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PRIMENESS AND PRIMITIVITY IN NEAR-RINGS

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Abstract. In near-ring theory several different types of primitivity exist. These all imply several different types of primeness. In case of near-rings with DCCN most of the types of primeness are known to imply primitivity of a certain kind. We are able to show that also so called 1-prime near-rings imply 1-primitivity. This enables us to classify maximal ideals in near-rings with chain condition with the concept of 1-primeness which leads to further results in the structure theory of near-rings.

1. Introduction

In what follows, we consider right near-rings, this means the right distributive law holds, but not necessarily the left distributive law. The notation is that of [9].

In the structure theory of near-rings several different concepts of primeness exist which all equal the definition of primeness when it comes that the near-ring under consideration is a ring. A near-ring $N(\neq \{0\})$ is called 0-prime provided that given two ideals I_1 and I_2 of N, $I_1 I_2 = \{0\}$ implies that $I_1 = \{0\}$ or $I_2 = \{0\}$. A near-ring $N(\neq \{0\})$ is called 1-prime if $L_1 L_2 = \{0\}$ implies $L_1 = \{0\}$ or $L_2 = \{0\}$, L_1, L_2 left ideals of N. In case we have $N_1 N_2 = \{0\}$ implies that $N_1 = \{0\}$ or $N_2 = \{0\}$, N_1 and N_2 being N-subgroups of the near-ring N, then N is called 2-prime. Here we follow the notation used in [9], [1], [2] and [4]. Various other types of primeness exist in the literature and care should be taken because there is no consistent notation. If we use the definition of v-prime as we do, then so called v-primitive near-rings, $v \in \{0, 1, 2\}$ are v-prime (see for example [1, Theorem 5.4.3]). The converse is known to hold under the presence of the DCCN if $v \in \{0, 2\}$ (see for example [1, Proposition 5.4.4]). It was posed as an open question in [4] if this is true also if $v = 1$, thus if a 1-prime near-ring with DCCN is 1-primitive. This will be answered to the positive in this paper. This result allows us to characterise maximal ideals in near-rings with chain condition as certain type of prime ideals. From this we

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will derive further results concerning simple near-rings, the Jacobson radicals of near-rings and when different types of primeness are equivalent.

2. Further definitions and notation

In our discussion we have to deal with N-groups of type v, $v \in \{0, 1, 2\}$ of a near-ring and the concept of v-primitivity. Thus, we give a brief overview of these definitions. Let N be a zero symmetric near-ring, this means that $n * 0 = 0$ for all $n \in N$ where $*$ is the near-ring multiplication. Let Γ be an N-group of the near-ring N. An N-ideal I of Γ is a normal subgroup of the group $(\Gamma, +)$ such that $\forall n \in N\ \forall \gamma \in \Gamma\ \forall \delta \in I : n(\gamma + \delta) - n\gamma \in I$. A left ideal L of a near-ring N is an N-ideal of the natural N-group N and in case N is zero symmetric, a left ideal is also an N-group. The left ideal L is an ideal, if $LN \subseteq L$. A non-zero N-group Γ of the near-ring N is of type 0 if there is an element $\gamma \in \Gamma$ such that $N\gamma = \Gamma$, γ is then called a generator of the N-group, and there are no non-trivial N-ideals in Γ. A non-zero N-group Γ is of type 1 if it is of type 0 and N acts strongly monogenic on Γ. N acting strongly monogenic on Γ means that $N\gamma = \Gamma$ or $N\gamma = \{0\}$ for all $\gamma \in \Gamma$.

Let U be a subgroup of the N-group Γ. U is called an N-subgroup of Γ if $NU \subseteq U$. The N-group Γ is called N-group of type 2 if $N\Gamma \neq \{0\}$ and there are no non-trivial N-subgroups in Γ. In case N has an identity element an N-group is of type 1 if and only if it is of type 2 (see [9, Proposition 3.7 and Proposition 3.4]).

Given an N-group Γ and a non-empty subset $S \subseteq \Gamma$ then $(0 : S) = \{n \in N | \forall s \in S : ns = 0\}$ is called the annihilator of S. Such annihilators always are left ideals of the near-ring N. In case S is an N-subgroup of Γ, $(0 : S)$ is an ideal of N. Γ is called faithful if $(0 : \Gamma) = \{0\}$. A near-ring is called v-primitive if it acts on a faithful N-group Γ of type v.

The Jacobson radicals of type v of a near-ring N are defined as the intersection of the annihilators of the N-groups of type v of the near-ring, so $J_v(N) := \bigcap_{\Gamma \text{of type } v}(0 : \Gamma)$. In our discussion we also need $J_{\frac{1}{2}}(N)$, which is the intersection of all 0-modular left ideals (see [9, Definition 3.28]) of N and we have that $J_0(N) \subseteq J_{\frac{1}{2}}(N) \subseteq J_1(N)$.

3. 1-prime implies 1-primitivity

We first prove a lemma on minimal left ideals, which guarantees the existence of certain non-zero ideals in a zero symmetric near-ring. It is understood that minimal always means non-zero. Our focus will be on minimal left ideals L which do not properly contain N-subgroups which are N-isomorphic to L. The next proposition shows that such a situation naturally occurs when studying near-rings with DCCN. To fix a notation, the symbol $\supset$ means a proper subset.

Proposition 3.1. *Let N be a zero symmetric near-ring. Let L be a minimal left ideal such that L satisfies the DCC on N-subgroups contained in L. Suppose*

$M \subseteq L$ is an N-subgroup such that $M \neq L$. Then, L and M cannot be N-isomorphic.

Proof. Suppose to the contrary that M and L are N-isomorphic, by the N-isomorphism f say. So, $M = f(L)$ and $M \neq L$. Consider the restriction of f to $f(L)$, so the map $f_1 : f(L) \longrightarrow f(f(L)), f(l) \mapsto f(f(l))$. It is straightforward to see that f_1 is an N-isomorphism between $f(L)$ and $f(f(L))$. Let $M_1 := f(f(L))$. Since $f(L) = M \subseteq L$, it is clear that $f(f(L)) \subseteq f(L) = M$. Therefore, M_1 is an N-subgroup of M. We now show that $M_1 \neq M$, implying that M_1 is a proper subgroup of M. We have that $M \neq L$, so there exists an element $l_1 \in L$ such that $l_1 \notin f(L) = M$. Assume that $M_1 = M$. Then we have the existence of an element $l_2 \in L$ such that $f(f(l_2)) = f(l_1)$. f being an isomorphism implies that $l_1 = f(l_2) \in f(L)$, contradicting that $l_1 \notin f(L)$. Consequently, $M_1 \neq M$ which means that M contains a proper subgroup M_1 which is N-isomorphic to M. In the same way we see that also M_1 contains a proper subgroup M_2 which is N-isomorphic to M_1 and so on. From that we get an infinite decreasing chain of N-isomorphic N-subgroups $L \supset M \supset M_1 \supset M_2 \supset \cdots$. This violates the DCC on N-subgroups contained in L and therefore we have that M and L are not N-isomorphic. $\square$

In particular, Proposition 3.1 applies to zero symmetric near-rings with DCCN. In a zero symmetric near-ring N a left ideal L always is an N-group. We introduce a notation for the set of non-generators, generators respectively, of this N-group L.

Definition 1. Let L be a left ideal of the zero symmetric near-ring N. Then, $\theta_0^L := \{l \in L \,|\, Nl \neq L\}$ and $\theta_1^L := \{l \in L \,|\, Nl = L\}$.

