J. Korean Math. Soc. **58** (2021), No. 2, pp. 265–281
https://doi.org/10.4134/JKMS.j190378
pISSN: 0304-9914 / eISSN: 2234-3008

INVARIANCE OF KNEADING MATRIX UNDER CONJUGACY

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Abstract. In the kneading theory developed by Milnor and Thurston, it is proved that the kneading matrix and the kneading determinant associated with a continuous piecewise monotone map are invariant under orientation-preserving conjugacy. This paper considers the problem for orientation-reversing conjugacy and proves that the former is not an invariant while the latter is. It also presents applications of the result towards the computational complexity of kneading matrices and the classification of maps up to topological conjugacy.

1. Introduction

One of the most complicated yet very important discrete dynamical systems emanates from a continuous piecewise monotone self-map on an interval [6, 8, 12, 13]. Milnor and Thurston have developed the kneading theory [10, 11] to analyse the iterates of such maps, which makes use of combinatorial techniques and advanced analysis. They associated each piecewise monotone map with a matrix and an unusual determinant called the kneading matrix and the kneading determinant, respectively. The kneading matrix of such a map with m turning points is an $m \times (m+1)$ matrix with entries from the ring of formal power series over integers, and the corresponding kneading determinant is closely related to the determinant obtained from this matrix after deleting a column. In some sense, this matrix contains most of the crucial combinatorial information of the map and its iterates. This theory has slight modifications made by Preston [14], wherein the kneading matrix is indeed a square matrix of order m, and the corresponding kneading determinant is the usual determinant.

In recent times, kneading theory has been developed in various aspects. Preston [14] extended this theory for piecewise monotone maps, which have

Received May 30, 2019; Revised August 24, 2020; Accepted January 15, 2021.
2010 *Mathematics Subject Classification.* Primary 37E05, 37C15; Secondary 15A24.
Key words and phrases. Dynamical system, piecewise monotone map, topological conjugacy, kneading matrix, kneading determinant.
The second author was supported by Science and Engineering Research Board (SERB), DST, Government of India, through the project $ECR/2017/000765$.

discontinuities at their turning points. Alves and Ramos [1], using a functorial approach to this theory, have given explicit methods to compute the lap numbers and periodic points of continuous piecewise monotone maps, and have also proved many of the results in [11]. Gopalakrishna and Veerapazham [7] have described a relation between the kneading matrices of maps and their iterates for a family of chaotic continuous piecewise monotone maps. Mendes and Ramos [9] have developed a kneading theory for two-dimensional triangular maps and thereby exhibited adequate techniques for rigorous computation of the topological entropy of such maps. Kneading theory is a very important tool and has plenty of applications, see for example, to Duffing equations [5] and adding machines [3]. The other advancements in this theory also include kneading theory for tree maps [2] and kneading with weights [15, 16].

It is proved in [11] that the kneading matrix and kneading determinant are invariant under orientation-preserving topological conjugacy. In this paper, by exhibiting an interesting relation between kneading matrices of conjugate maps, we prove that the kneading matrix is not an invariant under orientation-reversing conjugacy while the kneading determinant is. We also present two applications of our results: the reduction of computational complexity, and the nonexistence of topological conjugacy between continuous piecewise monotone maps.

2. Preliminaries

In the sequel, let $I = [a, b]$ be a compact interval in $\mathbb{R}$ and $\mathcal{C}(I)$ be the set of all continuous self-maps of I. As defined in [11], an element $f \in \mathcal{C}(I)$ is called *piecewise monotone* if there exists a partition $a = c_0 < c_1 < \cdots < c_m < c_{m+1} = b$ of I such that the restriction of f to subintervals $I_1 = [c_0, c_1], I_2 = [c_1, c_2], \ldots, I_{m+1} = [c_m, c_{m+1}]$ is strictly monotone. The set of all piecewise monotone maps in $\mathcal{C}(I)$ is denoted by $\mathcal{M}(I)$. Let $f \in \mathcal{M}(I)$ and suppose that the minimal choice for the c_i's is made so that f is not monotone in any neighbourhood of c_i for $1 \leq i \leq m$. Then the points $c_1, c_2, \ldots, c_m$ are called the *turning points* of f and the subintervals $I_j, j = 1, 2, \ldots, m+1$, the *laps* of f. An $f \in \mathcal{M}(I)$ with exactly one turning point is called a *unimodal map*. For each $f \in \mathcal{M}(I)$, let $T(f)$ denote the set of turning points of f, $|T(f)|$ the number of turning points of f, $L(f)$ the set of laps of f, and $l(f)$ the number of laps of f.

