, which shows that

$$
\begin{aligned}
&\Big|\Big(1+\frac{\alpha}{1+\frac{1-\epsilon_2}{\sqrt{1+\epsilon_1}}}\Big)W + \frac{n}{\sqrt{2n}(n-2)}\mathring{\mathrm{Ric}} \owedge g\Big| \\
&< \Big|(1+\alpha)W + \frac{n}{\sqrt{2n}(n-2)}\mathring{\mathrm{Ric}} \owedge g\Big|. \qquad (3.17)
\end{aligned}
$$

Therefore, if (1.7) is true, then (3.15) and (3.17) mean that M^n is Einstein.

Case three: When $-2 < \alpha < 0$. In this case, it is easy to see that there exists θ such that $f_\alpha(\theta,\epsilon_1,\epsilon_2) = 0$ provided that ϵ_1, ϵ_2 satisfy $(1-\epsilon_2)^2 - (1+\alpha)^2(1+\epsilon_1) = 0$ (that is, the discriminant of the quadratic function $\tilde{f}(\theta) = (1+\alpha)\theta^2 - 2(1-\epsilon_2)\theta + (1+\alpha)(1+\epsilon_1)$ with respect to θ is zero), and hence

(3.10) becomes

$$\frac{[4\epsilon_2(1-\epsilon_2)+\epsilon_1]\theta^2+\epsilon_1(1+\epsilon_1)}{\epsilon_1\epsilon_2[\theta^2-2(1-\epsilon_2)\theta+(1+\epsilon_1)]}\int_M |\mathring{R}_{ij}|^2|\nabla\phi_r|^2 \geq \int_M\Big(-\sqrt{\frac{n}{n-1}}|\mathring{R}_{ij}|+\frac{1}{n-1}R\Big)|\mathring{R}_{ij}|^2\phi_r^2, \tag{3.18}$$

which shows that M^n is Einstein provided that (1.8) occurs.

Furthermore, when (1.7) holds with $\alpha \geq -1+nC_n\sqrt{\frac{2}{(n-1)(n-2)}}$ or $\alpha \leq -1-nC_n\sqrt{\frac{2}{(n-1)(n-2)}}$, we have that M^n is Einstein and (1.7) becomes

$$|1+\alpha||W| \leq \sqrt{\frac{2}{(n-1)(n-2)}}\,R, \tag{3.19}$$

which shows

$$C_n|1+\alpha||W| \leq C_n\sqrt{\frac{2}{(n-1)(n-2)}}R \leq \frac{|1+\alpha|}{n}R. \tag{3.20}$$

This combining with (2.10) completes the proof of Theorem 1.1.

3.2. Proof of Theorem 1.3

Using (3.3), we obtain

$$\begin{aligned}\int_M |\nabla|\mathring{R}_{ij}||^2\phi_r^2 \leq & \int_M |\nabla\mathring{R}_{ij}|^2\phi_r^2 \\ \leq & \frac{1}{1-\epsilon_2}\int_M\Bigg[\sqrt{\frac{n-2}{2(n-1)}}\Big|(1+\alpha)W+\frac{n}{\sqrt{2n}(n-2)}\mathring{\mathrm{Ric}}\owedge g\Big| \\ & -\frac{1}{n-1}R\Bigg]|\mathring{R}_{ij}|^2\phi_r^2+\frac{1}{\epsilon_2(1-\epsilon_2)}\int_M |\mathring{R}_{ij}|^2|\nabla\phi_r|^2.\end{aligned} \tag{3.21}$$