Lemma 3.2. *Let N be a zero symmetric near-ring and let L be a minimal left ideal. Suppose that L does not contain N-subgroups properly contained in L and being N-isomorphic to L. Then, $L \subseteq (0 : \theta_0^L)$ and $(0 : \theta_0^L)$ is a non-zero ideal of N, containing L.*

Proof. As an annihilator, $(0 : \theta_0^L)$ is a left ideal of N. Let $l \in \theta_0^L$ and $n \in N$. Then $N(nl) \subseteq Nl \neq L$, so $nl \in \theta_0^L$. Let $a \in (0 : \theta_0^L)$, $n \in N$ and $l \in \theta_0^L$. Then $(an)l = a(nl) = 0$ because $nl \in \theta_0^L$. So, we have shown that $(0 : \theta_0^L)$ is an ideal of N.

It remains to show that L is contained in $(0 : \theta_0^L)$ and the ideal thus being non-zero. Let $l \in \theta_0^L$, so $Nl \neq L$. Suppose that $Ll \neq \{0\}$. Then $L \not\subseteq (0 : l)$. $(0 : l) \cap L$ is a left ideal contained in L and so minimality of L implies $(0 : l) \cap L = \{0\}$. Thus, the map $\psi_l : L \to Ll, j \mapsto jl$ is injective. ψ_l clearly is a surjective N-homomorphism between L and Ll and so we have that L and Ll are N-isomorphic N-groups. Since $Ll \subseteq L$, it follows from our assumption that $L = Ll$. Consequently, $L = Nl$ contradicting the fact that $Nl \neq L$. Thus, for all $l \in \theta_0^L$, $Ll = \{0\}$. This shows that $L \subseteq (0 : \theta_0^L)$. $\square$

We are now in a position to prove our main theorem on 1-prime near-rings. We recall from the introduction section:

Definition 2. A zero symmetric near-ring N, $N \neq \{0\}$, is called 1-prime if $L_1L_2 = \{0\}$ implies $L_1 = \{0\}$ or $L_2 = \{0\}$, L_1, L_2 left ideals of N.

Theorem 3.3. *Let $N \neq \{0\}$ be a zero symmetric 1-prime near-ring with DCCL. We further suppose that amongst the minimal left ideals there exists a minimal left ideal L such that L does not contain N-subgroups properly contained in L and being N-isomorphic to L. Then, N is a 1-primitive near-ring.*

Proof. Let L be the minimal left ideal in the near-ring N which does not contain N-subgroups properly contained in L and being N-isomorphic to L. Due to 1-primeness, $L^2 \neq \{0\}$ and there is an element $l \in L$ such that $Ll \neq \{0\}$. By minimality of L as a left ideal, this implies $L \cap (0 : l) = \{0\}$. Consequently, the map $\psi_l : L \to Ll, j \mapsto jl$ is injective and therefore an N-isomorphism between L and Ll. Thus, L and Ll are N-isomorphic and by assumption this implies $L = Ll$. So, we see that L has the generator l.

What is more, L contains an idempotent e which is a right identity in L. To see this, let $e \in L$ such that $el = l$. Such an e exists since $Ll = L$. Thus, $e^2l = el$ and consequently, $(e^2 - e)l = 0$. So, $e^2 - e \in L \cap (0 : l) = \{0\}$ and we see that $e = e^2$. Let $j \in L$. Then, $je = je^2$, so $(j - je)e = 0$. Hence, $j - je \in L \cap (0 : e)$. Since $e \in Le$ by idempotence of e we see that $Le \neq \{0\}$ and so, by minimality of L we have that $L \cap (0 : e) = \{0\}$. Hence, $j = je$ and e is a multiplicative right identity in L. From [9, Theorem 5.37] we know that $J_{\frac{1}{2}}(N)$ is a quasi regular left ideal. From [9, Proposition 3.38] we get that the idempotent e is not a quasiregular element and consequently $e \notin J_{\frac{1}{2}}(N)$. Minimality of the left ideal L now implies $L \cap J_{\frac{1}{2}}(N) = \{0\}$. Since $J_{\frac{1}{2}}(N)$ is the intersection of all the 0-modular left ideals of N, this implies the existence of a 0-modular left ideal L_m such that $L \not\subseteq L_m$. From definition of 0-modularity (see [9, Definition 3.28]) we have that L_m is a maximal left ideal in N and N/L_m is an N-group of type 0. $L \not\subseteq L_m$ thus implies $L_m + L = N$ and on the other hand, $L \cap L_m = \{0\}$ by minimality of L. Consequently, by [9, Theorem 2.8], $N/L_m = (L_m + L)/L_m \cong_N L/L_m \cap L = L/\{0\} \cong_N L$. This proves that L is an N-group of type 0.

Let $(0 : L)$ be the annihilator of L in N. Then, $(0 : L)L = \{0\}$ and 1-primeness of the near-ring implies that $(0 : L) = \{0\}$. This shows that N is acting 0-primitively on L. Note that $L = \theta_0^L \cup \theta_1^L$, so we have $\{0\} = (0 : L) = (0 : \theta_0^L) \cap (0 : \theta_1^L)$.

From Lemma 3.2 we have that $(0 : \theta_0^L)$ is a non-zero ideal. $(0 : \theta_1^L)$ is a left ideal and this implies that $(0 : \theta_0^L)(0 : \theta_1^L) \subseteq (0 : \theta_0^L) \cap (0 : \theta_1^L) = \{0\}$. Now the fact that $(0 : \theta_0^L) \neq \{0\}$ implies $(0 : \theta_1^L) = \{0\}$ by 1-primeness of N.

For each $\gamma \in \theta_1^L$ we have that $(0 : \gamma)$ is a modular left ideal, see [9, Proposition 3.23]. Since by [9, Proposition 3.10], $N/(0 : \gamma) \cong_N L$ and L is an N-group of type 0, this implies that $(0 : \gamma)$ is a 0-modular left ideal ([9, Definition 3.28]).

Thus $J_{\frac{1}{2}}(N)$, the intersection of all 0-modular left ideals of N, is contained in $(0 : \theta_1^L)$. So, we have that $J_{\frac{1}{2}}(N) \subseteq (0 : \theta_1^L) = \{0\}$. By [9, Theorem 5.39] (this is the place were we need the DCCL of the whole near-ring), this implies that the natural N-group ${}_N N$ is completely reducible. This means that each left ideal of N has a direct summand in N, see [9, Theorem 2.48 and 2.50] (or see [8, Lemma 8.7] for a more direct proof of this fact). Thus, the ideal $(0 : \theta_0^L)$ has a left ideal K, say as a direct summand. Consequently, $N = (0 : \theta_0^L) \dot{+} K$. But then $(0 : \theta_0^L)K \subseteq (0 : \theta_0^L) \cap K = \{0\}$. Since $(0 : \theta_0^L) \neq \{0\}$, 1-primeness implies that $K = \{0\}$ and so, $N = (0 : \theta_0^L)$.

We have already seen that N is acting 0-primitively on L. Since $N = (0 : \theta_0^L)$, we have that for each $n \in N$ and each non-generator $\delta \in \theta_0^L$ we have $n\delta = 0$. So, N acts strongly monogenic on L implying 1-primitivity of the action of N on L. □

As a consequence of Proposition 3.1 and Theorem 3.3 we have the following corollary which answers the question posed in [4, Chapter 6, Remark 2.5].

Corollary 3.4. *Let $N \neq \{0\}$ be a zero symmetric 1-prime near-ring with DCCN. Then N is a 1-primitive near-ring.*

One may ask if in Theorem 3.3 the condition that the minimal left ideal L does not contain proper N-subgroups which are N-isomorphic to the minimal left ideal L itself could somehow be removed or avoided by using other methods in the proof. The answer is no. There exist examples of zero symmetric near-rings which fulfill the DCCL, are 1-prime but not 1-primitive. We take an example of a near-ring multiplication from [5, Example on page 88], which was used there for other purposes, and prove that it works for our purposes.