The following proposition gives an equivalent definition for the turning point of a piecewise monotone map.

Proposition 2.1 ([17]). *Let $f \in \mathcal{M}(I)$. A point $c \in (a, b)$ is a turning point of f if and only if for every $\epsilon > 0$ there exist $x, y \in I$ with $x \neq y, |x - c| < \epsilon$ and $|y - c| < \epsilon$ such that $f(x) = f(y)$.*

For each $f \in \mathcal{C}(I)$ and $k \geq 0$, let f^k denote the k-th order iterate of f defined recursively by

$$f^0 = \mathrm{id}_I \text{ and } f^k = f \circ f^{k-1},$$

where id_I is the identity map on I. The set $\mathcal{M}(I)$ is closed with respect to composition of maps. In fact, if $f, g \in \mathcal{M}(I)$, then

$$T(f \circ g) = \big(T(g) \cup g^{-1}\big(T(f)\big)\big) \cap (a, b). \tag{1}$$

So, in particular, if $f \in \mathcal{M}(I)$, then $f^k \in \mathcal{M}(I)$. Moreover

$$T(f^k) = \{x \in (a, b) : f^l(x) \in T(f) \text{ for some } 0 \leq l \leq k-1\} \tag{2}$$

for each $k \in \mathbb{N}$. On the other hand, if $f, g \in \mathcal{C}(I)$ such that $f \circ g \in \mathcal{M}(I)$, then $g \in \mathcal{M}(I)$. In particular, if $f \in \mathcal{C}(I)$ such that $f^k \in \mathcal{M}(I)$ for some $k \in \mathbb{N}$, then $f \in \mathcal{M}(I)$ (see Lemma 2.2 in [17]).

Henceforth, throughout this section, let $f \in \mathcal{M}(I)$ with turning points c_1, $c_2, \ldots, c_m$ and laps $I_1 = [c_0, c_1], I_2 = [c_1, c_2], \ldots, I_{m+1} = [c_m, c_{m+1}]$. We recall several formal power series associated with the map f, which serves as raw ingredients to develop this kneading theory.

Let V be the $(m+1)$-dimensional vector space over the field of rationals $\mathbb{Q}$ with an ordered basis the set of formal symbols $I_1, I_2, \ldots, I_{m+1}$ and $V[[t]]$ be the $\mathbb{Q}[[t]]$-module consisting of all formal power series with coefficients in V. For each $x \in I$ and $k \geq 0$, let

$$A(f^k(x)) := \begin{cases} I_j & \text{if} \quad f^k(x) \in I_j \text{ and } f^k(x) \notin T(f),\ 1 \leq j \leq m+1, \\ C_i & \text{if} \quad f^k(x) = c_i,\ 1 \leq i \leq m, \end{cases}$$

where $C_i := \frac{1}{2}(I_i + I_{i+1})$ for $1 \leq i \leq m$, and let

$$A(x, f; t) := \sum_{k \geq 0} A(f^k(x)) t^k.$$

The symbol $A(x)$ is called the *address* of x, and for $k \geq 0$, we denote $A(f^k(x))$ by $A_k(x, f)$.

For any subinterval I' of I, we write $f \nearrow I'$ (resp. $f \searrow I'$) if f is strictly increasing (resp. strictly decreasing) on I'. For each symbol I_j, define the *sign* by

$$\epsilon(I_j) = \begin{cases} +1 & \text{if} \quad f \nearrow I_j, \\ -1 & \text{if} \quad f \searrow I_j, \end{cases}$$

and for each of the vector C_j corresponding to the turning point c_j, let $\epsilon(C_j) := 0$.

For each $x \in I$, let $\epsilon(x, f^k) := \epsilon(A_k(x, f))$ for $k \geq 0$, and

$$\theta(x, f^0) := A_0(x, f) \text{ and } \theta(x, f^k) := \left(\prod_{l=0}^{k-1} \epsilon(x, f^l)\right) A_k(x, f) \text{ for } k \geq 1.$$

The corresponding formal power series are defined by

$$\epsilon(x, f; t) = \sum_{k \geq 0} \epsilon_k(x, f) t^k \text{ and } \theta(x, f; t) = \sum_{k \geq 0} \theta_k(x, f) t^k,$$

where $\epsilon_k(x, f)$ and $\theta_k(x, f)$ denote $\epsilon(x, f^k)$ and $\theta(x, f^k)$, respectively.