Taking $v=|\mathring{R}_{ij}|\phi_r$ in (1.11) and noticing (3.21) yield

$$\begin{aligned}& Q_g(M)\Big(\int_M(|\mathring{R}_{ij}|\phi_r)^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}} \\ \leq & \int_M\Big(|\nabla(|\mathring{R}_{ij}|\phi_r)|^2+\frac{n-2}{4(n-1)}R|\mathring{R}_{ij}|^2\phi_r^2\Big) \\ \leq & (1+\epsilon_3)\int_M|\nabla|\mathring{R}_{ij}||^2\phi_r^2+\Big(1+\frac{1}{\epsilon_3}\Big)\int_M|\mathring{R}_{ij}|^2|\nabla\phi_r|^2 \\ & +\frac{n-2}{4(n-1)}\int_M R|\mathring{R}_{ij}|^2\phi_r^2 \\ \leq & \frac{1+\epsilon_3}{1-\epsilon_2}\sqrt{\frac{n-2}{2(n-1)}}\int_M\Big|(1+\alpha)W+\frac{n}{\sqrt{2n}(n-2)}\mathring{\mathrm{Ric}}\owedge g\Big||\mathring{R}_{ij}|^2\phi_r^2\end{aligned}$$

$$+\frac{1}{n-1}\Big[\frac{n-2}{4}-\frac{1+\epsilon_3}{1-\epsilon_2}\Big]\int_M R|\mathring{R}_{ij}|^2\phi_r^2$$
$$+(1+\epsilon_3)\Big[\frac{1}{\epsilon_3}+\frac{1}{\epsilon_2(1-\epsilon_2)}\Big]\int_M|\mathring{R}_{ij}|^2|\nabla\phi_r|^2, \tag{3.22}$$

where ϵ_3 is a positive constant. By virtue of the Hölder inequality

$$\int_M\Big|(1+\alpha)W+\frac{n}{\sqrt{2n}(n-2)}\mathring{\mathrm{Ric}}\owedge g\Big||\mathring{R}_{ij}|^2\phi_r^2$$
$$\leq\Big(\int_M\Big|(1+\alpha)W+\frac{n}{\sqrt{2n}(n-2)}\mathring{\mathrm{Ric}}\owedge g\Big|^{\frac{n}{2}}\Big)^{\frac{2}{n}}\Big(\int_M(|\mathring{R}_{ij}|\phi_r)^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}}$$

in (3.22) gives

$$\Bigg[Q_g(M)-\frac{1+\epsilon_3}{1-\epsilon_2}\sqrt{\frac{n-2}{2(n-1)}}\Big(\int_M\Big|(1+\alpha)W$$
$$+\frac{n}{\sqrt{2n}(n-2)}\mathring{\mathrm{Ric}}\owedge g\Big|^{\frac{n}{2}}\Big)^{\frac{2}{n}}\Bigg]\Big(\int_M(|\mathring{R}_{ij}|\phi_r)^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}}$$
$$\leq\frac{1}{n-1}\Big[\frac{n-2}{4}-\frac{1+\epsilon_3}{1-\epsilon_2}\Big]\int_M R|\mathring{R}_{ij}|^2\phi_r^2$$
$$+(1+\epsilon_3)\Big[\frac{1}{\epsilon_3}+\frac{1}{\epsilon_2(1-\epsilon_2)}\Big]\int_M|\mathring{R}_{ij}|^2|\nabla\phi_r|^2. \tag{3.23}$$

Case one: When $3\leq n\leq 6$ and $R\geq 0$, for all ϵ_2,ϵ_3, we always have

$$\frac{n-2}{4}-\frac{1+\epsilon_3}{1-\epsilon_2}<0. \tag{3.24}$$

On the other hand, the condition (1.12) is equivalent to

$$\frac{\Big(\int_M\Big|(1+\alpha)W+\frac{n}{\sqrt{2n}(n-2)}\mathring{\mathrm{Ric}}\owedge g\Big|^{\frac{n}{2}}\Big)^{\frac{2}{n}}}{\sqrt{\frac{2(n-1)}{n-2}}Q_g(M)}<1. \tag{3.25}$$