Example 3.5. Let N be an infinite simple group and i an isomorphism of N onto one of its proper subgroups N_1. An example of such a group has not been presented in [5], so we present one here. For example, let $N = PSL_2(K)$, the projective special linear group of 2×2 matrices over an infinite field K, thus being a simple group. Let K be such that it possesses a proper subfield K_1 being isomorphic as a field to K, via the isomorphism ϕ, say. Such infinite fields exist, for example let K be the field of rational functions in infinitely many variables $x_1, x_2, \ldots$ and K_1 be the proper subfield of functions depending only on $x_2, x_3, \ldots$. Then K and K_1 are isomorphic. Then the map $i : PSL_2(K) \to PSL_2(K_1), \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \begin{pmatrix} \phi(a) & \phi(b) \\ \phi(c) & \phi(d) \end{pmatrix}$ is an isomorphism of a kind we are looking for. Now (by straightforward computation, or see [5]), each non-zero element $n \in N$ has a unique representation of the form $n = i^k(f)$, where $k \geq 0$ is a non-zero integer and $f \in N \setminus N_1$. Define a multiplication on N by $n * 0 = 0$, $n \in N$ and if $i^k(f_1) = n_1 \in N \setminus \{0\}$ define $n * n_1 := i^k(n)$. Multiplication is well defined due to the unique representation of n_1. The multiplication is zero symmetric and it is right distributive due to the fact that the function i^k is a homomorphism. So we get a zero symmetric near-ring N. Note that each element in $N \setminus N_1$ is a multiplicative right identity, so $N^2 = N$. From the fact that i is not onto N we get an infinite decreasing chain of isomorphic proper subgroups of the form $i^k(N)$, k a natural number. These subgroups are in fact N-subgroups of the near-ring (see [5], or check by an easy computation). Now we see that for $k \in \mathbb{N}$ the map $i^k : N \to N, n \mapsto i^k(n)$ is an N-isomorphism between the N-groups N and $i^k(N)$. Clearly, it is a group isomorphism. Let $h, n \in N$, then $h = i^m(f)$ for some suitable non-zero integer m and $f \in N \setminus N_1$. So, $i^k(n * h) = i^k(i^m(n)) = i^{k+m}(n) = n * i^k(i^m(f)) = n * i^k(h)$. Due to the fact that the additive group N of the near-ring is a simple group, we have that the near-ring N itself is minimal as a left ideal, so N satisfies the DCCL. But for $k \in \mathbb{N}$, the N-subgroup $i^k(N)$ is a proper N-subgroup N-isomorphic to N. Now, N is minimal as a left ideal and $N^2 = N$, so N is a 1-prime near-ring. Suppose that N is a 1-primitive near-ring, acting 1-primitively on some N-group Γ with generator γ. Then, $\Gamma \cong_N N/(0 : \gamma) = N$, by minimality of N as a left ideal. But the N-group N of the near-ring is not of type 1 (only of type 0). Let $n \in i(N)$. Then, $\{0\} \neq N * n \subseteq i(N) \neq N$. So, the action of N is not strongly monogenic.

4. Maximal ideals in near-rings with DCCN

In the following sections of this paper we make applications of Theorem 3.3, Corollary 3.4 respectively, to the structure theory of near-rings. We restrict our discussion to near-rings with DCCN because then the assumptions in Theorem 3.3 are naturally fulfilled (see Proposition 3.1). We first study when an ideal in a near-ring is a maximal ideal.

To continue our discussion we have to introduce the notion of a v-prime ideal I in a near-ring, $v \in \{0, 1, 2\}$. An ideal I, $I \neq N$ in a near-ring N is said to be 1-prime (2-prime, 0-prime) if $HK \subseteq I$ implies $H \subseteq I$ or $K \subseteq I$, H, K left ideals (N-subgroups, ideals, respectively) of the near-ring (see [1, Definition 5.1.1], or [9, Remark 2.108] or [2, Definition 1] or [4]). Note that in this sense a zero symmetric near-ring is 1-prime if the zero ideal $\{0\}$ is a 1-prime ideal (see Definition 2).

It is straightforward to see that an ideal I is v-prime if and only if the near-ring N/I is v-prime (see [1, Remark 5.1.5]). Note that we want $I \neq N$ for a v-prime ideal in N, otherwise the whole near-ring N would always be a v-prime ideal, a trivial case we want to avoid. An ideal I of a near-ring N is called v-primitive, $v \in \{0, 1, 2\}$, if N/I is a v-primitive near-ring (see [9, Definition 4.2]). Note that if an ideal I is v-primitive, $v \in \{0, 1, 2\}$, then I is v-prime (see [1, Theorem 5.4.3]).

We obtain another corollary from Theorem 3.3 as an immediate consequence.

Corollary 4.1. *Let N be a zero symmetric near-ring with DCCN and let I be a 1-prime ideal of N. Then, N/I is a 1-primitive near-ring, I is a 1-primitive ideal and I is a maximal ideal.*

Proof. Since I is 1-prime, N/I is a 1-prime near-ring and the DCCN carries over to N/I (see [9, Theorem 2.35]). It follows from Corollary 3.4 that N/I is a 1-primitive near-ring. Thus, I is a 1-primitive ideal and by [9, Theorem 4.46], N/I is a simple near-ring which shows that I is a maximal ideal. □

Observe that it is not true that a 0-prime ideal I in a zero symmetric near-ring N with DCCN is necessarily maximal. The factor near-ring N/I will be 0-primitive but 0-primitive near-rings need not be simple near-rings (see [9, Remark 2.73] or see Example 6.4). Conversely, it is known that a maximal ideal I in a near-ring is 0-prime or $N^2 \subseteq I$ (see [9, Proposition 2.71]). We are now in a position to obtain a characterisation of maximal ideals in a near-ring with DCCN using the concept of 1-primeness. We need a technical lemma first, collecting some results on minimal ideals from [9].

Lemma 4.2. *Let N be a zero symmetric near-ring with DCCN and I a minimal ideal, $I^2 \neq \{0\}$. Then, each non-zero left ideal $L \subseteq I$ has the property that $L^2 \neq \{0\}$.*

Proof. Suppose there is a left ideal $L \subseteq I$ such that $L^2 = \{0\}$. Then by [9, Corollary 3.55], I is nilpotent. From [9, Proposition 3.53] we have that $I^2 = \{0\}$, contradicting our assumptions. □

Theorem 4.3. *Let N be a zero symmetric near-ring with DCCN. Then the following are equivalent*:

(1) *I is a 1-prime ideal.*
(2) *I is a maximal ideal and $N^2 \not\subseteq I$.*

Proof. (1) $\Rightarrow$ (2): Suppose I is a 1-prime ideal, so $I \neq N$. From Corollary 4.1 we have that I is maximal. Clearly we must have $N^2 \not\subseteq I$ due to 1-primeness of I and the fact that $I \neq N$.

(2) $\Rightarrow$ (1): Conversely, suppose that I is a maximal ideal and $N^2 \not\subseteq I$. Then $\overline{N} = N/I$ is a simple near-ring with DCC$\overline{N}$ and $\overline{N}^2 \neq \{0\}$. Let $\overline{L}$ and $\overline{J}$ be two left ideals in $\overline{N}$ and suppose that $\overline{L}\,\overline{J} = \{0\}$. Further suppose that $\overline{L} \neq \{0\}$. Thus, $\overline{L} \subseteq (0 : \overline{J})$ and so $(0 : \overline{J})$ is a non-zero ideal in $\overline{N}$. Simplicity of $\overline{N}$ implies $\overline{N} = (0 : \overline{J})$. Therefore, $\overline{J}\,\overline{J} = \{0\}$. Lemma 4.2 applied to the minimal ideal $\overline{N}$ implies $\overline{J} = \{0\}$. This shows that $\overline{N}$ is 1-prime. Hence, I is a 1-prime ideal. □

Note that $N^2 \not\subseteq I$ is naturally fulfilled if N contains a right identity for example, since then $N^2 = N$.

Probably interesting to note at this point is that in a zero symmetric near-ring N with DCCI and $N^2 = N$ we only have finitely many maximal ideals. To prove this result we need a theorem from [2].