Consider $V[[t]]$ in the formal power series topology in which the submodules $t^k V[[t]]$ form a basis for the neighbourhoods of zero. For each $x \in [a, b)$ and $k \geq 0$, let

$$x+ := \mathrm{id}_I(x+),\ A(f^k(x+)) := \lim_{y \to x^+} A_k(y, f),\ \epsilon_k(x+, f) := \lim_{y \to x^+} \epsilon_k(y, f)$$

and $\theta_k(x+, f) := \lim_{y \to x^+} \theta_k(y, f)$. Also, for each $x \in (a, b]$ and $k \geq 0$, let

$$x- := \mathrm{id}_I(x-),\ A(f^k(x-)) := \lim_{y \to x^-} A_k(y, f),\ \epsilon_k(x-, f) := \lim_{y \to x^-} \epsilon_k(y, f)$$

and $\theta_k(x-, f) := \lim_{y \to x^-} \theta_k(y, f)$. Then it follows that

$$\epsilon_k(x+, f) = \epsilon(A_k(x+, f)) \text{ for } x \in [a, b),\ k \geq 0,$$

and

$$\epsilon_k(x-, f) = \epsilon(A_k(x-, f)) \text{ for } x \in (a, b],\ k \geq 0,$$

where $A_k(x+, f)$ and $A_k(x-, f)$ denote $A(f^k(x+))$ and $A(f^k(x-))$, respectively. Moreover

$$A_k(c_i+, f) = A_k(c_i-, f) \tag{3}$$

for $1 \leq i \leq m$ and $k \in \mathbb{N}$. For each $x \in [a, b)$, let

$$\theta(x+, f) := \lim_{y \to x^+} \theta(y, f)$$

and for each $x \in (a, b]$, let

$$\theta(x-, f) := \lim_{y \to x^-} \theta(y, f).$$

Then

$$\theta(x+, f; t) = \sum_{k \geq 0} \theta_k(x+, f) t^k$$

for $x \in [a, b)$ and

$$\theta(x-, f; t) = \sum_{k \geq 0} \theta_k(x-, f) t^k$$

for $x \in (a, b]$.

Definition ([11])**.** For $1 \leq i \leq m$, the formal power series $\theta(c_i+, f; t) - \theta(c_i-, f; t)$ is called the i^{th} *kneading increment* $\nu(c_i, f; t)$ of f. The matrix $N(f; t) = [N_{ij}(f; t)]$ of order $m \times (m + 1)$, with entries in $\mathbb{Z}[[t]]$, obtained by setting

$$\nu(c_i, f; t) = N_{i1}(f; t) I_1 + N_{i2}(f; t) I_2 + \cdots + N_{i,m+1}(f; t) I_{m+1} \text{ for } 1 \leq i \leq m$$

is called the *kneading matrix* of f.

We can write the matrix $N(f;t)$ as a power series $\sum_{k\geq 0}[N^k_{ij}(f;t)]t^k$ where the coefficients $[N^0_{ij}(f;t)]$, $[N^1_{ij}(f;t)]$, ... are matrices with integer entries. For $k=0$, the matrix $[N^0_{ij}(f;t)]$ is given by

$$N^0_{ij}(f;t) = \begin{cases} -1 & \text{if } j=i,\ 1\leq i\leq m, 1\leq j\leq m+1, \\ 1 & \text{if } j=i+1,\ 1\leq i\leq m, 1\leq j\leq m+1, \\ 0 & \text{otherwise,} \end{cases}$$

and it is indeed independent of the map f. Let $N_k(f;t)$ denote the matrix $[N^k_{ij}(f;t)]$ for $k\geq 0$, and let $M(f;t)$ denote the matrix $\sum_{k\geq 1} N_k(f;t)t^k$.

For $1\leq j\leq m+1$, let $N^{(j)}(f;t)$ denote the $m\times m$ matrix obtained by deleting the j^{th} column of $N(f;t)$. Then the power series $(-1)^{j+1}\big(1-\epsilon(I_j)t\big)^{-1}\det\big(N^{(j)}(f;t)\big)$ is indeed independent of choice of j for $1\leq j\leq m+1$, and this common expression is denoted by $D(f;t)$, called the *kneading determinant* of f ([11]).

Note that the power series $D(f;t)$ has the leading coefficient $+1$, and hence is a unit in the ring $\mathbb{Z}[[t]]$. Moreover, it is proved in [11] that

$$D(f;t) = 1 + \sum_{k\geq 1}\left(\prod_{l=1}^{k}\epsilon_l(c+,f)\right)t^k,$$

whenever f is a unimodal map with turning point c.

