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PARAMETER MARCINKIEWICZ INTEGRAL AND ITS COMMUTATOR ON GENERALIZED ORLICZ-MORREY SPACES

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Abstract. The aim of this paper is to mainly establish the sufficient and necessary conditions for the boundedness of the commutator $\mathcal{M}^{\rho}_{\Omega,b}$ which is generated by the parameter Marcinkiwicz integral $\mathcal{M}^{\rho}_{\Omega}$ and the Lipschitz function b on generalized Orlicz-Morrey space $L^{\Phi,\varphi}(\mathbb{R}^d)$ in the sense of the Adams type result (or Spanne type result). Moreover, the necessary conditions for the parameter Marcinkiewizcz integral $\mathcal{M}^{\rho}_{\Omega}$ on the $L^{\Phi,\varphi}(\mathbb{R}^d)$, and the commutator $[b,\mathcal{M}^{\rho}_{\Omega}]$ generated by the $\mathcal{M}^{\rho}_{\Omega}$ and the space BMO on the $L^{\Phi,\varphi}(\mathbb{R}^d)$, are also obtained, respectively.

1. Introduction

In 1938, Marcinkiewicz [22] originally introduced the following operator

$$\mathcal{M}(f)(x)=\left(\int_0^{\pi}\frac{|F(x+t)+F(x-t)-2F(x)|^2}{t^3}\mathrm{d}t\right)^{\frac{1}{2}},\ x\in[0,2\pi],$$

where $F(x)=\int_0^x f(t)\mathrm{d}t$. For convenience, the operator is called the Marcinkiewicz integral. After a few years, Stein [25] generalized the above Marcinkiewicz integral into the higher-dimensional case, respectively, its definition as follows

$$(1)\qquad \mathcal{M}_{\Omega}(f)(x)=\left(\int_0^{\infty}\left|\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{d-1}}f(y)\mathrm{d}y\right|^2\frac{\mathrm{d}t}{t^3}\right)^{\frac{1}{2}},\quad x\in\mathbb{R}^d.$$

And obtained some results for the $\mathcal{M}_{\Omega}$ on the Lebesgue space, if $\Omega\in \mathrm{Lip}_{\alpha}(\mathbb{S}^{d-1})$ for some $\alpha\in(0,1]$. Since then, many people have paid much attention to study the properties of the higher-dimensional Marcinkiewicz integral in various of function spaces. For example, Fan and Sato in [12] proved that if

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$\Omega \in L\log L(\mathbb{S}^{d-1})$, then $\mathcal{M}_\Omega$ is bounded from Lebesgue space $L^1(\mathbb{R}^d)$ into weak Lebesgue space $L^{1,\infty}(\mathbb{R}^d)$. In 2002, Al-Salman et al. [1] got the $L^p(\mathbb{R}^d)$-boundedness of the $\mathcal{M}_\Omega$ for $p \in (1,\infty)$ if $\Omega \in L(\log L)^{\frac{1}{2}}(\mathbb{S}^{d-1})$. The more development and properties about the operator $\mathcal{M}_\Omega$, we refer the readers to see [10,13,21,28,29,31] and the references therein. On the other hand, Calderón [2] first introduced the definition of commutator

$$[b,T]f := bT(f) - T(bf),$$

and proved that the commutator $[b,H]$ which was generated by the Hilbert transform H and the $b \in \mathrm{BMO}(\mathbb{R}^d)$ is bounded on $L^2(\mathbb{R}^d)$. In 1976, Coifman, Rochberg and Weiss got that the commutator, which was generated by the Calderón-Zygmund operator T and the $\mathrm{BMO}(\mathbb{R}^d)$ function, is bounded on $L^p(\mathbb{R}^d)$ for $p \in (1,\infty)$ when $b \in \mathrm{BMO}(\mathbb{R}^d)$ (see [5]). After that, the properties of the commutators generated by the different operators and the functions b have been widely focused. For example, in 1990, Torchinsky and Wang [26] obtained the boundedness of the commutator which was generated by the Marcinkiewicz integral $\mathcal{M}_\Omega$ and the $\mathrm{BMO}(\mathbb{R}^d)$ function on $L^p(\mathbb{R}^d)$ for $p \in (1,\infty)$ when $\Omega \in \mathrm{Lip}_\alpha(\mathbb{S}^{d-1})$ for some $\alpha \in (0,1)$. The further research about the commutator on different spaces, we refer the readers to see [3, 4, 16, 19, 27, 32] and so on.

However, in 1960, Hörmander [15] first introduced the parameter Littlewood-Paley operator, and proved that it is bounded on the Lebesgue space. Since then, the boundedness of the parameter Marcinkiewicz integral and its corresponding commutator on different kinds of the function spaces have been obtained. Such as Sakamoto and Yabuta in [24] proved that the parameter Littlewood-Paley operators and their commutators are bounded on $L^p(\mathbb{R}^d)$ for $p \in (1,\infty)$. In addition, in 2006, Ding et al. [11] obtained the existence and boundedness for the parameterized Marcinkiewicz integral with rough kernel on Campanato spaces. Recently, the research about the parameter Marcinkiewicz integral and its commutator on different function spaces is still widely studied (see [18, 20, 23]).

Before giving out the organization of the paper, we need to recall some necessary definitions and notions. The following definition of the parameter Marcinkiewicz integral is from [15].

Definition 1.1. Let $\mathbb{S}^{d-1} := \{x \in \mathbb{R}^d : |x| = 1\}(d \geq 2)$ be the unit sphere in $\mathbb{R}^d$ equipped with the Lebesgue measure $\mathrm{d}\sigma$. Suppose that Ω satisfies the following three conditions.

(i) Ω is the homogeneous function of degree zero on $\mathbb{R}^d \setminus \{0\}$, namely,

$$\Omega(\lambda x) = \Omega(x) \quad \text{for any } \lambda > 0,\ x \in \mathbb{R}^d \setminus \{0\}. \tag{2}$$

(ii) Ω has mean zero on $\mathbb{S}^{d-1}$, that is,

$$\int_{\mathbb{S}^{d-1}} \Omega(x')\mathrm{d}\sigma(x') = 0. \tag{3}$$

(iii) $\Omega \in \mathrm{Lip}(\mathbb{S}^{d-1})$, namely, for any $x', y' \in \mathbb{S}^{d-1}$, there exists a constant $C > 0$, such that

$$|\Omega(x') - \Omega(y')| \le C|x' - y'|. \tag{4}$$

Then, for any $x \in \mathbb{R}^d$ and $\rho \in (0, \infty)$, the parameter Marcinkiewicz integral $\mathcal{M}^{\rho}_{\Omega}(f)$ is defined by

$$\mathcal{M}^{\rho}_{\Omega}(f)(x) = \left(\int_0^{\infty} \left| \frac{1}{t^{\rho}} \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{d-\rho}} f(y) \mathrm{d}y \right|^2 \frac{\mathrm{d}t}{t} \right)^{\frac{1}{2}}. \tag{5}$$

Remark 1.2. If we take $\rho = 1$ as in (5), then the parameter Marcinkiewicz integral as in (5) is just the classical Marcinkiewicz integral operator as in (1).