Proposition 4.4 ([2, Theorem 2.2])**.** *Let N be a zero symmetric near-ring and P a 0-prime ideal in N. Let $X_1, \ldots, X_n$ be finitely many ideals in N. Then $X_1 \cdot \ldots \cdot X_n \subseteq P \Rightarrow X_i \subseteq P$ for some $i \in \{1, \ldots, n\}$.*

Theorem 4.5. *Let N be a zero symmetric near-ring with DCCI and $N^2 = N$. Then there exist only finitely many maximal ideals in N.*

Proof. Let C be the collection of all finite intersections of maximal ideals of N. If C is empty, there is nothing to prove. Otherwise, C is a non-empty set of ideals of N which contains minimal ideals with respect to $\subseteq$, due to the DCCI. Let $M_1 \cap \cdots \cap M_n$ be an element of C which is minimal in C. Let M be a maximal ideal in N. So, N/M is a simple near-ring and since $N^2 = N$ we cannot have $(N/M)^2 = \{0\}$. Since we do not have any non-trivial ideals in N/M, this shows that the zero ideal in N/M is a 0-prime ideal. So, N/M is a 0-prime near-ring and therefore (see [9, Proposition 2.67]), M is a 0-prime ideal.

Clearly, $(M_1 \cap \cdots \cap M_n) \cap M \subseteq M_1 \cap \cdots \cap M_n$. But $(M_1 \cap \cdots \cap M_n) \cap M$ is a finite intersection of maximal ideals, so contained in C and by minimality of $M_1 \cap \ldots \cap M_n$ in C we must have $M_1 \cap \cdots \cap M_n = (M_1 \cap \cdots \cap M_n) \cap M \subseteq M$. Since M_1 and the intersection $M_2 \cap \cdots \cap M_n$ are ideals in N, we have that $M_1 \cdot (M_2 \cap \cdots \cap M_n) \subseteq M_1 \cap (M_2 \cap \cdots \cap M_n) \subseteq M$. By Proposition 4.4, there is an $i \in \{1, \ldots, n\}$ such that $M_i \subseteq M$. Since M_i is maximal, $M_i = M$. This shows that each maximal ideal in N must be one of the ideals M_i, $i \in \{1, \ldots, n\}$. $\square$

We now collect some more facts concerning the link between v-prime and v-primitive ideals, their link to N-groups of type v and their link to maximal ideals.

Note that the content of the following Lemma 4.6 has been previously known for the case $v \in \{0, 2\}$ (see [1, Theorem 5.4.3] and [1, Proposition 5.4.4]). Here we are able to complete this description by considering the case $v = 1$.

Lemma 4.6. *Let I be an ideal in a zero symmetric near-ring N with DCCN, $v \in \{0, 1, 2\}$. Then the following are equivalent*:

(1) *There exists an N-group Γ of type v such that $I = (0 : \Gamma)$.*
(2) *I is a v-primitive ideal.*
(3) *I is a v-prime ideal.*

Proof. (1) $\Leftrightarrow$ (2) is a standard result in near-ring theory, proven in [9, Proposition 4.3]. From [1, Theorem 5.4.3] we have that a v-primitive ideal is a v-prime ideal, $v \in \{0, 1, 2\}$, showing (2) $\Rightarrow$ (3). From [1, Proposition 5.4.4] we have that v-primeness of an ideal, $v \in \{0, 2\}$, implies v-primitivity of the ideal. From Corollary 4.1 we have that 1-primeness of an ideal implies that the ideal is a 1-primitive ideal. So, we also have (3) $\Rightarrow$ (2). $\square$

With the help of Lemma 4.6 we obtain a corollary which shows the equivalence between 1-primitive ideals, 1-prime ideals and maximal ideals in case of zero symmetric near-rings with DCCN.

Corollary 4.7. *Let N be a zero symmetric near-ring with DCCN and I an ideal of N such that $N^2 \not\subseteq I$. Then the following are equivalent*:

(1) *I is 1-primitive.*

(2) *I is 1-prime.*
(3) *I is maximal.*

Proof. (1) $\Leftrightarrow$ (2) was shown in Lemma 4.6 and (2) $\Leftrightarrow$ (3) follows from Theorem 4.3. □

Immediately from Corollary 4.7 we get a description of simple near-rings, 1-primitive near-rings respectively, with DCCN in terms of primeness of the near-ring.

Corollary 4.8. *Let N be a zero symmetric near-ring with DCCN and $N^2 \neq \{0\}$. Then the following are equivalent*:

(1) *N is a 1-primitive near-ring.*
(2) *N is a 1-prime near-ring.*
(3) *N is a simple near-ring.*

Proof. The statements follow from Corollary 4.7 for the case $I = \{0\}$. □

In [2] we find a theorem giving a kind of element wise characterisation of 1-prime ideals. To fix a notation let $b \in N$, then $\langle b \rangle_l$ is the left ideal generated by b.

Theorem 4.9 ([2, Theorem 2.1]). *Let N be a zero symmetric near-ring. Then P is a 1-prime ideal if and only if for $a, b \in N$, $a\langle b \rangle_l \subseteq P \Rightarrow a \in P$ or $b \in P$.*

We can use Theorem 4.9 to give a kind of element wise characterisation for simple near-rings, 1-primitive near-rings respectively. This shows a possible application of how our results about 1-primeness can be used to characterise simple near-rings, 1-primitive near-rings respectively.

Corollary 4.10. *Let N be a zero symmetric near-ring with DCCN and $N^2 \neq \{0\}$. Then the following are equivalent*:

(1) *N is a 1-primitive near-ring, a simple near-ring respectively.*
(2) $\forall a, b \in N : a \neq 0$ *and* $b \neq 0 \Rightarrow a\langle b \rangle_l \neq \{0\}$

Proof. Corollary 4.8 shows the equivalence of N being simple and N being 1-primitive. We consider 1-primitive near-rings in the following.

(1) $\Rightarrow$ (2): From Lemma 4.6 we have that $\{0\}$ is a 1-prime ideal. Theorem 4.9 shows that this is equivalent to $\forall a, b \in N$: $a\langle b \rangle_l = \{0\} \Rightarrow a = 0$ or $b = 0$ which is equivalent to statement (2).

(2) $\Rightarrow$ (1): From Theorem 4.9 we have that $\{0\}$ is a 1-prime ideal. Thus, Corollary 3.4 shows that N is a 1-primitive near-ring. □

5. 0-prime ideals which are maximal

Theorem 4.3 shows that a 0-prime ideal I in a zero symmetric near-ring with DCCN is a maximal ideal only in case it is already 1-prime (as long as $N^2 \not\subseteq I$). We will now discuss when it happens that a 0-prime ideal is maximal. Therefore we need the following result, to be found in [9].

Proposition 5.1 ([9, Theorem 5.34])**.** *Let N be a zero symmetric near-ring with DCCL. Suppose that N is* 1*-semisimple* (*i.e.,* $J_1(N) = \{0\}$) *and let* Γ *be an N-group of type* 0*. Then* Γ *is of type* 1*.*

Theorem 5.2. *Let N be a zero symmetric near-ring with DCCN. Then the following are equivalent*:

(1) $J_1(N)$ *is nilpotent.*
(2) *For all ideals $I \subseteq N : I$ is* 1 *prime* $\Leftrightarrow$ *I is* 0*-prime.*
(3) *Each N-group of type* 0 *is of type* 1*.*

Proof. (1) $\Rightarrow$ (3): From [9, Theorem 5.37] we have that $J_1(N) \subseteq J_0(N)$ in case $J_1(N)$ is nilpotent and therefore $J_1(N) = J_0(N)$. Let Γ be an N-group of type 0. Then, $J_0(N) = J_1(N) \subseteq (0 : \Gamma)$. Consequently, by [9, Proposition 3.14], Γ is an $N/J_1(N)$-group of type 0. From [9, Theorem 5.16] we have that $N/J_1(N)$ is 1-semisimple. Thus, by Proposition 5.1, Γ is an $N/J_1(N)$-group of type 1. Again by [9, Proposition 3.14], this shows that Γ is an N-group of type 1.