Example 2.2. For the *Tent map* $T:[0,1]\to[0,1]$ defined by $T(x)=1-|1-2x|$, we have

$$D(T;t) = 1-t-t^2-t^3-\cdots$$

and $N(T;t) = [N_{11}(T;t),\ N_{12}(T;t)]$, where $N_{11}(T;t) = -1+2t^2+2t^3+\cdots$ and $N_{12}(T;t)=1-2t$.

Example 2.3. Consider the unimodal map $f:[-2,2]\to[-2,2]$ defined by

$$f(x) = \frac{x^2-4}{2},\ \forall x\in[-2,2].$$

Then $D(f;t)=(1+t)^{-1}$ and $N(f;t)=[N_{11}(f;t),\ N_{12}(f;t)]$, where $N_{11}(f;t) = -1+2t-2t^2+\cdots$ and $N_{12}(f;t)=1$.

3. Topological conjugacy and kneading theory

Let $f\in\mathcal{C}(I)$ and $g\in\mathcal{C}(J)$, where $J=[c,d]$, a compact interval in $\mathbb{R}$. As in [8], we say that f is *topologically h-conjugate* (or simply *conjugate*) to g if there exists a homeomorphism $h:I\to J$ such that $h\circ f = g\circ h$. In this case h is called a *topological conjugacy.* One of the main results in kneading theory is the following.

Theorem 3.1 ([11])**.** *Let $f\in\mathcal{M}(I)$ and $g\in\mathcal{M}(J)$. If f is h-conjugate to g, where h preserves orientation, then $N(f;t)=N(g;t)$ and $D(f;t)=D(g;t)$.*

In this section, we consider if the above theorem is true when h is orientation-reversing. Henceforth, for the entirety of this paper, unless otherwise stated, let $f \in \mathcal{M}(I)$ and $g \in \mathcal{M}(J)$ with $T(f) = \{c_1, c_2, \ldots, c_m\}$, $T(g) = \{d_1, d_2, \ldots, d_n\}$, $L(f) = \{I_1, I_2, \ldots, I_{m+1}\}$ and $L(g) = \{J_1, J_2, \ldots, J_{n+1}\}$, where

$$I_j = [c_{j-1}, c_j] \text{ for } 1 \le j \le m+1 \text{ and } J_i = [d_{i-1}, d_i] \text{ for } 1 \le i \le n+1$$

with $c_0 = a, c_{m+1} = b, d_0 = c$ and $d_{n+1} = d$.

Lemma 3.2. *If f is h-conjugate to g, then $|T(f)| = |T(g)|$.*

Proof. To prove $n = m$, it suffices to show that

$$T(g) = \{h(c_1), h(c_2), \ldots, h(c_m)\}.$$

Consider any $i \in \{1, 2, \ldots, m\}$. Let $\epsilon > 0$ be arbitrary and $U := (h(c_i) - \epsilon, h(c_i) + \epsilon)$. Choose $\delta > 0$ such that $(c_i - \delta, c_i + \delta) \subseteq h^{-1}(U)$. Since $c_i \in T(f)$, by Proposition 2.1, there exist $x, y \in I$ with $c_i - \delta < x < c_i < y < c_i + \delta$ such that $f(x) = f(y)$, implying that $g\big(h(x)\big) = (g \circ h)(x) = (h \circ f)(x) = h\big(f(x)\big) = h\big(f(y)\big) = (h \circ f)(y) = (g \circ h)(y) = g\big(h(y)\big)$. Since $x, y \in (c_i - \delta, c_i + \delta) \subseteq h^{-1}(U)$, we have $h(x), h(y) \in U$. Further, either $h(x) < h(c_i) < h(y)$ or $h(y) < h(c_i) < h(x)$ according as h preserves or reverses orientation, respectively. Thus in any case, there exist $u, v \in J$ with $h(c_i) - \epsilon < u < h(c_i) < v < h(c_i) + \epsilon$ such that $g(u) = g(v)$. Hence, by Proposition 2.1, we have $h(c_i) \in T(g)$.

On the other hand, let $w \in T(g)$ and suppose that $w \neq h(c_i)$ for $1 \le i \le m$. Then $h^{-1}(w) \neq c_i$ for $1 \le i \le m$, implying that $h^{-1}(w) \notin T(f)$. So by Proposition 2.1, there exists $\epsilon > 0$ such that f is strictly monotone on $W := \big(h^{-1}(w) - \epsilon, h^{-1}(w) + \epsilon\big)$. Let $f \nearrow W$. A proof for the case that $f \searrow W$ is similar.