In this case, there exist ϵ_2,ϵ_3 small enough such that

$$\frac{\Big(\int_M\Big|(1+\alpha)W+\frac{n}{\sqrt{2n}(n-2)}\mathring{\mathrm{Ric}}\owedge g\Big|^{\frac{n}{2}}\Big)^{\frac{2}{n}}}{\sqrt{\frac{2(n-1)}{n-2}}Q_g(M)}=\frac{1-\epsilon_2}{1+\epsilon_3} \tag{3.26}$$

which, from (3.23), shows that

$$0\leq\frac{1}{n-1}\Big[\frac{n-2}{4}-\frac{1+\epsilon_3}{1-\epsilon_2}\Big]\int_M R|\mathring{R}_{ij}|^2\phi_r^2$$
$$+(1+\epsilon_3)\Big[\frac{1}{\epsilon_3}+\frac{1}{\epsilon_2(1-\epsilon_2)}\Big]\int_M|\mathring{R}_{ij}|^2|\nabla\phi_r|^2$$
$$\leq(1+\epsilon_3)\Big[\frac{1}{\epsilon_3}+\frac{1}{\epsilon_2(1-\epsilon_2)}\Big]\int_M|\mathring{R}_{ij}|^2|\nabla\phi_r|^2\to 0 \tag{3.27}$$

as $r \to \infty$. Hence, M^n is Einstein. In particular, for the Einstein metric with α satisfying (1.13), then (1.12) becomes

$$|1+\alpha|\Big(\int_M |W|^{\frac{n}{2}}\Big)^{\frac{2}{n}} < \sqrt{\frac{2(n-1)}{n-2}}Q_g(M) \le |1+\alpha|F_n Q_g(M), \tag{3.28}$$

which, combining with Lemma 2.2, shows that it is of constant sectional curvature.

Case two: When $n \ge 7$, there exist ϵ_2, ϵ_3 depending only on the dimension n such that

$$\frac{1+\epsilon_3}{1-\epsilon_2} = \frac{n-2}{4}. \tag{3.29}$$

In this case, (3.23) becomes

$$\begin{aligned} 0 \le & \Bigg[Q_g(M) - \frac{n-2}{4}\sqrt{\frac{n-2}{2(n-1)}}\Big(\int_M \Big|(1+\alpha)W \\ & + \frac{n}{\sqrt{2n}(n-2)}\mathring{\mathrm{Ric}} \owedge g\Big|^{\frac{n}{2}}\Big)^{\frac{2}{n}}\Bigg]\Big(\int_M (|\mathring{R}_{ij}|\phi_r)^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}} \\ & \le (1+\epsilon_3)\Big[\frac{1}{\epsilon_3} + \frac{1}{\epsilon_2(1-\epsilon_2)}\Big]\int_M |\mathring{R}_{ij}|^2|\nabla\phi_r|^2 \to 0 \end{aligned} \tag{3.30}$$

as $r \to \infty$ from the assumption (1.15), which shows that M^n is Einstein. In particular, for an Einstein metric with α satisfying (1.16), using the similar arguments as in the proof of case one, we can finish the proof of Theorem 1.3.

3.3. Proof of Theorem 1.5

We need the following lemma (see Lemma 2.1 in [5]):

Lemma 3.3. *For any manifold M^n, it holds that*

$$\begin{aligned} \frac{1}{2}\Delta|\mathring{R}_{ijkl}|^2 = & |\nabla\mathring{R}_{ijkl}|^2 + 2\mathring{R}_{ijkl}(2\mathring{R}_{ipkq}\mathring{R}_{pjql} - \frac{1}{2}\mathring{R}_{klpq}\mathring{R}_{pqij}) \\ & + 2\mathring{R}_{ijkp}\mathring{R}_{ijkq}\mathring{\mathrm{R}}_{pq} + 2\mathring{R}_{ijkl}\mathring{R}_{ijkp,pl} + \frac{2}{n}R|\mathring{R}_{ijkl}|^2 \\ & - \frac{4}{n(n-1)}R|\mathring{R}_{ij}|^2 + \frac{4}{n(n-1)}\mathring{R}_{ij}R_{,ij}. \end{aligned} \tag{3.31}$$