(3) $\Rightarrow$ (2): Under the assumptions of (3) we have from Lemma 4.6 that for each ideal $I \subseteq N$ we have the equivalence I is 1-prime $\Leftrightarrow$ I is 0-prime.

(2) $\Rightarrow$ (1): By [9, Theorem 5.2], we have that $J_v(N)$ equals the intersection of all the v-primitive ideals in a near-ring, $v \in \{0, 1\}$. By Lemma 4.6 it follows that $J_0(N)$ equals the intersection of all the 0-prime ideals in a near-ring with DCCN. By our assumption it follows that $J_0(N)$ equals the intersection of all the 1-prime ideals in a near-ring. Again from Lemma 4.6 we have that $J_0(N)$ equals the intersection of all the 1-primitive ideals in a near-ring and so, $J_1(N) = J_0(N)$. It follows from [9, Theorem 5.40] that $J_1(N)$ is nilpotent. $\square$

Theorem 5.2 can also be seen as a characterisation of the situation when the radical $J_1(N)$ of a zero symmetric near-ring N with DCCN is nilpotent. This is precisely the case when each 0-prime ideal is a 1-prime ideal. For zero symmetric near-rings with $N^2 = N$ and DCCN, we can also formulate this as $J_1(N)$ is nilpotent precisely when each 0-prime ideal is maximal.

At this point we quickly want to discuss what happens when $J_2(N)$ of a zero symmetric near-ring N with DCCN is nilpotent. From [9, Theorem 5.48] we have that $J_2(N)$ is nilpotent precisely when each 0-prime ideal is a 2-primitive ideal or equivalently when each N-group of type 1 is of type 2. This result is due to D. Ramakotaiah. Note that in the light of Lemma 4.6 an ideal is 2-primitive precisely when it is 2-prime (in a near-ring with DCCN).

Our knowledge concerning the link between 1-prime and 1-primitive ideals allows us to study the Jacobson radical of type 1 of a near-ring more closely. Let N be a zero symmetric near-ring with DCCN and containing an identity element. Then [9, Theorem 5.42] tells us that $J_2(N) = J_1(N) = M(N)$, where $M(N)$ is the intersection of all maximal ideals of N. We can generalise this result now.

Theorem 5.3. *Let N be a zero symmetric near-ring with DCCN and $N^2 = N$. Then, $J_1(N) = M(N)$, where $M(N)$ is the intersection of all maximal ideals of N.*

Proof. Since we have that $N^2 = N$ we know from Theorem 4.3 that an ideal I is maximal precisely when it is 1-prime. Lemma 4.6 shows that this is equivalent to I being a 1-primitive ideal. Thus, the 1-primitive ideals are precisely the maximal ideals. Since, by [9, Theorem 5.2], $J_1(N)$ equals the intersection of all the 1-primitive ideals in a near-ring we have $J_1(N) = M(N)$. □

Theorem 5.3 holds for example in zero symmetric near-rings with DCCN containing a multiplicative right identity element. Note that in such near-rings we usually do not have $J_2(N) = J_1(N)$. Also note that as a consequence of Theorem 4.5, the intersection of all maximal ideals in N is a finite intersection. We obtain an immediate corollary concerning the algebraic structure of maximal ideals modulo the Jacobson 1 radical of the near-ring.

Corollary 5.4. *Let N be a zero symmetric near-ring with DCCN and $N^2 = N$ and I a maximal ideal. Then, $J_1(N) \subseteq I$. In particular, every nilpotent left ideal of N is contained in I. $I/J_1(N)$ is zero or a finite distributive sum of 1-primitive near-rings.*

Proof. From Theorem 5.3 we have that $J_1(N) \subseteq M(N) \subseteq I$. By [9, Theorem 5.37], $J_1(N)$ contains every nilpotent left ideal of N in case N has the DCCN. $I/J_1(N)$ is an ideal in the 1-semisimple near-ring $N/J_1(N)$ with DCC on $N/J_1(N)$-subgroups. Thus, $I/J_1(N)$ is zero or a finite distributive sum of 1-primitive near-rings, by [9, Theorems 5.31 and 5.32]. □

6. 0-prime ideals which are not maximal

We now study the situation when a 0-prime ideal is not maximal, which means that it is not 1-prime in the light of Theorem 4.3. We will determine the ideal which in the ideal lattice of a zero symmetric near-ring with DCCN sits directly above a 0-prime ideal in such a case. We need the following Lemma, to be found in [9].

Lemma 6.1 ([9, Lemma 1 of Theorem 3.54])**.** *Let N be a zero symmetric near-ring with DCCN and Γ be a faithful N-group. Let K be a minimal N-ideal of Γ. Let $\{0\} \neq L \subseteq (K : \Gamma)$ be a left ideal such that $\forall \gamma \in \Gamma : N\gamma = \Gamma$ or $L\gamma = \{0\}$. Then L is a finite direct sum of N-isomorphic minimal left ideals of N.*

Lemma 6.1 is one of the key lemmas in Stuart Scott's proof that a minimal ideal in a zero symmetric near-ring N with DCCN decomposes as a direct sum of minimal left ideals. The complete proof of this result can be found in [9].

Before proving the next theorem we collect and discuss some notation and elementary facts to be used in the proof of the theorem. Let Γ be an N-group of type 0 and $\overline{N} := N/(0 : \Gamma)$. The action $(n + (0 : \Gamma))\gamma := n\gamma$ turns Γ into an $\overline{N}$-group. We will use the notation $(0:S)_{\overline{N}}$ for annihilators of subsets S of the $\overline{N}$-group Γ. Let θ_1 be the set of generators of the N-group Γ and θ_0 be the set of non-generators, so $\theta_1 := \{\gamma \in \Gamma \mid N\gamma = \Gamma\}$ and $\theta_0 := \{\gamma \in \Gamma \mid N\gamma \neq \Gamma\}$. From the definition of the action of $\overline{N}$ it is clear that Γ has the same set of generators θ_1 and non-generators θ_0 also as an $\overline{N}$-group. Now consider the annihilator $(0:\theta_0)$ in N. Clearly, it is a left ideal in N. Moreover, for $\delta \in \theta_0$ and $n \in N$ we have $N(n\delta) \subseteq N\delta \neq \Gamma$ and so we see that $n\delta \in \theta_0$. Thus, whenever $a \in (0:\theta_0)$, $\delta \in \theta_0$ and $n \in N$, then $(an)\delta = a(n\delta) = 0$. So we see that $(0:\theta_0)$ is an ideal in N. Clearly, $(0:\Gamma) \subseteq (0:\theta_0)$ and so we have that $(0:\theta_0)/(0:\Gamma)$ is an ideal in $\overline{N} = N/(0:\Gamma)$. From the action of $\overline{N}$ on Γ it is immediate to see that $(0:\theta_0)/(0:\Gamma) = (0:\theta_0)_{\overline{N}}$.

Note that when Γ happens to be an N-group of type 1, then N acts strongly monogenic on Γ which means that $(0:\theta_0) = N$. If Γ is of type 0 but not of type 1, then $(0:\theta_0) \neq N$.

We need another proposition for the proof of the next theorem in this section.