Claim: g is strictly monotone on the neighbourhood $h(W)$ of w in J.

Since h^{-1} is continuous, clearly $h(W)$ is a neighbourhood of w in J. Consider any $u, v \in h(W)$ with $u < v$. Then $u = h(x)$ and $v = h(y)$ for some $x, y \in W$.

If h preserves orientation, then we have $x < y$, implying that $f(x) < f(y)$, because $f \nearrow W$. Now $g(u) = g\big(h(x)\big) = (g \circ h)(x) = (h \circ f)(x) = h\big(f(x)\big) < h\big(f(y)\big) = (h \circ f)(y) = (g \circ h)(y) = g\big(h(y)\big) = g(v)$, and therefore $g \nearrow h(W)$. In the case that h reverses orientation, we have $x > y$, which implies that $f(x) > f(y)$, because $f \nearrow W$. Therefore $g(u) = g\big(h(x)\big) = (g \circ h)(x) = (h \circ f)(x) = h\big(f(x)\big) < h\big(f(y)\big) = (h \circ f)(y) = (g \circ h)(y) = g\big(h(y)\big) = g(v)$, so that $g \nearrow h(W)$. Thus the claim holds, which implies that there exists $\delta > 0$ such that g is strictly monotone on $(w - \delta, w + \delta)$, a contradiction to our assumption that $w \in T(g)$. So $w = h(c_i)$ for some $i, 1 \le i \le m$. Hence $h(c_1), h(c_2), \ldots, h(c_m)$ are precisely the turning points of g. □

Lemma 3.3. *Let f be h-conjugate to g.*

(i) *If h is orientation-preserving, then $d_i = h(c_i)$ for $1 \le i \le m$ and $J_j = h(I_j)$ for $1 \le j \le m+1$.*

(ii) *If h is orientation-reversing, then* $d_i = h(c_{m+1-i})$ *for* $1 \le i \le m$ *and* $J_j = h(I_{m+2-j})$ *for* $1 \le j \le m+1$.

Proof. Let f be h-conjugate to g. Then by Lemma 3.2, we have $n = m$ and $T(g) = \{h(c_1), h(c_2), \ldots, h(c_m)\}$. If h preserves orientation, then $h(c_1) < h(c_2) < \cdots < h(c_m)$, implying that $d_i = h(c_i)$ for $1 \le i \le m$. Also, by intermediate value theorem, we have $J_j = h(I_j)$ for $1 \le j \le m+1$. If h reverses orientation, then $h(c_m) < h(c_{m-1}) < \cdots < h(c_1)$, implying that $d_i = h(c_{m+1-i})$ for $1 \le i \le m$. Again, by intermediate value theorem, it follows that $J_j = h(I_{m+2-j})$ for $1 \le j \le m+1$. □

Lemma 3.4. *Let f be h-conjugate to g. Then the following statements are true.*

(i) *If h is orientation-reversing, then* $\epsilon(J_j) = \epsilon(I_{m+2-j})$ *for* $1 \le j \le m+1$.
(ii) *If h is orientation-preserving, then* $\epsilon(J_j) = \epsilon(I_j)$ *for* $1 \le j \le m+1$.
(iii) *For* $1 \le i \le m$ *and* $k \ge 0$,

$$\epsilon(A((h \circ f^k)(c_i+))) = \epsilon_k(c_i+, f) \tag{4}$$

and

$$\epsilon(A((h \circ f^k)(c_i-))) = \epsilon_k(c_i-, f). \tag{5}$$

Proof. Since f is h-conjugate to g, by Lemma 3.2, we have $n = m$ and $T(g) = \{h(c_1), h(c_2), \ldots, h(c_m)\}$.