Using the inequality proved by Li and Zhao [17] and Huisken [15], we have

$$\begin{aligned} & \Big|2\mathring{R}_{ijkl}\mathring{R}_{ipkq}\mathring{R}_{pjql} - \frac{1}{2}\mathring{R}_{ijkl}\mathring{R}_{klpq}\mathring{R}_{pqij}\Big| \\ \le & 2|\mathring{R}_{ijkl}\mathring{R}_{ipkq}\mathring{R}_{pjql}| + \frac{1}{2}|\mathring{R}_{ijkl}\mathring{R}_{klpq}\mathring{R}_{pqij}| \\ \le & \Big[\frac{n-2}{\sqrt{n(n-1)}} + \frac{n^2-n-4}{2\sqrt{(n-2)(n-1)n(n+1)}}\Big]|\mathring{R}_{ijkl}|^3, \end{aligned} \tag{3.32}$$

and

$$|\mathring{R}_{ijkp}\mathring{R}_{ijkq}\mathring{\mathrm{R}}_{pq}| \leq \sqrt{\frac{n-1}{n}}|\mathring{\mathrm{R}}_{ij}||\mathring{R}_{ijkl}|^2 \leq \frac{1}{2}\sqrt{\frac{(n-1)(n-2)}{n}}|\mathring{R}_{ijkl}|^3, \tag{3.33}$$

where we used (2.6) in (3.33). Noticing that R is constant and inserting (3.32) and (3.33) into (3.31) give

$$\frac{1}{2}\Delta|\mathring{R}_{ijkl}|^2 \geq |\nabla\mathring{R}_{ijkl}|^2 - B_n|\mathring{R}_{ijkl}|^3 + 2\mathring{R}_{ijkl}\mathring{R}_{ijkp,pl} + A_nR|\mathring{R}_{ijkl}|^2, \tag{3.34}$$

where B_n is given by (1.19) and

$$A_n = \begin{cases} \frac{1}{n-1}, & \text{if } R > 0; \\ 0, & \text{if } R = 0; \\ \frac{2}{n}, & \text{if } R < 0. \end{cases} \tag{3.35}$$

Let $u = |\mathring{R}_{ijkl}|$. By virtue of (3.34) and the Kato inequality $|\nabla\mathring{R}_{ijkl}|^2 \geq |\nabla|\mathring{R}_{ijkl}||^2$, it is easy to see

$$u\Delta u \geq -B_nu^3 + 2\mathring{R}_{ijkl}\mathring{R}_{ijkp,pl} + A_nRu^2, \tag{3.36}$$

and hence

$$-\int_M \phi_r^2 u\Delta u \leq \int_M \phi_r^2[B_nu^3 - 2\mathring{R}_{ijkl}\mathring{R}_{ijkp,pl} - A_nRu^2]. \tag{3.37}$$

Inserting

$$\begin{aligned} -\int_M \phi_r^2 u\Delta u &= \int_M u_i(\phi_r^2u)_i = \int_M |\nabla u|^2\phi_r^2 + 2\int_M uu_i\phi_r(\phi_r)_i \\ &\geq (1-\epsilon_4)\int_M |\nabla u|^2\phi_r^2 - \frac{1}{\epsilon_4}\int_M |\nabla\phi_r|^2u^2 \end{aligned} \tag{3.38}$$

and

$$\begin{aligned} -2\int_M \mathring{R}_{ijkl}\mathring{R}_{ijkp,pl}\phi_r^2 &= 2\int_M (\mathring{R}_{ijkl}\phi_r^2)_{,l}\mathring{R}_{ijkp,p} \\ &\leq 2(1+\epsilon_5)\int_M |\mathring{R}_{ijkp,p}|^2\phi_r^2 + \frac{2}{\epsilon_5}\int_M |\nabla\phi_r|^2u^2 \end{aligned} \tag{3.39}$$