Given a function $b \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. The commutator $\mathcal{M}^{\rho}_{\Omega,b}$ generated by the function b and the parameter Marcinkiewicz integral $\mathcal{M}^{\rho}$ is defined by, respectively,

$$\mathcal{M}^{\rho}_{\Omega,b} f(x) = \left(\int_0^{\infty} \left| \frac{1}{t^{\rho}} \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{d-\rho}} (b(x) - b(y)) \right|^2 \frac{\mathrm{d}t}{t} \right)^{\frac{1}{2}}. \tag{6}$$

We now recall the definition of the Young function from [14].

Definition 1.3. A function $\Phi : [0, \infty) \to [0, \infty)$ is called a Young function if Φ is convex, left-continuous, $\lim_{r \to 0^+} \Phi(r) = \Phi(0) = 0$ and $\lim_{r\to\infty} \Phi(r) = \infty$.

Remark 1.4. (I) From the convexity and $\Phi(0) = 0$ in Definition 1.3, it is not difficult to obtain that the Young function is increasing. Moreover, if there exists $s \in (0, \infty)$ such that $\Phi(s) = \infty$, then $\Phi(r) = \infty$ for $r \ge s$.

(II) The set of Young functions satisfying the following condition

$$0 < \Phi(r) < \infty \qquad \text{for} \quad 0 < r < \infty$$

will be denoted by $\mathcal{Y}$. If $\Phi \in \mathcal{Y}$, then Φ is absolutely continuous on every closed interval in $[0, \infty)$ and bijective from $[0, \infty)$ to itself.

(III) For a Young function Φ and $0 \le s \le \infty$, set

$$\Phi^{-1}(s) = \inf\{r \ge 0 : \Phi(r) \ge s\}.$$

If $\Phi \in \mathcal{Y}$, then Φ^{-1} is the inverse function of Φ. Furthermore, we note that

$$\Phi(\Phi^{-1}(r)) \le r \le \Phi^{-1}(\Phi(r)) \qquad \text{for} \quad 0 \le r < \infty.$$

(IV) Define $\widetilde{\Phi}(r)$ by

$$\widetilde{\Phi}(r) = \begin{cases} \sup\{rs - \Phi(s) : s \in [0, \infty)\}, & r \in [0, \infty) \\ \infty, & r = \infty, \end{cases}$$

it is not difficult to see that $r \le \Phi^{-1}(r)\widetilde{\Phi}^{-1}(r) \le 2r$ holds for $r \ge 0$.

A Young function Φ is said to satisfy the Δ_2-condition, if there exists some $k > 1$, such that, for all $r > 0$,

$$\Phi(2r) \leq k\Phi(r). \tag{7}$$

Moreover, if Φ satisfies the Δ_2-condition, then $\Phi \in \mathcal{Y}$. A Young function Φ is said to belong to the ∇_2-condition, if there exists some $k > 1$, such that, for all $r \geq 0$,

$$\Phi(r) \leq \frac{1}{2k}\Phi(kr). \tag{8}$$

The following Orlicz space $L^{\Phi}(\mathbb{R}^d)$ is from [14].

Definition 1.5. Let Φ be a Young function, the Orlicz space $L^{\Phi}(\mathbb{R}^d)$ is defined by

$$L^{\Phi}(\mathbb{R}^d) := \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^d) : \int_{\mathbb{R}^d} \Phi(k|f(x)|)\mathrm{d}x < \infty \ \text{ for some } \ k > 0 \right\}. \tag{9}$$

Also, the space $L^{\Phi}_{\mathrm{loc}}(\mathbb{R}^d)$ is defined as the set of all functions f such that $f\chi_B \in L^{\Phi}(\mathbb{R}^d)$ for all balls $B \subset \mathbb{R}^d$.

Moreover, the Orlicz space $L^{\Phi}(\mathbb{R}^d)$ is a Banach space with respect to the norm

$$\|f\|_{L^{\Phi}(\mathbb{R}^n)} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^d} \Phi\left(\frac{|f(x)|}{\lambda}\right)\mathrm{d}x < 1 \right\}.$$

Notice that

$$\int_{\mathbb{R}^d} \Phi\left(\frac{|f(x)|}{\lambda}\right)\mathrm{d}x < 1. \tag{10}$$

Remark 1.6. If we take $\Phi(r) = r^p$ in (9) with $1 \leq p < \infty$, then the Orlicz space $L^{\Phi}(\mathbb{R}^d)$ is just the Lebesgue space $L^p(\mathbb{R}^d)$. Moreover, if we take $\Phi(r) = 0$ for $0 \leq r \leq 1$, and $\Phi(r) = \infty$ with $r > 1$, then $L^{\Phi}(\mathbb{R}^d) = L^{\infty}(\mathbb{R}^d)$.

Now we recall the definition of the generalized Orlicz-Morrey space $L^{\Phi,\varphi}(\mathbb{R}^d)$ from [14].

Definition 1.7. Let $\Phi(r)$ be a positive measurable function on $(0, \infty)$ and Φ be any Young function. The generalized Orlicz-Morrey space $L^{\Phi,\varphi}(\mathbb{R}^d)$ is defined by

$$L^{\Phi,\varphi}(\mathbb{R}^d) = \{f \in L^{\Phi}_{\mathrm{loc}}(\mathbb{R}^d) : \ \|f\|_{L^{\Phi,\varphi}(\mathbb{R}^d)} < \infty\},$$

where

$$\|f\|_{L^{\Phi,\varphi}(\mathbb{R}^d)} := \sup_{x \in \mathbb{R}^d, \ r > 0} [\varphi(r)]^{-1}\Phi^{-1}(r^{-n})\|f\|_{L^{\Phi}(B(x,r))}. \tag{11}$$

Definition 1.8. A function $\varphi : \ (0, \infty) \to (0, \infty)$ is said to be almost decreasing (respectively, almost decreasing) if there exists a positive constant C such that for any $r \leq s$,

$$\varphi(r) \leq C\varphi(s) \qquad (\text{respectively, } \ \varphi(r) \geq C\varphi(s)).$$

For a Young function Φ, $\mathcal{G}_\Phi$ is denoted by the set of all almost decreasing functions $\varphi:\ (0,\infty)\to(0,\infty)$ such that the following mapping

$$t\mapsto \frac{1}{\Phi^{-1}(t^{-n})}\varphi(t),\qquad t\in(0,\infty),$$

is almost increasing.