Proposition 6.2. *Let N be a zero symmetric near-ring and let Γ_1 and Γ_2 be N-isomorphic N-groups of type 0. Let θ_0^1 be the set of non-generators of Γ_1 and θ_0^2 be the set of non-generators of Γ_2. Then, $(0:\theta_0^1) = (0:\theta_0^2)$.*

Proof. Let $\psi : \Gamma_1 \to \Gamma_2$ be an N-isomorphism. Let $a \in (0:\theta_0^1)$ and $\delta_2 \in \theta_0^2$. So, there is $\delta_1 \in \Gamma_1$ such that $\psi(\delta_1) = \delta_2$. So, $\Gamma_2 \neq N\delta_2 = N\psi(\delta_1) = \psi(N\delta_1)$. Since ψ is a bijection, we see that $N\delta_1 \neq \Gamma_1$. Thus, $\delta_1 \in \theta_0^1$. Then, $0 = \psi(a\delta_1) = a\psi(\delta_1) = a\delta_2$. This proves that $(0:\theta_0^1) \subseteq (0:\theta_0^2)$. In a similar way, using the N-isomorphism ψ^{-1}, we see that $(0:\theta_0^2) \subseteq (0:\theta_0^1)$. □

Now we present the theorem concerning 0-prime ideals which are not maximal ideals. To fix a notation, when we have a subdirectly irreducible near-ring N, then we will call its smallest non-zero ideal H the heart of N (according to the notation in [1]).

Theorem 6.3. *Let N be a zero symmetric near-ring with DCCN and I a 0-prime ideal which is not maximal as an ideal. Then there exists an N-group Γ of type 0 which is not of type 1 such that $I = (0:\Gamma) \subset (0:\theta_0) \subset N$, the containments being proper. Moreover, N/I is subdirectly irreducible with heart $(0:\theta_0)/I$.*

Proof. From Lemma 4.6 we have the existence of an N-group of type 0 such that $I = (0:\Gamma)$. Since I is 0-prime, $N^2 \not\subseteq I$. By assumption, I is not a maximal ideal, so I is not 1-prime by Theorem 4.3. Therefore, again by Lemma 4.6, Γ is not of type 1 and consequently, $(0:\theta_0) \neq N$. By [9, Proposition 3.14], $\overline{N} := N/(0:\Gamma)$ is 0-primitive on Γ (via $(n + (0:\Gamma))\gamma = n\gamma$), hence $\overline{N}$ is a 0-prime near-ring (see [1, Theorem 5.4.3]). Thus, we have from [9, Corollary 2.107] that $\overline{N}$ is subdirectly irreducible. Hence, there exists a smallest non-zero ideal in $\overline{N}$, $\overline{H}$ say. Since $(0:\Gamma)$ is not maximal, $\overline{N}$ is not a simple near-ring, so we follow that $\overline{H} \neq \overline{N}$. Let H be the ideal in N, containing $(0:\Gamma)$ such that

$\overline{H} = H/(0:\Gamma)$. Consequently, we have $(0:\Gamma) \subset H \subset N$, the containments being proper. To complete our proof it remains to show that $H = (0:\theta_0)$.

Since N has the DCCN, so has $\overline{N}$ (see [9, Theorem 2.35]). Consequently, $\overline{H}$ is a minimal ideal in the near-ring $\overline{N}$ satisfying the DCC$\overline{N}$. Therefore, we know from [9, Theorem 3.54] that $\overline{H}$ is a finite direct sum of minimal left ideals of the near-ring $\overline{N}$, $\overline{H} = \sum_{i=1}^{k} \overline{L}_i$, $k \in \mathbb{N}$, say. Let $\gamma \in \theta_1$, hence $N\gamma = \Gamma$. Suppose that $H\gamma = \{0\}$. Then, $H\Gamma = H(N\gamma) \subseteq H\gamma = \{0\}$ and $H \subseteq (0:\Gamma)$ which cannot be because $\overline{H} = H/(0:\Gamma) \neq \{0\}$. Therefore, $H\gamma$ is a non-zero N-ideal in the N-group Γ of type 0, which implies $H\gamma = \Gamma$. From the action of $\overline{N}$ on Γ it follows that $\overline{H}\gamma = \Gamma$. Thus, there is an $i \in \{1, \ldots, k\}$ such that $\overline{L}_i\gamma \neq \{0\}$. Γ is an $\overline{N}$-group of type 0 and $\overline{L}_i$ is a minimal left ideal such that $\overline{L}_i \not\subseteq (0:\gamma)_{\overline{N}}$ and therefore, by [9, Proposition 3.10], $\overline{L}_i \cong_{\overline{N}} \Gamma$.

From Lemma 3.2, applied to the near-ring $\overline{N}$, we see that $\overline{L}_i \subseteq (0:\theta_0^{\overline{L}_i})_{\overline{N}}$ (for the definition of $\theta_0^{\overline{L}_1}$ see Definition 1). So, we have that $(0:\theta_0^{\overline{L}_i})_{\overline{N}}$ is a non-zero ideal in $\overline{N}$. Since $\overline{L}_i \cong_{\overline{N}} \Gamma$ it follows from Proposition 6.2 that $(0:\theta_0^{\overline{L}_i})_{\overline{N}} = (0:\theta_0)_{\overline{N}}$.

Γ is a faithful $\overline{N}$-group of type 0, so Γ is minimal as an $\overline{N}$-ideal. We now apply Lemma 6.1. We let $K := \Gamma$ and $\{0\} \neq L := (0:\theta_0)_{\overline{N}}$. So, $(K:\Gamma)_{\overline{N}} = \overline{N}$ and $\{0\} \neq (0:\theta_0)_{\overline{N}} \subseteq (K:\Gamma)_{\overline{N}}$. Let $\gamma \in \theta_1$. Then, $\overline{N}\gamma = \Gamma$. If $\gamma \in \theta_0$, then $(0:\theta_0)_{\overline{N}}\gamma = \{0\}$. So, Lemma 6.1 applies to $(0:\theta_0)_{\overline{N}}$ and we get that $(0:\theta_0)_{\overline{N}}$ is a finite direct sum of $\overline{N}$-isomorphic minimal left ideals of $\overline{N}$. Let $(0:\theta_0)_{\overline{N}} = \sum_{i=1}^{s} \overline{J}_i$, $s \in \mathbb{N}$, $\overline{J}_i$, $i \in \{1, \ldots, s\}$ being minimal left ideals of $\overline{N}$. Let $j \in \{1, \ldots, s\}$. By faithfulness of Γ as an $\overline{N}$-group and the fact that $\overline{J}_j\theta_0 = \{0\}$, there is an element $\gamma_j \in \theta_1$ such that $\overline{J}_j \not\subseteq (0:\gamma_j)_{\overline{N}}$. By [9, Proposition 3.10], we get $\overline{J}_j \cong_{\overline{N}} \Gamma$.

Suppose a non-trivial ideal $\overline{I}$ of the near-ring $\overline{N}$ is properly contained in $(0:\theta_0)_{\overline{N}}$. Since we know that $(0:\theta_0)_{\overline{N}} = \sum_{i=1}^{s} \overline{J}_i$, $s \in \mathbb{N}$ there must be a $j \in \{1, \ldots, s\}$ such that $\overline{J}_j \not\subseteq \overline{I}$ in such a case. By minimality of the left ideal $\overline{J}_j$ we get $\overline{J}_j \cap \overline{I} = \{0\}$ and hence $\overline{I}\ \overline{J}_j \subseteq \overline{J}_j \cap \overline{I} = \{0\}$. By $\overline{N}$-isomorphism of $\overline{J}_j$ and Γ this now implies $\overline{I}\Gamma = \{0\}$, contradicting the faithfulness of Γ. Hence, $(0:\theta_0)_{\overline{N}}$ is a minimal ideal in $\overline{N}$ and by subdirect irreducibility of $\overline{N}$ it is the smallest non-zero ideal in $\overline{N}$. Thus $\overline{H} = (0:\theta_0)_{\overline{N}}$.