with any positive constant ϵ_5 into (3.37) gives

$$\begin{aligned} \int_M |\nabla u|^2\phi_r^2 \leq{}& \frac{1}{1-\epsilon_4}\int_M \phi_r^2[B_nu^3 - A_nRu^2] + \frac{2(1+\epsilon_5)}{1-\epsilon_4}\int_M |\mathring{R}_{ijkp,p}|^2\phi_r^2 \\ &+ \frac{1}{1-\epsilon_4}\Big(\frac{2}{\epsilon_5} + \frac{1}{\epsilon_4}\Big)\int_M |\nabla\phi_r|^2u^2, \end{aligned} \tag{3.40}$$

provided $\epsilon_4 \in (0,1)$.

On the other hand, since the α-Bach tensor is flat, we have

$$0 = \frac{\alpha}{n-2}\int_M W_{ikjl}\mathring{\mathrm{R}}_{kl}\mathring{\mathrm{R}}_{ij}\phi_r^2 + \frac{1}{n-3}\int_M W_{ikjl,lk}\mathring{\mathrm{R}}_{ij}\phi_r^2$$

$$
\begin{aligned}
&= \frac{\alpha}{n-2}\int_M W_{ikjl}\mathring{\mathrm{R}}_{kl}\mathring{\mathrm{R}}_{ij}\phi_r^2 + \frac{1}{n-2}\int_M \mathring{R}_{ikjl,lk}\mathring{\mathrm{R}}_{ij}\phi_r^2 \\
&= \frac{\alpha}{n-2}\int_M W_{ikjl}\mathring{\mathrm{R}}_{kl}\mathring{\mathrm{R}}_{ij}\phi_r^2 - \frac{1}{n-2}\int_M \mathring{R}_{ikjl,l}\mathring{\mathrm{R}}_{ij,k}\phi_r^2 \\
&\quad - \frac{2}{n-2}\int_M \mathring{R}_{ikjl,l}\mathring{\mathrm{R}}_{ij}\phi_r(\phi_r)_k \\
&\le \frac{\alpha}{n-2}\int_M W_{ikjl}\mathring{\mathrm{R}}_{kl}\mathring{\mathrm{R}}_{ij}\phi_r^2 - \frac{1}{2(n-2)}\int_M |\mathring{R}_{ijkp,p}|^2\phi_r^2 \\
&\quad + \frac{1}{n-2}\int_M (\epsilon_6|\mathring{R}_{ijkp,p}|^2\phi_r^2 + \frac{1}{\epsilon_6}|\mathring{\mathrm{R}}_{ij}|^2|\nabla\phi_r|^2) \\
&= \frac{\alpha}{n-2}\int_M W_{ikjl}\mathring{\mathrm{R}}_{kl}\mathring{\mathrm{R}}_{ij}\phi_r^2 - \frac{1-2\epsilon_6}{2(n-2)}\int_M |\mathring{R}_{ijkp,p}|^2\phi_r^2 \\
&\quad + \frac{1}{(n-2)\epsilon_6}\int_M |\mathring{\mathrm{R}}_{ij}|^2|\nabla\phi_r|^2. \qquad (3.41)
\end{aligned}
$$

Using the following inequality which was first proved by Huisken (cf. [15, Lemma 3.4]):

$$
|W_{ikjl}\mathring{\mathrm{R}}_{ij}\mathring{\mathrm{R}}_{kl}| \le \sqrt{\frac{n-2}{2(n-1)}}|W_{ijkl}||\mathring{\mathrm{R}}_{ij}|^2 \qquad (3.42)
$$

which shows

$$
|W_{ikjl}\mathring{\mathrm{R}}_{ij}\mathring{\mathrm{R}}_{kl}| \le \frac{n-2}{4}\sqrt{\frac{n-2}{2(n-1)}}u^3 \qquad (3.43)
$$

by using (2.6), and hence (3.41) yields

$$
\begin{aligned}
0 &\le \frac{|\alpha|}{4}\sqrt{\frac{n-2}{2(n-1)}}\int_M u^3\phi_r^2 - \frac{1-2\epsilon_6}{2(n-2)}\int_M |\mathring{R}_{ijkp,p}|^2\phi_r^2 \\
&\quad + \frac{1}{4\epsilon_6}\int_M u^2|\nabla\phi_r|^2. \qquad (3.44)
\end{aligned}
$$