The organization of the paper is sated as follows. In Section 2, we will recall some known results to prove the main theorems in the latter of the paper. Deringoz et al. in [6] gave out the characterizations for the Riesz potential and its commutator on the generalized Orlicz-Morrey space in the sense of Spanne and Adams. One year later, Deringoz et al. [7] obtained a characterization for strong (or weak) Adams-type boundedness of the fractional maximal operator on generalized Orlicz-Morrey spaces. In 2018, Guliyev and Deringoz [14] established some characterizations of the Lipschitz spaces via the boundedness of commutators associated with the fractional maximal operator, Calderón-Zygmund operator and Riesz potential on the generalized Orlicz-Morrey space. Motivated by these, in Section 3, we will mainly establish the necessary and sufficient conditions for the commutator of the parameter Marcinkiewicz on the generalized Orlicz-Morrey space. In Section 4, we try to get the necessary condition for the parameter Marcinkiewicz integral on the generalized Orlicz-Morrey space, meanwhile, the necessary condition for the commutator generated by the parameter Marcinkiewicz integral and $b\in \mathrm{BMO}(\mathbb{R}^d)$ on the generalized Orlicz-Morrey space is obtained, too.

Throughout the whole paper, C represents a positive constant which is independent of the main parameters. For any subset E of $\mathcal{X}$, we use χ_E to denote its characteristic function. Given any $q\in(1,\infty)$, let $q':=q/(q-1)$ denote its conjugate index.

2. Preliminaries

To prove the main theorems in the latter of the paper, we need to recall some necessary theorems and lemmas. The following results for the fractional integral and its commutator are given in [6].

Theorem 2.1 ([6], Adams type result). *Let $0<\alpha<d$, $\Phi\in\mathcal{Y}$, $\gamma\in(0,1)$ and $\eta(t)=[\varphi(t)]^\gamma$ and $\Psi(t)=\Phi(t^{1/\gamma})$.*

1. *If $\Phi\in\nabla_2$ and $\Phi(t)$ satisfies*

$$\sup_{r<t<\infty}\Phi^{-1}(t^{-n})\operatorname*{ess\,inf}_{t<s<\infty}\frac{\phi(s)}{\Phi^{-1}(s^{-n})}\le C\varphi(r),\tag{12}$$

then the condition

$$t^\alpha\varphi(t)+\int_t^\infty r^\alpha\varphi(r)\frac{\mathrm{d}r}{r}\le C[\varphi(t)]^\gamma$$

for all $t>0$, where $C>0$ does not rely on t, is sufficient for the boundedness of I_α form $L^{\Phi,\varphi}(\mathbb{R}^d)$ to $L^{\Psi,\eta}(\mathbb{R}^d)$. Here and in what follows, I_α represents the *fractional integral operator, its definition is as follows*

$$I_\alpha f(x) = \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-\alpha}} \mathrm{d}y, \quad x \in \mathbb{R}^d. \tag{13}$$

2. *If $\varphi \in \mathcal{G}_\Phi$, then the condition*

$$t^\alpha \varphi(t) \le C[\varphi(t)]^\gamma \tag{14}$$

for all $t > 0$, where $C > 0$ does not depend on t, is necessary for the boundedness of I_α as in (13) *from $L^{\Phi,\varphi}(\mathbb{R}^d)$ into $L^{\Psi,\eta}(\mathbb{R}^d)$.*

3. *Let $\Phi \in \nabla_2$. If $\varphi \in \mathcal{G}_\Phi$ satisfying the regularity condition*

$$\int_t^\infty r^\alpha \varphi(r) \frac{\mathrm{d}r}{r} \le C t^\alpha \varphi(t)$$

for all $t > 0$, where $C > 0$ does not depend on t, then the condition (14) *is necessary and sufficient for the boundedness of I_α from $L^{\Phi,\varphi}(\mathbb{R}^d)$ into $L^{\Psi,\eta}(\mathbb{R}^d)$.*

Theorem 2.2 ([6], Spanne type result)**.** *Let $\Phi, \Psi \in \mathcal{Y}$ and $0 < \alpha < d$.*

1. *Let $\Phi \in \Delta_2$. If the functions (Φ, Ψ) satisfy the following condition*

$$r^\alpha \Phi^{-1}(r^{-n}) + \int_r^\infty \Phi^{-1}(t^{-n}) t^\alpha \frac{\mathrm{d}t}{t} \le C \Psi^{-1}(r^{-n}), \tag{15}$$

then the condition

$$\int_t^\infty \operatorname*{ess\,inf}_{r<s<\infty} \frac{\varphi_1(s)}{\Phi^{-1}(s^{-n})} \Psi^{-1}(r^{-n}) \frac{\mathrm{d}r}{r} \le C \varphi_2(t) \tag{16}$$

for all $t > 0$, where $C > 0$ does not depend on t, is sufficient for the boundedness of I_α from $L^{\Phi,\varphi_1}(\mathbb{R}^d)$ into $L^{\Psi,\varphi_2}(\mathbb{R}^d)$.

2. *If the function $\varphi_1 \in \mathcal{G}_\Phi$, then the condition*

$$t^\alpha \varphi_1(t) \le C \varphi_2(t) \tag{17}$$

for all $t > 0$, where $C > 0$ does not depend on t, is sufficient for the boundedness of I_α from $L^{\Phi,\varphi_1}(\mathbb{R}^d)$ into $L^{\Psi,\varphi_2}(\mathbb{R}^d)$.

3. *Let $\Phi \in \Delta_2$. Let also the functions (Φ, Ψ) satisfy the condition* (15)*. If $\varphi_1 \in \mathcal{G}_\Phi$ satisfies the regularity type condition*

$$\int_t^\infty \frac{\Psi^{-1}(r^{-n})}{\Phi^{-1}(r^{-n})} \varphi_1(r) \frac{\mathrm{d}r}{r} \le C t^\alpha \varphi_1(t)$$

for all $t > 0$, where $C > 0$ does not depend on $t > 0$ does not depend on t, then the condition (17) *is necessary and sufficient for the boundedness of I_α from $L^{\Phi,\varphi_1}(\mathbb{R}^d)$ into $L^{\Psi,\varphi_2}(\mathbb{R}^d)$.*

The following lemma is from [8].

Lemma 2.3 ([8])**.** *Let Φ be a Young function and B be a ball in $\mathbb{R}^d$. Then the following inequality is valid*

$$\|f\|_{L^1(B)} \le 2|B|\Phi^{-1}(|B|^{-1}) \|f\|_{L^\Phi(B)}, \tag{18}$$

where $\|f\|_{L^\Phi(B)} = \|f\chi_B\|_{L^\Phi}$.

Finally, we recall the following lemma from [6].

Lemma 2.4 ([6])**.** *Let B be a ball in $\mathbb{R}^d$. If $\varphi \in \mathcal{G}_\Phi$, then there exists a constant $C > 0$ such that*

$$\frac{1}{\varphi(r_B)} \leq \|\chi_B\|_{L^{\Phi,\varphi}(\mathbb{R}^d)} \leq \frac{C}{\varphi(r_B)}, \tag{19}$$

here r_B represents the radius of the ball B.

3. Lipschitz estimate for parameter Marcinkiewicz integral

In this section, we will study some characterizations for the commutator M_b^ρ generated by the parameter Marcinkiewicz integral $\mathcal{M}^\rho$ and the $b \in \dot{\Lambda}_\beta(\mathbb{R}^d)$ in the Adams type (or Spanne type) on the generalized Orlicz-Morrey space $L^{\Phi,\varphi}$. First, we recall the definition of the Lipschitz space [30].