Since there is a bijection between ideals in N containing $(0:\Gamma)$ and ideals in $\overline{N}$ (see [9, Theorem 1.30]), it follows from $\overline{H} = H/(0:\Gamma) = (0:\theta_0)_{\overline{N}} = (0:\theta_0)/(0:\Gamma)$ that $H = (0:\theta_0)$. □

We will now illustrate Theorem 6.3 by examples which show that, in the language of the theorem, $(0:\theta_0)$ can be a maximal ideal (Example 6.5), it can also be a 0-prime ideal which is not maximal (Example 6.4) and it can be an ideal which is not even 0-prime (Example 6.6). When computing the examples, which are chosen such that one can compute these examples by hand, we will also take the opportunity to highlight some of the results of Theorems 5.2 and 4.3.

We introduce a notation to be used in the examples. Let Γ be an additively written group. Then $M_0(\Gamma)$ is the near-ring of all zero preserving functions, so $M_0(\Gamma) := \{f : \Gamma \to \Gamma \mid f(0) = 0\}$ with pointwise addition of functions and function composition as near-ring operations.

Of course, examples of 0-prime near-rings which are not 1-prime are known (see for example [1, Exercise 5.5.4]). In the light of Corollary 3.4 it is easy to find more such examples, namely 0-primitive near-rings which are not 1-primitive. We note that in case we have a zero symmetric near-ring N with DCCN which acts 1-primitively on some N-group Γ, then by [9, Theorem 4.46] each N-group of type 0 is of type 1 and N-isomorphic to Γ. This implies that when we have a zero symmetric near-ring with DCCN which acts 0-primitively on some N-group Γ but not 1-primitively, then N cannot act 1-primitively possibly on some other N-group. We start with an example of a near-ring N which is 0-prime but not 1-prime and contains a non-trivial ideal which is 0-prime and not 1-prime. In the light of Theorem 5.2 we must have that $J_1(N)$ is not nilpotent, as we will see in the example.

Example 6.4. Let $\Gamma := \mathbb{Z}_8$ and $S := \{0, 2, 4, 6\}$ and $K := \{0, 4\}$ be the subgroups of order 4 and 2 in Γ. Consider the near-ring $N := \{f \in M_0(\Gamma) \mid f(\{2, 6\}) \subseteq S$ and $f(4) \subseteq K\}$ (that N is indeed additively closed follows from the fact that S and K are additively closed). N acts faithfully on Γ and the elements in the set $\{1, 3, 5, 7\}$ generate Γ as an N-group. Both, S and K are N-subgroups of Γ. Suppose S or K is an N-ideal in Γ. Then, for each $\gamma \in \Gamma$ and each $f \in N$ we must have $f(\gamma + 4) - f(\gamma) \in S$, K respectively. Let $g : \Gamma \to \Gamma, g(5) = 5, g(\gamma) = 0$ else. $g \in N$ and $g(1 + 4) - g(1) = 5 \notin S$, K respectively. This shows that neither S nor K are N-ideals. So we see that N acts 0-primitively on Γ, with the set of non-generators $\theta_0 = S$. Thus, N is a 0-prime near-ring (see [1, Theorem 5.4.3]) and $\{0\}$ is a 0-prime ideal. N does not act 1-primitively on Γ because the action is not strongly monogenic. For example, $2 \in \theta_0$ but $N(2) \neq \{0\}$. So, $\{0\}$ is not a maximal ideal, because N is not 1-primitive (see Corollary 4.8). Theorem 6.3 shows that $(0 : S)$ is the next ideal strictly above $\{0\}$ in the ideal lattice of the near-ring. Since K is an N-subgroup of Γ we have that $(0 : K)$ is also an ideal in N. Note that N has the identity function as the identity element, so clearly, $N \neq (0 : K)$ and as $K \subseteq S$ we have $(0 : \theta_0) = (0 : S) \subseteq (0 : K)$. Consider the function $h : \Gamma \to \Gamma, h(0) = h(4) = 0, h(\gamma) = 6$ else. Then, $h \in (0 : K)$ but $h \notin (0 : S)$. Thus we have that the ideal $(0 : \theta_0)$ is not a maximal ideal, it is strictly contained in the non-trivial ideal $(0 : K)$.

N contains an identity element, so $N = N^2$. In the light of Theorem 4.3, since $(0 : S)$ is not maximal and $N = N^2 \not\subseteq (0 : S)$ we cannot have that $(0 : S)$ is a 1-prime ideal. So there must exist left ideals $L_1 \not\subseteq (0 : S)$ and $L_2 \not\subseteq (0 : S)$ such that $L_1 L_2 \subseteq (0 : S)$. Let $L_1 := (0 : K)$ and $L_2 := (0 : \{2, 6\})$. The function $j : \Gamma \to \Gamma, j(4) = 4, j(\gamma) = 0$ else, is contained in L_2 but not in $(0 : S)$, also we have already shown that $L_1 \not\subseteq (0 : S)$. Now, $L_1L_2(2) = L_1L_2(6) = 0$ and since $L_2(4) \subseteq \{0, 4\}$ by definition of functions in N we also have $L_1L_2(4) = 0$. Thus, $L_1L_2 \subseteq (0 : S)$ which shows that $(0 : S)$ is not 1-prime.

We now will show that $(0 : S)$ is a 0-prime ideal in N. First we will show that S is an N-group of type 0. The only non-trivial subgroup contained in S is K. Let $2 \in S$ and consider the function $h : \Gamma \to \Gamma, h(6) = 6, h(\gamma) = 0$ else. $h \in N$ and $h(2 + 4) - h(2) = 6 \notin K$, so K is not an N-ideal in S. But 2 is a generator of the N-group S. Let $x \in S$ and define the function $f_x : \Gamma \to \Gamma, f_x(2) = x, f_x(\gamma) = 0$ else. $f_x \in N$ and we see that $S = \bigcup_{x\in S} f_x(2)$. So, 2 is a generator of the N-group S. Therefore, S is an N-group of type 0. Consequently, $N/(0 : S)$ is a 0-primitive and hence 0-prime near-ring. So, $(0 : S)$ is a 0-prime ideal. $(0 : S)$ is not maximal, so we can apply Theorem 6.3 which shows that $(0 : \{0, 4\}) = (0 : K)$ is the next ideal strictly above $(0 : S)$ in the ideal lattice of N. Since K is an N-group of type 2 (4 is the generator of K and there are no non-trivial N-subgroups in K), $J_1(N) \subseteq (0 : K)$ (we have that $J_1(N) = J_2(N)$ because N has an identity element).

Now we compute the radical $J_1(N)$. Let $L := \{f \in N \,|\, f(\{1, 3, 4, 5, 7\}) = \{0\}$ and $f(\{2, 6\}) \subseteq K\}$. Since K is an N-subgroup, it is straightforward to see that L is a non-zero subnear-ring of N. In fact, it is an N-subgroup of N. To see this, let $l \in L$ and $n \in N$. Then for $\gamma \in \{1, 3, 4, 5, 7\}$, $nl(\gamma) = \{0\}$. For $\delta \in \{2, 6\}$ we have $nl(\delta) \subseteq n(\{0, 4\}) \subseteq \{0, 4\}$. What is more, L is a nilpotent N-subgroup of N. To see this, let $l \in L$ and $\gamma \in \Gamma$. Then $l^2(\gamma) = l(l(\gamma)) \subseteq l(\{0, 4\} = \{0\}$. Consequently, $L^2 = \{0\}$ and therefore, we have that $\{0\} \neq L \subseteq J_2(N)$ (see [9, Corollary 5.45]). By Theorem 6.3, N is subdirectly irreducible with heart $(0 : S)$, so $(0 : S) \subseteq J_2(N)$ and therefore, $L + (0 : S) \subseteq J_2(N) = J_1(N)$. Since $L \cap (0 : S) = \{0\}$ (by faithfulness of N acting on Γ), we have $(0 : S) \neq J_1(N)$. Due to the fact that $(0 : K)$ is the next ideal strictly above $(0 : S)$ in the ideal lattice of N, we must have $J_1(N) = (0 : K)$.