(i) Consider any $j \in \{1, 2, \ldots, m+1\}$. In the case that $\epsilon(I_{m+2-j}) = +1$, we need to show that $g \nearrow J_j$. So, let $u, v \in J_j$ with $u < v$. Since $h : I \to J$ is an injective map, by result (ii) of Lemma 3.3, it follows that $h : I_{m+2-j} \to J_j$ is a bijective map. Let $x, y \in I_{m+2-j}$ such that $h(x) = u$ and $h(y) = v$. Since $h \searrow I_{m+2-j}$, we have $y < x$, and hence $f(y) < f(x)$, because $f \nearrow I_{m+2-j}$. This implies that $(h \circ f)(x) < (h \circ f)(y)$, since h reverses orientation. Therefore $g(u) = (h \circ f \circ h^{-1})(u) = (h \circ f)(h^{-1}(u)) = (h \circ f)(x) < (h \circ f)(y) = (h \circ f)(h^{-1}(v)) = (h \circ f \circ h^{-1})(v) = g(v)$. Thus $g \nearrow J_j$, and hence $\epsilon(J_j) = +1$. The proof of the equality $\epsilon(J_j) = \epsilon(I_{m+2-j})$ in the case that $\epsilon(I_{m+2-j}) = -1$ is similar.

(ii) This is similar to (i).

(iii) First we consider the case that h reverses orientation. Let $1 \le i \le m$ and $k \ge 0$.

Claim: $A((h \circ f^k)(c_i+)) = J_j$ whenever $j \in \{1, 2, \ldots, m+1\}$ such that $A_k(c_i+, f) = I_{m+2-j}$.

Let $A_k(c_i+, f) = I_{m+2-j}$, where $j \in \{1, 2, \ldots, m+1\}$. Choose $\delta > 0$ such that $A(f^k(x)) = I_{m+2-j}$ for $c_i < x < c_i + \delta$. Then $f^k(x) \in I_{m+2-j}$, implying that $(h \circ f^k)(x) \in h(I_{m+2-j})$ for $c_i < x < c_i + \delta$. By result (ii) of Lemma 3.3, we have $h(I_{m+2-j}) = J_j$. So $(h \circ f^k)(x) \in J_j$, and hence $A((h \circ f^k)(x)) = J_j$ for $c_i < x < c_i + \delta$, implying that $A((h \circ f^k)(c_i+)) = J_j$. This proves the claim. By result (i), we have $\epsilon(J_j) = \epsilon(I_{m+2-j})$. So by the claimed fact, it follows that $\epsilon(A((h\circ f^k)(c_i+)))=\epsilon(A_k(c_i+,f))$, proving (4). The proofs of (5), and (4) and (5) in the case that h is orientation-preserving are similar.

This proves the lemma. □

Corollary 3.5. *Let m be a positive even integer. If $f\nearrow I_1$ and $g\searrow J_1$, then f is not conjugate to g.*

Proof. Since $f\nearrow I_1$ and $g\searrow J_1$, we have $\epsilon(I_1)=+1$ and $\epsilon(J_1)=-1$. Suppose that there exists a conjugacy h of f and g. If h preserves orientation, then by result (ii) of Lemma 3.4, we have $\epsilon(J_1)=\epsilon(I_1)=+1$, which is a contradiction. If h reverses orientation, then by result (i) of Lemma 3.4, $\epsilon(J_{m+1})=\epsilon(I_1)=+1$. This implies that $\epsilon(J_j)=+1$ for every odd $j\in\{1,2,\dots,m+1\}$, because $m+1$ is odd. So, in particular $\epsilon(J_1)=+1$, again a contradiction. Hence f is not conjugate to g. □

We now prove our main results.

Theorem 3.6. *Let f be h-conjugate to g, given that h is orientation-reversing. Then*

$$N(g;t)=-S_mN(f;t)S_{m+1},$$

where $S_m=[s_{ij}]_{m\times m}$ with

$$s_{ij}=\begin{cases}1 & \text{if } i+j=m+1,\\ 0 & \text{otherwise.}\end{cases}$$

Proof. Consider any $i\in\{1,2,\dots,m\}$ and $l\in\mathbb{N}$. We have

$$A((g^l\circ h)(c_{m+1-i}-))=\lim_{y\to c^-_{m+1-i}}A((g^l\circ h)(y))=\lim_{y\to c^-_{m+1-i}}A(g^l(h(y))). \tag{6}$$

Since h is an orientation-reversing homeomorphism, it follows that $y\to c^-_{m+1-i}$ if and only if $h(y)\to h(c_{m+1-i})^+$. Therefore

$$\begin{aligned}\lim_{y\to c^-_{m+1-i}}A(g^l(h(y)))&=\lim_{h(y)\to h(c_{m+1-i})^+}A(g^l(h(y)))\\&=A(g^l(h(c_{m+1-i})+)),\end{aligned}$$

implying by (6) that

$$A((g^l\circ h)(c_{m+1-i}-))=A(g^l(h(c_{m+1-i})+)).$$

Since h reverses orientation, by result (ii) of Lemma 3.3, we have $d_i=h(c_{m+1-i})$, and therefore