Definition 3.1. Let $0 < \beta < 1$. A function b is said to belong to the Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R}^d)$ if, there exists a constant $C > 0$ such that for all $x,\ y \in \mathbb{R}^d$,

$$|b(x) - b(y)| \leq C|x-y|^\beta. \tag{20}$$

The smallest such constant C is called the $\dot{\Lambda}_\beta(\mathbb{R}^d)$ norm of b and is denoted by $\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^d)}$.

Given a function $b \in \dot{\Lambda}_\beta(\mathbb{R}^d)$. The commutator $\mathcal{M}^\rho_{\Omega,b}$ associated with the parameter Marcinkiewicz integral as in (4) is defined by

$$\mathcal{M}^\rho_{\Omega,b}(f)(x) = \left(\int_0^\infty \left|\frac{1}{t^\rho}\int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{d-\rho}} f(y)\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t}\right)^{\frac{1}{2}}. \tag{21}$$

Now we recall the following characterization of the Lipschitz space from [9].

Lemma 3.2 ([9])**.** *Let $0 < \beta < 1$, we have*

$$\|f\|_{\dot{\Lambda}_\beta(\mathbb{R}^d)} \approx \sup_B \frac{1}{|B|^{1+\frac{\beta}{d}}} \int_B |f(x) - f_B|\mathrm{d}x, \tag{22}$$

where f_B represents the mean value of the function f on ball B, that is,

$$f_B := \frac{1}{|B|}\int_B f(y)\mathrm{d}y.$$

The main theorems of this section are stated as follows.

Theorem 3.3 (Adams type result)**.** *Let $0 < \beta,\ \gamma < 1$, $\Phi \in \mathcal{Y}$, $b \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, $\eta(t) = [\varphi(t)]^\gamma$ and $\Psi(t) = \Phi(t^{1/\gamma})$.*

(1) *If $\Omega \in \mathrm{Lip}(\mathbb{S}^{d-1})$, $\Phi \in \nabla_2$ and $\Phi(t)$ satisfies* (12)*,*

$$\int_t^\infty r^\beta \varphi(r)\frac{\mathrm{d}r}{r} \leq Ct^\beta\varphi(t) \tag{23}$$

and

$$t^\beta \varphi(t) \leq C[\varphi(t)]^\gamma \tag{24}$$

for all $t > 0$, where $C > 0$ does not rely on t, then the condition $b \in \dot{\Lambda}_\beta(\mathbb{R}^d)$ is sufficient for the boundedness of the commutator $\mathcal{M}^\rho_{\Omega,b}$ form $L^{\Phi,\varphi}(\mathbb{R}^d)$ to $L^{\Psi,\eta}(\mathbb{R}^d)$.

(2) *If $\varphi \in \mathcal{G}_\Phi$, Ω satisfy* (2), (3) *and*

$$|\Omega(x') - \Omega(y')| \leq \frac{C}{(\log(2/|x'-y'|))^\tau},\ \tau > 1,\ x',\ y' \in \mathbb{S}^{d-1}, \tag{25}$$

and the condition

$$[\varphi(t)]^\gamma \leq Ct^\beta \varphi(t) \tag{26}$$

for all $t > 0$, where $C > 0$ does not depend on t, then the condition $b \in \dot{\Lambda}_\beta(\mathbb{R}^d)$ is necessary for the boundedness of the commutator $\mathcal{M}^\rho_{\Omega,b}$ from $L^{\Phi,\varphi}(\mathbb{R}^d)$ into $L^{\Psi,\eta}(\mathbb{R}^d)$.

(3) *If $\Phi \in \nabla_2$, $\varphi \in \mathcal{G}_\Phi$, condition* (23) *holds and $[\varphi(t)]^\gamma \approx t^\beta \varphi(t)$, then the condition* (14) *is necessary and sufficient for the boundedness of the commutator $\mathcal{M}^\rho_{\Omega,b}$ from $L^{\Phi,\varphi}(\mathbb{R}^d)$ into $L^{\Psi,\eta}(\mathbb{R}^d)$.*

Theorem 3.4 (Spanne type result)**.** *Let $0 < \beta < 1$, $b \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and $\Phi, \Psi \in \mathcal{Y}$*

1. *Let $\Phi \in \Delta_2$,* (16) *holds and the functions (Φ, Ψ) satisfy the following condition*

$$r^\alpha \Phi^{-1}(r^{-n}) + \int_r^\infty \Phi^{-1}(t^{-n}) t^\alpha \frac{\mathrm{d}t}{t} \leq C\Psi^{-1}(r^{-n}) \tag{27}$$

for all $r > 0$, where $C > 0$ does not depend on r. Then the condition $b \in \dot{\Lambda}_\beta(\mathbb{R}^d)$ is sufficient for the boundedness of the commutator $\mathcal{M}^\rho_{\Omega,b}$ from $L^{\Phi,\varphi_1}(\mathbb{R}^d)$ into $L^{\Psi,\varphi_2}(\mathbb{R}^d)$.

2. *If the function $\varphi_1 \in \mathcal{G}_\Phi$ and the condition*

$$\varphi_2(t) \leq Ct^\beta \varphi_1(t) \tag{28}$$

for all $t > 0$, where $C > 0$ does not depend on t, then the condition $b \in \dot{\Lambda}_\beta(\mathbb{R}^d)$ is sufficient for the boundedness of the commutator $\mathcal{M}^\rho_{\Omega,b}$ from $L^{\Phi,\varphi_1}(\mathbb{R}^d)$ into $L^{\Psi,\varphi_2}(\mathbb{R}^d)$.

3. *Let $\Phi \in \Delta_2$, the condition* (27) *holds and $\varphi_2(t) \approx t^\beta \varphi_1(t)$. If $\varphi_1 \in \mathcal{G}_\Phi$ satisfies the regularity type condition*

$$\int_t^\infty \frac{\Psi^{-1}(r^{-n})}{\Phi^{-1}(r^{-n})} \varphi_1(r) \frac{\mathrm{d}r}{r} \leq Ct^\beta \varphi_1(t)$$

for all $t > 0$, where $C > 0$ does not depend on $t > 0$ does not depend on t, then the condition $b \in \dot{\Lambda}_\beta(\mathbb{R}^d)$ is necessary and sufficient for the boundedness of the commutator $\mathcal{M}^\rho_{\Omega,b}$ from $L^{\Phi,\varphi_1}(\mathbb{R}^d)$ into $L^{\Psi,\varphi_2}(\mathbb{R}^d)$.

Now we give out the proof of Theorems 3.3 and 3.4 as follows.