It is easy to see that $J_1(N) = (0 : K)$ is not nilpotent. For example, the function $j : \Gamma \to \Gamma, j(1) = 1, j(\gamma) = 0$ else is in $(0 : K)$ but $j^2 = j$. Note that the non-nilpotency of $J_1(N)$ also follows from Theorem 5.2 because the ideals $\{0\}$ and $(0 : S)$ are 0-prime but not 1-prime.

Of course $(0 : \theta_0)$ as in Theorem 6.3 can be a maximal ideal, for example when θ_0 is an N-group of type 2.

Example 6.5. Let $\Gamma := \mathbb{Z}_4$ and $S := \{0, 2\}$ the subgroup of order 2 in Γ. Consider the near-ring $N := \{f \in M_0(\Gamma) \,|\, f(\{2\}) \subseteq S\}$. As in Example 6.4 it is straightforward to see that N is a near-ring which acts 0-primitively on Γ. S is an N-group of type 2 (but not an N-ideal) and $\theta_0 = S$. N does not act 1-primitively on Γ (since $N(2) \neq \{0\}$). So, we have from Corollary 3.4 that $\{0\}$ is a 0-prime ideal which is not 1-prime. So we have $\{0\} \subset (0 : \theta_0) \subset N$ from Theorem 6.3. From the fact that S is an N-group of type 2 we have that $N/(0 : S)$ is a finite 2-primitive and therefore a simple near-ring (see [9, Theorem 4.46]). So, $(0 : \theta_0) = (0 : S)$ is a maximal ideal.

The next example shows that $(0 : \theta_0)$ as in Theorem 6.3 need not be a 0-prime ideal.

Example 6.6. Let $\Gamma := \mathbb{Z}_{12}$ and $S_1 := \{0, 4, 8\}$ and $S_2 := \{0, 6\}$ the subgroups of order 3, order 2 respectively in Γ. Consider the near-ring $N := \{f \in M_0(\Gamma) \mid f(S_1) \subseteq S_1 \text{ and } f(S_2) \subseteq S_2\}$. N acts faithfully on Γ with set of generators $\theta_1 := \{1, 2, 3, 5, 7, 9, 10, 11\}$ and non-generators $\theta_0 := \{0, 4, 6, 8\} = S_1 \cup S_2$. The only possible candidates for non-trivial N-ideals in Γ are S_1 and S_2. Let $f : \Gamma \to \Gamma, f(5) = 5, f(\gamma) = 0$ else. Then $f \in N$ but $f(1 + 4) - f(1) = 5 \notin S_1$. So, S_1 is not an N-ideal. In a similar way we see that S_2 is not an N-ideal. Thus, N acts 0-primitively on Γ, so $\{0\}$ is a 0-prime ideal. N does not act 1-primitively on Γ because, for example, $N(4) \neq \{0\}$. Thus, in the light of Corollary 3.4, $\{0\}$ is not a 1-prime and consequently, by Theorem 4.3, not a maximal ideal in N. From Theorem 6.3 we have that $(0 : \theta_0)$ is the heart of the subdirectly irreducible near-ring N. Let $g : \Gamma \to \Gamma, f(6) = 6, f(\gamma) = 0$ else. Then, $g \in (0 : S_1)$ but $g \notin (0 : \theta_0)$. Consequently, $(0 : S_1) \not\subseteq (0 : \theta_0)$ and in a similar way we see that $(0 : S_2) \not\subseteq (0 : \theta_0)$. S_1 and S_2 are N-subgroups of N and so we have that $(0 : S_1)$ and $(0 : S_2)$ are ideals in N. Therefore, $(0 : S_1)(0 : S_2) \subseteq (0 : S_1) \cap (0 : S_2) = (0 : \theta_0)$. Therefore, $(0 : \theta_0)$ is not a 0-prime ideal.

7. Two natural situations when 1-prime implies 2-prime

In the final section of this paper we discuss two natural situations when 1-primeness of an ideal implies 2-primeness of the ideal. 2-primeness of an ideal implies 1-primeness of the ideal in zero symmetric near-rings (see for example [1, Proposition 5.4.1]), but in general not conversely (for examples and a thorough discussion of questions of this kind see [1]). We now are able to show the converse in two interesting situations.

Theorem 7.1. *Let N be a zero symmetric near-ring and I a 1-prime ideal. Let N satisfy the DCCN. Then the following hold*:

(1) *If N contains an identity element, then I is a 2-prime ideal.*
(2) *If N is distributively generated (d.g.), then I is 2-prime ideal.*

Proof. Since I is a 1-prime ideal, N/I is a 1-prime near-ring. N/I satisfies the DCC on N-subgroups contained in N/I (see [9, Theorem 2.35]). Thus, according to Corollary 3.4, N/I is 1-primitive, so $J_1(N/I) = \{0\}$ and following from [9, Theorem 4.46], N/I is a simple near-ring. If N contains an identity element, then so does N/I and therefore (see [9, Proposition 5.3]) $J_1(N/I) = J_2(N/I) = \{0\}$. In case N is d.g., N/I as a homomorphic image of N is d.g., also (see [9, Theorem 6.9]). From [6] and [3] we know that in a d.g. near-ring M we have $J_1(M) = J_2(M)$, so we again have $J_1(N/I) = J_2(N/I) = \{0\}$. Thus in both situations in question, N/I is a J_2-semisimple near-ring which satisfies the DCC on N/I-subgroups contained in N/I. Thus, by [9, Theorem 5.31] N/I must be a finite distributive sum of ideals which are 2-primitive near-rings. By simplicity of N/I, there is only one summand in this distributive sum of ideals and this shows that N/I is 2-primitive. Therefore, N/I is a 2-prime near-ring (see for example [1, Theorem 5.4.3]) and consequently, I is a 2-prime ideal (see [1, Remark 5.1.5]). $\square$

Note that different to the situation in Theorem 5.2 we do not have that $J_2(N)$ of a near-ring N has to be nilpotent to imply the equivalence between 1- and 2-prime ideals. Near-rings with identity or d.g. near-rings usually have non-nilpotent J_2-radicals. What happens with the prime ideals when $J_2(N)$ is nilpotent has been discussed in Section 5.

Remark 7.2. The fact that in a d.g. near-ring N we have that $J_1(N) = J_2(N)$ is not available in the monographs [9] and [8] and also not in [1]. A proof of it can be found in [3, Proposition 9], even under more general assumptions than the d.g. assumption, whereas in [6] we find only the fact mentioned. For self containment of the paper we offer a quick sketch of this proof, following the ideas of [3]. Let N be d.g. and let Γ be a monogenic N-group. Let $Ann_\Gamma(N) := \{\gamma \in \Gamma \mid N\gamma = \{0\}\}$. Using that each $n \in N$ is a sum of distributive elements, it is a routine calculation to see that $Ann_\Gamma(N)$ is an N-ideal of Γ. If Γ is of type 1, we therefore must have that $Ann_\Gamma(N) = \{0\}$. This implies that each non-zero element in Γ is a generator of the N-group Γ and so we see that Γ is of type 2. Hence, in case of a d.g. near-ring N, N-groups of type 1 and type 2 coincide.

We know that for near-rings which are rings the concept of 0-primeness, 1-primeness and 2-primeness of an ideal I are equivalent (that 1- and 2-primeness are equivalent is immediate because N-groups and left ideals are the same when the near-ring is a ring; that 0- and 1-primeness are equivalent in the ring situation can be derived for example from [7, Proposition 10.2]). From Example 6.4 we see that 0-primeness of an ideal does not imply 1-primeness of an ideal, even if the near-ring in question has an identity element. What happens in case the near-ring under consideration is d.g.? From Theorem 5.2 we know that each 0-prime ideal in the near-ring is already 1-prime if and only if the near-ring has a nilpotent J_1-radical. But (finite) d.g. near-rings which do not have nilpotent J_1-radicals are known (see for example [9, Remark 6.35a] or the appendix of [9] for references and examples). Therefore 0-primeness of an ideal does not imply 1-primeness in case of d.g. near-rings.

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