$$A_l(d_i+,g)=A((g^l\circ h)(c_{m+1-i}-)). \tag{7}$$

Since f is h-conjugate to g, we have $g^l\circ h=h\circ f^l$, implying that

$$A((g^l\circ h)(c_{m+1-i}-))=A((h\circ f^l)(c_{m+1-i}-)). \tag{8}$$

Using (8) in (7), we have

$$A_l(d_i+,g)=A((h\circ f^l)(c_{m+1-i}-)), \tag{9}$$

and hence

$$\epsilon_l(d_i+, g) = \epsilon(A((h \circ f^l)(c_{m+1-i}-))). \tag{10}$$

Now

$$\begin{aligned} A((g^l \circ h)(c_{m+1-i}+)) &= \lim_{y \to c^+_{m+1-i}} A(g^l(h(y))) \\ &= \lim_{h(y) \to h(c_{m+1-i})^-} A(g^l(h(y))) \\ &= A(g^l(h(c_{m+1-i})-)) = A_l(d_i-, g). \end{aligned} \tag{11}$$

Also, by (3) we have $A_l(d_i-, g) = A_l(d_i+, g)$. Therefore (11) implies that

$$\begin{aligned} A((g^l \circ h)(c_{m+1-i}+)) &= A_l(d_i+, g) \\ &= A((g^l \circ h)(c_{m+1-i}-)) \text{ (by using (7))}. \end{aligned}$$

Thus

$$A((h \circ f^l)(c_{m+1-i}-)) = A((h \circ f^l)(c_{m+1-i}+)), \tag{12}$$

and hence

$$\epsilon(A((h \circ f^l)(c_{m+1-i}-))) = \epsilon(A((h \circ f^l)(c_{m+1-i}+))). \tag{13}$$

Moreover, by (4) we have

$$\epsilon(A((h \circ f^l)(c_{m+1-i}+))) = \epsilon(A(f^l(c_{m+1-i}+))), \tag{14}$$

and

$$\epsilon(A(h(c_{m+1-i}-))) = \epsilon(A(c_{m+1-i}-)) = -\epsilon(A(c_{m+1-i}+)), \tag{15}$$

where the last equality in (15) is true, because $A(c_{m+1-i}-)$ and $A(c_{m+1-i}+)$ are two consecutive laps of f. Thus, from (10) and (9) we obtain

$$\begin{aligned} \theta_k(d_i+, g) = &\left(\prod_{l=0}^{k-1} \epsilon_l(d_i+, g)\right) A_k(d_i+, g) \\ = &\left(\prod_{l=0}^{k-1} \epsilon(A((h \circ f^l)(c_{m+1-i}-)))\right) A((h \circ f^k)(c_{m+1-i}-)) \\ = &- \epsilon(A(c_{m+1-i}+)) \left(\prod_{l=1}^{k-1} \epsilon(A((h \circ f^l)(c_{m+1-i}+)))\right) \\ &A((h \circ f^k)(c_{m+1-i}+)), \end{aligned}$$

for each $k \in \mathbb{N}$, where the last equality follows from (15), (13) and (12). Using (14) in this equation, we have

$$\theta_k(d_i+, g) = -\epsilon(A(c_{m+1-i}+)) \left(\prod_{l=1}^{k-1} \epsilon(A(f^l(c_{m+1-i}+)))\right) A(h(f^k(c_{m+1-i}+)))$$

$$
\begin{aligned}
&= -\left(\prod_{l=0}^{k-1} \epsilon(A(f^l(c_{m+1-i}+)))\right) A(h(f^k(c_{m+1-i}+))) \\
&= -\left(\prod_{l=0}^{k-1} \epsilon_l(c_{m+1-i}+, f)\right) A(h(f^k(c_{m+1-i}+))) \tag{16}
\end{aligned}
$$

for every $k \in \mathbb{N}$.