Proof of Theorem 3.3. (1) For any $b \in \dot{\Lambda}_\beta(\mathbb{R}^d)$ and $f \in L^{\Phi,\varphi}(\mathbb{R}^d)$, by applying (2), (3), (20) and Minkowski inequality, we can get

$$\begin{aligned}
\mathcal{M}^{\rho}_{\Omega,b}(f)(x) &\leq \int_{\mathbb{R}^d} \frac{|\Omega(x-y)|}{|x-y|^{d-\rho}}|b(x)-b(y)||f(y)|\bigg(\int_{|x-y|}^{\infty} \frac{\mathrm{d}t}{t^{1+2\rho}}\bigg)^{\frac{1}{2}}\mathrm{d}y \\
&\leq \int_{\mathbb{R}^d} \frac{|\Omega((x-y)')|}{|x-y|^{d-\rho}}|b(x)-b(y)||f(y)|\bigg(\int_{|x-y|}^{\infty} \frac{\mathrm{d}t}{t^{1+2\rho}}\bigg)^{\frac{1}{2}}\mathrm{d}y \\
&\leq C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^d)} \int_{\mathbb{R}^d} \frac{|f(y)|}{|x-y|^{d-\rho}} \frac{|x-y|^{\beta}}{|x-y|^{\rho}}\mathrm{d}y \\
&\leq C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^d)} I_\beta(|f|)(x),
\end{aligned}$$

where I_β represents the fractional integral operator as in (13). Furthermore, combining the result of Theorem 2.1, we complete the proof of first statement.

(2) Now we turn to prove the second part. Suppose that (26) and $\mathcal{M}^{\rho}_{\Omega,b}$ is bounded from $L^{\Phi,\varphi}(\mathbb{R}^d)$ into $L^{\Psi,\eta}(\mathbb{R}^d)$. We want to prove that, for any ball $B = B(x_0, r)$ with $x_0 \in \mathbb{R}^d$ and $r \in (0, \infty)$, the following inequality

$$\frac{1}{|B(x_0,r)|^{1+\frac{\beta}{d}}} \int_{B(x_0,r)} |b(y) - c_0|\mathrm{d}y \leq C \tag{29}$$

holds, here and in what follows, $c_0 = \frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} b(y)\mathrm{d}y$.

We will use the same idea introduced by Janson in [17]. Choose $z_0 \in \mathbb{R}^d$ and $\delta > 0$ such that in the neighborhood $\{z : \ |z - z_0| < \sqrt{d}\delta\}$, function $|z|^{d-\beta}$ can be represented as a Fourier series with absolute convergence, namely, its expression is as follows

$$|z|^{d-\beta} = \sum_{k=0}^{\infty} a_k e^{i\nu_k \cdot z}.$$

Let $z_1 = \frac{z_0}{\delta}$, $y_0 = x_0 - 2rz_1$ and $\widetilde{B} = B(y_0, r)$. Then, for any $x \in B$ and $y \in \widetilde{B}$, we can get

$$\left|\frac{x-y}{2r} - z_1\right| = \left|\frac{x-y}{2r} - \frac{x_0-y_0}{2r}\right| \leq \left|\frac{x-x_0}{2r}\right| + \left|\frac{y-y_0}{2r}\right| \leq 1.$$

Let $f(y) = [\mathrm{sgn}(b(y) - c_0)]\chi_{B(x_0,r)}(y)$. With an argument similar to that used in the proof of Theorem 3.4 in [14], we have

$$\int_B |b(x) - b_{B'}|\mathrm{d}x \approx r^{-\beta} \sum_{k=0}^{\infty} a_k \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{b(x)-b(y)}{|x-y|^{d-\beta}} e^{i\nu_k\cdot\frac{\delta}{2r}(x-y)} f(x)\chi_{B'}(y)\mathrm{d}y\mathrm{d}x.$$

Further, taking $g_k = e^{-i(\delta/2r)\nu_k\cdot y}\chi_{B'}(y)$, and $h_k(x) = e^{-i(\delta/2r)\nu_k\cdot x}f(x)$. Thus, by applying the Minkowski inequality, Hölder inequality and (21), we can deduce that

$$\int_B |b(x) - b_{B'}|$$

$$
\begin{aligned}
&\approx r^{-\beta}\sum_{k=0}^{\infty}a_k\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{b(x)-b(y)}{|x-y|^{d-\beta}}g_k(y)h_k(x)\mathrm{d}y\mathrm{d}x\\
&\le Cr^{-\beta}\sum_{k=0}^{\infty}|a_k|\int_B\left|\int_{\mathbb{R}^d}\frac{b(x)-b(y)}{|x-y|^{d-\beta}}g_k(y)\mathrm{d}y\right|\mathrm{d}x\\
&\le Cr^{-\beta}\sum_{k=0}^{\infty}|a_k|\int_B\frac{1}{|x-y_0|^{\rho-\beta}}\left|\int_{\mathbb{R}^d}\frac{b(x)-b(y)}{|x-y|^{d-\rho}}g_k(y)\mathrm{d}y\right|\mathrm{d}x\\
&\le Cr^{-\beta}\frac{(\log(2/A))^{\tau}}{(\log(2/A))^{\tau}}\sum_{k=0}^{\infty}|a_k|\int_B\frac{1}{|x-y_0|^{\rho-\beta}}\left|\int_{\mathbb{R}^d}\frac{b(x)-b(y)}{|x-y|^{d-\rho}}g_k(y)\mathrm{d}y\right|\mathrm{d}x\\
&\le Cr^{-\beta}\sum_{k=0}^{\infty}|a_k|\int_B\frac{1}{|x-y_0|^{\rho-\beta}}\left|\frac{1}{(\log(2/A))^{\tau}}\int_{\mathbb{R}^d}\frac{b(x)-b(y)}{|x-y|^{d-\rho}}g_k(y)\mathrm{d}y\right|\mathrm{d}x\\
&\le Cr^{-\beta}\sum_{k=0}^{\infty}|a_k|\int_B\frac{1}{|x-y_0|^{\rho-\beta}}\left|\int_{\mathbb{R}^d}\frac{\Omega((x-y)')}{|x-y|^{d-\rho}}(b(x)-b(y))g_k(y)\mathrm{d}y\right|\mathrm{d}x\\
&\le Cr^{-\beta}\sum_{k=0}^{\infty}|a_k|\int_B|x-y_0|^{\beta}\left|\int_{\mathbb{R}^d}\frac{\Omega(x-y)}{|x-y|^{d-\rho}}(b(x)-b(y))g_k(y)\mathrm{d}y\right|\\
&\quad\times\left(\int_{\substack{|x-y_0|\le t\\|x-y|\le t}}^{\infty}\frac{\mathrm{d}t}{t^{1+2\rho}}\right)\left(\int_{\substack{|x-y_0|\le t\\|x-y|\le t}}^{\infty}\frac{\mathrm{d}t}{t^{1+2\rho}}\right)^{-\frac{1}{2}}\mathrm{d}x\\
&\le Cr^{-\beta}\sum_{k=0}^{\infty}|a_k|\int_B|x-y_0|^{\beta}\left(\int_{|x-y_0|}^{\infty}\left|\int_{\mathbb{R}^d}\frac{\Omega(x-y)}{|x-y|^{d-\rho}}(b(x)-b(y))g_k(y)\right.\right.\\
&\quad\times\left.\left.\chi_{\{|x-y|\le t\}}(y)\mathrm{d}y\right|\frac{\mathrm{d}t}{t^{1+2\rho}}\right)\mathrm{d}x\left(\int_{\substack{|x-y_0|\le t\\|x-y|\le t}}^{\infty}\frac{\mathrm{d}t}{t^{1+2\rho}}\right)^{-\frac{1}{2}}\\
&\le Cr^{-\beta}\sum_{k=0}^{\infty}|a_k|\int_B|x-y_0|^{\beta}\left(\int_{|x-y_0|}^{\infty}\left|\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{d-\rho}}\right.\right.\\
&\quad\times\left.\left.(b(x)-b(y))g_k(y)\mathrm{d}y\right|^2\frac{\mathrm{d}t}{t^{1+2\rho}}\right)^{\frac{1}{2}}\mathrm{d}x\\
&\le Cr^{-\beta}\sum_{k=0}^{\infty}|a_k|\int_B|x-y_0|^{\beta}\left(\int_{0}^{\infty}\left|\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{d-\rho}}\right.\right.\\
&\quad\times\left.\left.(b(x)-b(y))g_k(y)\mathrm{d}y\right|^2\frac{\mathrm{d}t}{t^{1+2\rho}}\right)^{\frac{1}{2}}\mathrm{d}x\\
&\le Cr^{-\beta}\sum_{k=0}^{\infty}|a_k|\int_B|x-y_0|^{\beta}\mathcal{M}_{\Omega}^{\rho}(g_k)(x)\mathrm{d}x,
\end{aligned}
$$