It follows from (3.44) that

$$
\begin{aligned}
\int_M |\mathring{R}_{ijkp,p}|^2\phi_r^2 &\le \frac{(n-2)|\alpha|}{2(1-2\epsilon_6)}\sqrt{\frac{n-2}{2(n-1)}}\int_M u^3\phi_r^2 \\
&\quad + \frac{n-2}{2\epsilon_6(1-2\epsilon_6)}\int_M |\nabla\phi_r|^2u^2 \qquad (3.45)
\end{aligned}
$$

provided $\epsilon_6 \in (0, \frac{1}{2})$. Now, inserting (3.45) into (3.40) gives

$$
\begin{aligned}
\int_M |\nabla u|^2\phi_r^2 \le & -\frac{A_n}{1-\epsilon_4}\int_M R\phi_r^2u^2 \\
& + \frac{1}{1-\epsilon_4}\Big[\frac{(n-2)(1+\epsilon_5)|\alpha|}{(1-2\epsilon_6)}\sqrt{\frac{n-2}{2(n-1)}} + B_n\Big]\int_M u^3\phi_r^2
\end{aligned}
$$

$$+\frac{1}{1-\epsilon_4}\Big[\frac{(n-2)(1+\epsilon_5)}{\epsilon_6(1-2\epsilon_6)}+\frac{2}{\epsilon_5}+\frac{1}{\epsilon_4}\Big]\int_M|\nabla\phi_r|^2u^2. \tag{3.46}$$

Thus, replacing v in (1.11) with $\phi_r u$, we have

$$\begin{aligned}
&Q_g(M)\Big(\int_M|\phi_r u|^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}}\\
\le\;&\int_M|\nabla(\phi_r u)|^2+\frac{n-2}{4(n-1)}R\phi_r^2u^2\\
\le\;&(1+\epsilon_7)\int_M|\nabla u|^2\phi_r^2+\Big(1+\frac{1}{\epsilon_7}\Big)\int_M u^2|\nabla\phi_r|^2+\frac{n-2}{4(n-1)}\int_M R\phi_r^2u^2\\
\le\;&\Big[\frac{n-2}{4(n-1)}-\frac{A_n(1+\epsilon_7)}{1-\epsilon_4}\Big]\int_M R\phi_r^2u^2\\
&+\frac{1+\epsilon_7}{1-\epsilon_4}\Big[\frac{(n-2)(1+\epsilon_5)|\alpha|}{(1-2\epsilon_6)}\sqrt{\frac{n-2}{2(n-1)}}+B_n\Big]\int_M u^3\phi_r^2\\
&+(1+\epsilon_7)\Big[\frac{1}{1-\epsilon_4}\Big(\frac{(n-2)(1+\epsilon_5)}{\epsilon_6(1-2\epsilon_6)}+\frac{2}{\epsilon_5}+\frac{1}{\epsilon_4}\Big)+\frac{1}{\epsilon_7}\Big]\int_M|\nabla\phi_r|^2u^2,
\end{aligned} \tag{3.47}$$

where ϵ_7 is a positive constant and we used (3.46) in the last inequality. Using the Hölder inequality, we obtain