where we have used the fact, namely, since Ω is homogeneous function of degree zero and has mean zero, and satisfies (25), then there exists a positive constant A with $0 < A < 1$, for $x,\ y \in \mathbb{R}^d$ with $x \neq y$,

$$\Omega(x-y) = \Omega\bigg(\frac{x-y}{|x-y|}\bigg) = \Omega((x-y)') \geq \frac{C}{(\log(2/A))^{\tau}}.$$

Further, to estimate the above inequality, we need to consider the following cases.

Case I. If $y_0 \in B = B(x_0, r)$. Since $x \in B(x_0, r)$, then we have $|x - y_0| \leq r$. By applying Lemma 2.3 and Definition 1.7, we obtain

$$\begin{aligned}\int_B |b(x) - b_{B'}|\mathrm{d}x &\leq Cr^{-\beta} \sum_{k=0}^{\infty} |a_k| \int_B |x-y_0|^{\beta} \mathcal{M}_{\Omega}^{\rho}(g_k)(x)\mathrm{d}x \\ &\leq C \sum_{k=0}^{\infty} |a_k| \int_B |\mathcal{M}_{\Omega}^{\rho}(g_k)(x)|\mathrm{d}x \\ &\leq C \sum_{k=0}^{\infty} |a_k||B|\Psi^{-1}(|B|^{-1})\|\mathcal{M}_{\Omega}^{\rho}(g_k)\|_{L^{\Psi}(B)} \\ &\leq C \sum_{k=0}^{\infty} |a_k||B|\eta(r)\|\mathcal{M}_{\Omega}^{\rho}(g_k)\|_{L^{\Psi,\eta}(\mathbb{R}^d)} \\ &\leq C \sum_{k=0}^{\infty} |a_k||B|[\varphi(r)]^{\gamma}\|g_k\|_{L^{\Phi,\varphi}(\mathbb{R}^d)} \\ &\leq C \sum_{k=0}^{\infty} |a_k||B|r^{\beta}\varphi(r)[\varphi(r)]^{-1} \\ &\leq C \sum_{k=0}^{\infty} |a_k| r^{d+\beta}.\end{aligned}$$

Thus, we get

$$\frac{1}{|B|^{1+\frac{\beta}{d}}} \int_B |b(x) - c_0|\mathrm{d}x \leq \frac{2}{|B|^{1+\frac{\beta}{d}}} \int_B |b(x) - b_{B'}|\mathrm{d}x \leq C \sum_{k=0}^{\infty} |a_k| \leq C.$$

Case II. If $y_0 \notin B = B(x_0, r)$ and $|x_0 - y_0| \leq 2r$. Then, with an argument that used in **Case I**, it is not difficult to obtain that

$$\frac{1}{|B|^{1+\frac{\beta}{d}}} \int_B |b(x) - c_0|\mathrm{d}x \leq \frac{2}{|B|^{1+\frac{\beta}{d}}} \int_B |b(x) - b_{B'}|\mathrm{d}x \leq C \sum_{k=0}^{\infty} |a_k| \leq C.$$

Case III. If $y_0 \notin B = B(x_0, r)$ and $|x_0 - y_0| > 2r$. Since $y_0 = x_0 - 2rz_1$ and $z_1 = \frac{z_0}{\delta}$, then $\frac{|x_0 - y_0|}{2r} = |z_1| = \frac{|z_0|}{\delta} < \sqrt{d}$, that is, $|x_0 - y_0| \leq 2\sqrt{d}r$. Thus, by using Lemma 1.8, Definition 1.9 and Lemma 2.5, then we can deduce that

$$\begin{aligned}\int_B |b(x)-b_{B'}|\mathrm{d}x &\le Cr^{-\beta}\sum_{k=0}^{\infty}|a_k|\int_B |x-y_0|^{\beta}\mathcal{M}_\Omega^{\rho}(g_k)(x)\mathrm{d}x\\ &\le Cr^{-\beta}\sum_{k=0}^{\infty}|a_k||x_0-y_0|^{\beta}\int_B |\mathcal{M}_\Omega^{\rho}(g_k)(x)|\mathrm{d}x\\ &\le C\sum_{k=0}^{\infty}|a_k|\|\mathcal{M}_\Omega^{\rho}(g_k)\|_{L^1(B)}\\ &\le C\sum_{k=0}^{\infty}|a_k||B|\Psi^{-1}(|B|^{-1})\|f\|_{L^{\Psi}(B)}\\ &\le C\sum_{k=0}^{\infty}|a_k||B|\eta(r)\|f\|_{L^{\Psi,\eta}(\mathbb{R}^d)}\\ &\le C\sum_{k=0}^{\infty}|a_k|r^{d+\beta}.\end{aligned}$$

Thus, we have

$$\frac{1}{|B|^{1+\frac{\beta}{d}}}\int_B |b(x)-c_0|\mathrm{d}x \le \frac{2}{|B|^{1+\frac{\beta}{d}}}\int_B |b(x)-b_{B'}|\mathrm{d}x \le C\sum_{k=0}^{\infty}|a_k|\le C.$$

Which, combing the above three cases, (29) is finished. Thus, the second result is finished.

(3) The third result of the theorem follows from the first and second results of the theorem. □

Now we turn to the proof of Theorem 3.4.

Proof of Theorem 3.4. With a sight modified argument similar to that used in the proof of Theorem 3.3 in this paper, we can also obtain Theorem 3.4. Thus, here we omit the process of the proof. □

4. BMO estimate for parameter Marcinkiewicz integral

In this section, we shall give out the proof of the necessary condition for the parameter Marcinkiewicz integral $\mathcal{M}_\Omega^{\rho}$ and the commutator $[b,\mathcal{M}_\Omega^{\rho}]$ generated by the $\mathcal{M}_\Omega^{\rho}$ and BMO($\mathbb{R}^d$) on the generalized Orlicz-Morrey space $L^{\Phi,\varphi}(\mathbb{R}^d)$. However, the sufficient conditions for the parameter Marcinkiewicz integral $\mathcal{M}_\Omega^{\rho}$ and the commutator $[b,\mathcal{M}_\Omega^{\rho}]$ on $L^{\Phi,\varphi}(\mathbb{R}^d)$ is not got in this paper, maybe there exists a certain difficulty. We first need to recall the following definition of the bounded mean oscillation space (=BMO) given in [5].

Definition 4.1. A function $b \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ is said to be in the space $\mathrm{BMO}(\mathbb{R}^d)$ if

$$\|b\|_* := \sup_{B\subset\mathbb{R}^d} \frac{1}{|B|}\int_B |b(x)-b_B|\mathrm{d}x. \tag{30}$$

Given a function $b \in \mathrm{BMO}(\mathbb{R}^d)$. The commutator $[b,\mathcal{M}^\rho_\Omega]$ generated by the parameter Marcinkiewicz integral $\mathcal{M}^\rho_\Omega$ and the $b \in \mathrm{BMO}(\mathbb{R}^d)$ is defined by

(31)

$$[b,\mathcal{M}^\rho_\Omega](f)(x) := \left(\int_0^\infty \left|\frac{1}{t^\rho}\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{d-\rho}}(b(x)-b(y))f(y)\mathrm{d}y\right|^2\frac{\mathrm{d}t}{t}\right)^{\frac{1}{2}}.$$

Now we state the main theorems as follows.

Theorem 4.2. *Let $\Phi \in \mathcal{Y}$, $\gamma \in (0,1)$ and $\eta(t) \equiv [\varphi(t)]^\gamma$ and $\Psi(t) \equiv \Phi(t^{1/\gamma})$. Suppose that Ω satisfies* (2), (3) *and*

$$|\Omega(x')-\Omega(y')| \le \frac{C}{(\log(2/|x'-y'|))^\tau},\ C>0,\ \tau>1, x',\ y' \in \mathbb{S}^{d-1}. \tag{32}$$

If $\varphi \in \mathcal{G}_\varphi$, then the following condition

$$[\varphi(t)]^\gamma \le C\varphi(t) \tag{33}$$

for all $t>0$, where $C>0$ does not depend on t, is necessary for the boundedness of $\mathcal{M}^\rho_\Omega$ from $L^{\Phi,\varphi}(\mathbb{R}^d)$ into $L^{\Psi,\eta}(\mathbb{R}^d)$.

Theorem 4.3. *Let $b \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, $\Phi \in \mathcal{Y}$, $\eta(t) \equiv [\varphi(t)]^\gamma$ and $\Psi(t) = \Phi(t^{\frac{1}{\gamma}})$. Suppose that Ω satisfies* (2), (3) *and* (32). *If $\varphi \in \mathcal{G}_\varphi$ and* (33), *then the condition $b \in \mathrm{BMO}(\mathbb{R}^d)$ is necessary for the boundedness of $[b,\mathcal{M}^\rho_\Omega]$ from $L^{\Phi,\varphi}(\mathbb{R}^d)$ into $L^{\Psi,\eta}(\mathbb{R}^d)$.*

Remark 4.4. Chen et al. in [4] have proved that the necessary condition of the boundedness for the commutator $[b,\mathcal{M}_\Omega]$ on generalized Morrey space $L^{p,\varphi}(\mathbb{R}^d)$ is $b \in \mathrm{BMO}(\mathbb{R}^d)$. Similarly, by applying the idea of the proof, we will prove that the necessary condition of the boundedness of the commutator $[b,\mathcal{M}^\rho_b]$ on generalized Orlicz-Morrey space $L^{\Phi,\varphi}(\mathbb{R}^d)$ is also $b \in \mathrm{BMO}(\mathbb{R}^d)$. Thus, we only give out the different process of Theorem 4.3 in this paper.

Proof of Theorem 4.2. Without loss of generality, we set $f(x) = \chi_{B(x_0,r)}(x)$ and $B := B(x_0,r)$. Since $\mathcal{M}^\rho_\Omega$ is bounded from $L^{\Phi,\varphi}(\mathbb{R}^d)$ into $L^{\Psi,\eta}(\mathbb{R}^d)$, then, by applying Lemma 2.3, write

$$\begin{aligned}\|\mathcal{M}^\rho_\Omega\chi_{B(x_0,r)}\|_{L^{\Psi,\eta}} &\ge [\eta(r)]^{-1}\Psi^{-1}(|B|^{-1})\|\mathcal{M}^\rho_\Omega\chi_{B(x_0,r)}\|_{L^\Psi(B)}\\ &\ge \frac{\|\mathcal{M}^\rho_\Omega\chi_{B(x_0,r)}\|_{L^1(B)}}{2|B|\eta(r)}\\ &= \frac{1}{2|B|\eta(r)}\int_B \mathcal{M}^\rho_\Omega\chi_{B(x_0,r)}(x)\mathrm{d}x.\end{aligned}$$

To estimate the above inequality, we only need to estimate $\mathcal{M}_\Omega^\rho \chi_{B(x_0,r)}(x)$. By applying the Hölder inequality and (32), we can deduce that, for any $x \in B$,

$$\begin{aligned}
\mathcal{M}_\Omega^\rho \chi_{B(x_0,r)}(x) &\geq \bigg(\int_{|x-x_0|}^\infty \bigg| \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{d-\rho}} \chi_{B(x_0,r)}(y)\mathrm{d}y \bigg|^2 \frac{\mathrm{d}t}{t^{1+2\rho}} \bigg)^{\frac{1}{2}} \\
&\geq \bigg(\int_{|x-x_0|}^\infty \bigg| \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{d-\rho}} \chi_{B(x_0,r)}(y)\mathrm{d}y \bigg| \frac{\mathrm{d}t}{t^{1+2\rho}} \bigg) \\
&\quad \times \bigg(\int_{|x-x_0|}^\infty \frac{\mathrm{d}t}{t^{1+2\rho}} \bigg)^{-\frac{1}{2}} \\
&\geq |x-x_0|^\rho \bigg(\int_{|x-x_0|}^\infty \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{d-\rho}} \chi_{B(x_0,r)}(y)\mathrm{d}y \frac{\mathrm{d}t}{t^{1+2\rho}} \bigg) \\
&\geq |x-x_0|^\rho \int_{\mathbb{R}^d} \frac{\Omega(x-y)}{|x-y|^{d-\rho}} \chi_{B(x_0,r)}(y) \bigg(\int_{|x-x_0|}^\infty \frac{\mathrm{d}t}{t^{1+2\rho}} \bigg) \mathrm{d}y \\
&\approx \frac{1}{|x-x_0|^d} \int_{B(x_0,r)} \Omega(x-y)\mathrm{d}y \\
&\geq C \frac{r^d}{|x-x_0|^d},
\end{aligned}$$

where we have use the following fact (see [4])

$$\Omega(x-y) = \Omega\bigg(\frac{x-y}{|x-y|}\bigg) \geq C/(\log(2/A))^\gamma \quad \text{with } 0 < A < 1.$$