$$\begin{aligned}
&\Big\{Q_g(M)-\frac{1+\epsilon_7}{1-\epsilon_4}\Big[\frac{(n-2)(1+\epsilon_5)|\alpha|}{(1-2\epsilon_6)}\sqrt{\frac{n-2}{2(n-1)}}+B_n\Big]\Big(\int_M u^{\frac{n}{2}}\Big)^{\frac{2}{n}}\Big\}\Big(\int_M|\phi_r u|^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}}\\
\le\;&\Big[\frac{n-2}{4(n-1)}-\frac{A_n(1+\epsilon_7)}{1-\epsilon_4}\Big]\int_M R\phi_r^2u^2\\
&+(1+\epsilon_7)\Big[\frac{1}{1-\epsilon_4}\Big(\frac{(n-2)(1+\epsilon_5)}{\epsilon_6(1-2\epsilon_6)}+\frac{2}{\epsilon_5}+\frac{1}{\epsilon_4}\Big)+\frac{1}{\epsilon_7}\Big]\int_M|\nabla\phi_r|^2u^2.
\end{aligned} \tag{3.48}$$

Case one: When $R>0$. Then (3.48) becomes

$$\begin{aligned}
&\Big\{Q_g(M)-\frac{1+\epsilon_7}{1-\epsilon_4}\Big[\frac{(n-2)(1+\epsilon_5)|\alpha|}{(1-2\epsilon_6)}\sqrt{\frac{n-2}{2(n-1)}}+B_n\Big]\Big(\int_M u^{\frac{n}{2}}\Big)^{\frac{2}{n}}\Big\}\Big(\int_M|\phi_r u|^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}}\\
\le\;&\frac{1}{n-1}\Big[\frac{n-2}{4}-\frac{1+\epsilon_7}{1-\epsilon_4}\Big]\int_M R\phi_r^2u^2\\
&+(1+\epsilon_7)\Big[\frac{1}{1-\epsilon_4}\Big(\frac{(n-2)(1+\epsilon_5)}{\epsilon_6(1-2\epsilon_6)}+\frac{2}{\epsilon_5}+\frac{1}{\epsilon_4}\Big)+\frac{1}{\epsilon_7}\Big]\int_M|\nabla\phi_r|^2u^2.
\end{aligned} \tag{3.49}$$

If $3\le n\le 6$, then we have

$$\frac{n-2}{4}-\frac{1+\epsilon_7}{1-\epsilon_4}<0 \tag{3.50}$$

for all $\epsilon_7 > 0$ and $\epsilon_4 \in (0,1)$. In particular, if the condition (1.18) holds, then there exist $\epsilon_4, \epsilon_5, \epsilon_6, \epsilon_7$ small enough such that

$$Q_g(M) - \frac{1+\epsilon_7}{1-\epsilon_4}\Big[\frac{(n-2)(1+\epsilon_5)|\alpha|}{(1-2\epsilon_6)}\sqrt{\frac{n-2}{2(n-1)}} + B_n\Big]\Big(\int_M u^{\frac{n}{2}}\Big)^{\frac{2}{n}} = 0. \tag{3.51}$$

Thus, we deduce from (3.49) that $u = 0$ and hence it is of constant sectional curvature.

If $n \geq 7$, then there exist ϵ_4, ϵ_7 such that

$$\frac{n-2}{4} - \frac{1+\epsilon_7}{1-\epsilon_4} = 0 \tag{3.52}$$

and hence (3.49) becomes

$$\begin{aligned}&\Big\{Q_g(M) - \frac{n-2}{4}\Big[\frac{(n-2)(1+\epsilon_5)|\alpha|}{(1-2\epsilon_6)}\sqrt{\frac{n-2}{2(n-1)}} + B_n\Big]\Big(\int_M u^{\frac{n}{2}}\Big)^{\frac{2}{n}}\Big\}\Big(\int_M |\phi_r u|^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}}\\ &\leq (1+\epsilon_7)\Big[\frac{1}{1-\epsilon_4}\Big(\frac{(n-2)(1+\epsilon_5)}{\epsilon_6(1-2\epsilon_6)} + \frac{2}{\epsilon_5} + \frac{1}{\epsilon_4}\Big) + \frac{1}{\epsilon_7}\Big]\int_M |\nabla\phi_r|^2 u^2.\end{aligned} \tag{3.53}$$

Similarly, if (1.20) is true, we can conclude the proof of (2) in Theorem 1.5 by using the similar arguments.