Further, by Lemma 2.5, we can get

$$\begin{aligned}
\frac{\widetilde{C}}{\varphi(r)} &\geq \widetilde{C}\|\chi_{B(x_0,r)}\|_{L^{\Phi,\varphi}} \geq \|\mathcal{M}_\Omega^\rho \chi_{B(x_0,r)}\|_{L^{\Psi,\eta}} \\
&\geq \frac{C}{2|B|\eta(r)} \int_B \frac{r^d}{|x-x_0|^d}\mathrm{d}x \\
&\geq \frac{C}{\eta(r)} \approx \frac{C}{[\varphi(r)]^\gamma}.
\end{aligned}$$

Hence we have $[\varphi(t)]^\gamma \leq C\varphi(t)$. So, the proof of Theorem 4.2 is finished. □

Proof of Theorem 4.3. Without loss of generality, we may assume that the $\|[b, \mathcal{M}_\Omega^\rho]\|_{L^{\Psi,\eta} \to L^{\Phi,\varphi}} = 1$. We want to prove that, for any $x_0 \in \mathbb{R}^d$ and $r \in (0,\infty)$, the following inequality

$$H := \frac{1}{|B(x_0,r)|} \int_{B(x_0,r)} |b(y) - c_0|\mathrm{d}y \leq C$$

holds. Since $[b - c_0, \mathcal{M}_\Omega^\rho] = [b, \mathcal{M}_\Omega^\rho]$, we may set that $c_0 = 0$. Let

$$f(y) = [\mathrm{sgn}(b(y)) - a_0]\chi_{B(x_0,r)}, \tag{34}$$

where $a_0 = \frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} \mathrm{sgn}(b(z))\mathrm{d}z$. Because of $\frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} b(y)\mathrm{d}y = c_0 = 0$, it is not difficult to get $|a_0| < 1$. So, f satisfies the following properties

$$|f(y)| \le 2, \quad \mathrm{supp}(f) \subset B(x_0, r), \quad \int_{\mathbb{R}^d} f(y)\mathrm{d}y = 0,$$

and

$$f(y)b(y) > 0, \ y \in B(x_0, r), \quad \frac{1}{|B(x_0,r)|}\int_{\mathbb{R}^d} f(y)b(y)\mathrm{d}y = H.$$

Let $\Lambda := \{x' \in \mathbb{S}^{d-1} : \ \Omega(x') \ge \frac{2C}{(\log(2/A))^\tau}\}$ with $0 < A < 1$. With a slight modified argument similar to that used in the estimates for I_1 and I_2 in Theorem 1.2 of [4]. It is not difficult to get that, for any $x \in G = \{x \in \mathbb{R}^d : |x - x_0| \ge Cr, (x - x_0)' \in \Lambda\}$,

$$\begin{aligned}
[b, \mathcal{M}_\Omega^\rho]f(x) &\ge \bigg(\int_0^\infty \bigg|\frac{1}{t^\rho}\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{d-\rho}}b(y)f(y)\mathrm{d}y\bigg|^2\frac{\mathrm{d}t}{t}\bigg)^{\frac{1}{2}} \\
&\quad - |b(x)|\bigg(\int_0^\infty \bigg|\frac{1}{t^\rho}\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{d-\rho}}f(y)\mathrm{d}y\bigg|^2\frac{\mathrm{d}t}{t}\bigg)^{\frac{1}{2}} \\
&\ge CH\frac{r^d}{|x-x_0|^d} - \widetilde{C}\frac{|b(x)|r^d}{|x-x_0|^d}\bigg(\log\frac{|x-x_0|}{r}\bigg)^{-\tau}. \qquad (35)
\end{aligned}$$

Further, let

$$K = \bigg\{x \in G : \ |b(x)| > CH\bigg(\log\frac{|x-x_0|}{r}\bigg)^{\tau}, \ |x - x_0| < H^{\frac{1}{d}}r\bigg\}. \qquad (36)$$

We may assume that $H > C > 1$, otherwise, we obtain the desired result. By applying Lemma 2.3, Definition 1.7, (35) and (36), we can get

$$\begin{aligned}
C\|f\|_{L^{\Phi,\varphi}} &\ge [\eta(r)]^{-1}\Psi^{-1}(|B|^{-1})\|[b, \mathcal{M}_\Omega^\rho](f)\|_{L^\Psi(B(x_0,r))} \\
&\ge \frac{[\eta(r)]^{-1}\Psi^{-1}(|B|^{-1})}{2|B|\Psi^{-1}(|B|^{-1})}\|[b, \mathcal{M}_\Omega^\rho](f)\|_{L^1(B(x_0,r))} \\
&\ge r^{-d}[\eta(r)]^{-1}\int_{|x-x_0|<r}|[b, \mathcal{M}_\Omega^\rho](f)(x)|\mathrm{d}x \\
&\ge r^{-d}[\eta(r)]^{-1}\int_{(G\backslash K)\bigcap\{|x-x_0|<r\}}|[b, \mathcal{M}_\Omega^\rho](f)(x)|\mathrm{d}x \\
&\ge r^{-d}[\eta(r)]^{-1}\int_{(G\backslash K)\bigcap\{|x-x_0|<r\}}\frac{Hr^d}{|x-x_0|^d}\mathrm{d}x \\
&\ge H[\eta(r)]^{-1}\int_{(G\backslash K)\bigcap\{|x-x_0|<r\}}\frac{1}{|x-x_0|^d}\mathrm{d}x \\
&\ge H[\eta(r)]^{-1}\int_{G\bigcap\{C(|K|+(\widetilde{C}r)^d)^{\frac{1}{d}}<|x-x_0|<r\}}\frac{1}{|x-x_0|^d}\mathrm{d}x \\
&\ge H[\eta(r)]^{-1}\frac{C(|K|+(\widetilde{C}r)^d)}{r^d},
\end{aligned}$$

Thus,

$$H \le \frac{r^d \eta(r)\|f\|_{L^{\Phi,\varphi}}}{C(|K| + (\widetilde{C}r)^d)},$$

Furthermore, since $f(y) = [\mathrm{sgn}(b(y)) - a_0]\chi_{B(x_0,r)}$, by Lemma 2.5 and (33), then we can get

$$\begin{aligned} H &\le \frac{r^d \eta(r)\|f\|_{L^{\Phi,\varphi}}}{C(|K| + (\widetilde{C}r)^d)} \\ &\le C\frac{r^d \eta(r)\|\chi_{B(x_0,r)}\|_{L^{\Phi,\varphi}}}{C(|K| + (\widetilde{C}r)^d)} \\ &\le C\frac{r^d \eta(r)}{C(|K| + (\widetilde{C}r)^d)}\frac{1}{\varphi(r)} \\ &\le C[\varphi(r)]^{\gamma-1}\frac{r^d}{|K| + (\widetilde{C}r)^d} \le C. \end{aligned}$$

Which, the proof of Theorem 4.3 is completed. □

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