Case two: When $R < 0$. Then (3.48) becomes

$$\begin{aligned}&\Big\{Q_g(M) - \frac{1+\epsilon_7}{1-\epsilon_4}\Big[\frac{(n-2)(1+\epsilon_5)|\alpha|}{(1-2\epsilon_6)}\sqrt{\frac{n-2}{2(n-1)}} + B_n\Big]\Big(\int_M u^{\frac{n}{2}}\Big)^{\frac{2}{n}}\Big\}\Big(\int_M |\phi_r u|^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}}\\ &\leq \frac{2}{n}\Big[\frac{n(n-2)}{8(n-1)} - \frac{1+\epsilon_7}{1-\epsilon_4}\Big]\int_M R\phi_r^2 u^2\\ &\quad + (1+\epsilon_7)\Big[\frac{1}{1-\epsilon_4}\Big(\frac{(n-2)(1+\epsilon_5)}{\epsilon_6(1-2\epsilon_6)} + \frac{2}{\epsilon_5} + \frac{1}{\epsilon_4}\Big) + \frac{1}{\epsilon_7}\Big]\int_M |\nabla\phi_r|^2 u^2.\end{aligned} \tag{3.54}$$

If $n \geq 10$, then there exist ϵ_4, ϵ_7 such that

$$\frac{n(n-2)}{8(n-1)} - \frac{1+\epsilon_7}{1-\epsilon_4} = 0 \tag{3.55}$$

and hence (3.54) becomes

$$\begin{aligned}&\Big\{Q_g(M) - \frac{n(n-2)}{8(n-1)}\Big[\frac{(n-2)(1+\epsilon_5)|\alpha|}{(1-2\epsilon_6)}\sqrt{\frac{n-2}{2(n-1)}} + B_n\Big]\Big(\int_M u^{\frac{n}{2}}\Big)^{\frac{2}{n}}\Big\}\Big(\int_M |\phi_r u|^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}}\\ &\leq (1+\epsilon_7)\Big[\frac{1}{1-\epsilon_4}\Big(\frac{(n-2)(1+\epsilon_5)}{\epsilon_6(1-2\epsilon_6)} + \frac{2}{\epsilon_5} + \frac{1}{\epsilon_4}\Big) + \frac{1}{\epsilon_7}\Big]\int_M |\nabla\phi_r|^2 u^2.\end{aligned} \tag{3.56}$$

Similarly, the condition (1.21) concluding the proof of (3) in Theorem 1.5.

Case three: When $R = 0$. Then (3.48) becomes

(3.57)
$$\begin{aligned}&\Big\{Q_g(M) - \frac{1+\epsilon_7}{1-\epsilon_4}\Big[\frac{(n-2)(1+\epsilon_5)|\alpha|}{(1-2\epsilon_6)}\sqrt{\frac{n-2}{2(n-1)}} + B_n\Big]\Big(\int_M u^{\frac{n}{2}}\Big)^{\frac{2}{n}}\Big\}\Big(\int_M |\phi_r u|^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}} \\ &\le (1+\epsilon_7)\Big[\frac{1}{1-\epsilon_4}\Big(\frac{(n-2)(1+\epsilon_5)}{\epsilon_6(1-2\epsilon_6)} + \frac{2}{\epsilon_5} + \frac{1}{\epsilon_4}\Big) + \frac{1}{\epsilon_7}\Big]\int_M |\nabla\phi_r|^2 u^2,\end{aligned}$$

which concludes the proof of (4) in Theorem 1.5 provided (1.18) by using the similar arguments.

Therefore, we complete the proof of Theorem 1.5.

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