Now for a fixed $k \in \mathbb{N}$,

$$A(h(f^k(c_{m+1-i}+))) = J_{m+2-j}$$

whenever $A(f^k(c_{m+1-i}+)) = I_j$ for some $j \in \{1, 2, \ldots, m+1\}$ and conversely. Thus from (16), it follows that the coefficient of J_j in $\theta_k(d_i+, g)$ is equal to negative of the coefficient of I_{m+2-j} in $\theta_k(c_{m+1-i}+, f)$. This holds for every $k \in \mathbb{N}$. By a similar argument, it follows that

$$\theta_k(d_i-, g) = -\left(\prod_{l=0}^{k-1} \epsilon_l(c_{m+1-i}-, f)\right) A(h(f^k(c_{m+1-i}-)))$$

for $1 \leq i \leq m$ and $k \in \mathbb{N}$, and the coefficient of J_j in $\theta_k(d_i-, g)$ is equal to negative of the coefficient of I_{m+2-j} in $\theta_k(c_{m+1-i}-, f)$. Therefore

$$
\begin{aligned}
M(g;t) = [M_{ij}(g;t)] &= [-M_{m+1-i,m+2-j}(f;t)] \\
&= -[M_{m+1-i,m+2-j}(f;t)]. \tag{17}
\end{aligned}
$$

Now, an easy computation shows that

$$[M_{m+1-i,m+2-j}(f;t)] = S_m[M_{ij}(f;t)]S_{m+1} = S_m M(f;t) S_{m+1}. \tag{18}$$

Also, $N_0(g;t) = -S_m N_0(f;t) S_{m+1}$. Thus, from (17) and (18) we have

$$
\begin{aligned}
N(g;t) &= N_0(g;t) + M(g;t) \\
&= -S_m N_0(f;t) S_{m+1} - S_m[M_{ij}(f;t)]S_{m+1} \\
&= -S_m N(f;t) S_{m+1}.
\end{aligned}
$$

This completes the proof. □

Theorem 3.7. *Let f be h-conjugate to g, given that h is orientation-reversing. Then $D(g;t) = D(f;t)$.*

Proof. First, consider the case that $f \nearrow I_1$. We have

$$
\begin{aligned}
D(f;t) &= (-1)^{1+1}\big(1 - \epsilon(I_1)t\big)^{-1} \det\big(N^{(1)}(f;t)\big) \\
&= (1-t)^{-1} \det\big(N^{(1)}(f;t)\big), \tag{19}
\end{aligned}
$$

and

$$
\begin{aligned}
D(g;t) &= (-1)^{(m+1)+1}\big(1 - \epsilon(J_{m+1})t\big)^{-1} \det\big(N^{(m+1)}(g;t)\big) \\
&= (-1)^{m+2}\big(1 - \epsilon(J_{m+1})t\big)^{-1} \det\big(N^{(m+1)}(g;t)\big). \tag{20}
\end{aligned}
$$

Since f is h-conjugate to g, where h is orientation-reversing, by Theorem 3.6, we have $N(g;t) = -S_m N(f;t) S_{m+1}$, and therefore

$$N^{(m+1)}(g;t) = -S_m N^{(1)}(f;t) S_m,$$

implying that

$$\begin{aligned}\det\left(N^{(m+1)}(g;t)\right) &= (-1)^m (\det S_m)^2 \det\left(N^{(1)}(f;t)\right)\\ &= (-1)^m \det\left(N^{(1)}(f;t)\right),\end{aligned}$$

where the last equality follows by noting that $\det S_m = (-1)^{\lfloor \frac{m}{2} \rfloor}$. Hence, from (19) and (20) we obtain

$$D(g;t) = \left(1 - \epsilon(J_{m+1})t\right)^{-1}(1-t)D(f;t). \tag{21}$$

From (i) of Lemma 3.4, $\epsilon(J_{m+1}) = \epsilon(I_1)$, and therefore $\epsilon(J_{m+1}) = +1$, because $\epsilon(I_1) = +1$. Then (21) implies that $D(g;t) = (1-t)^{-1}(1-t)D(f;t) = D(f;t)$.

Next, in the case that $f \searrow I_1$, we have $g \nearrow J_1$. Also, g is h^{-1}-conjugate to f such that h^{-1} is orientation-reversing. Therefore, by a similar argument as in the above case, we obtain $D(g;t) = D(f;t)$. □

Remark 3.8. (i) Since $-S_m N(f;t) S_{m+1} \neq N(f;t)$, by Theorem 3.6 we have $N(g;t) \neq N(f;t)$ whenever f is h-conjugate to g, given that h is orientation-reversing. Therefore the kneading matrix is not an invariant under orientation-reversing conjugacy. However, since the matrices S_m and S_{m+1} are invertible, Theorems 3.1 and 3.6 together imply that the matrices $N(f;t)$ and $N(g;t)$ are equivalent whenever f and g are topologically conjugates.

(ii) The converses of Theorems 3.6 and 3.7 are not true in general. For example, consider the maps f, $g : [0,1] \to [0,1]$ defined by

$$f(x) = \begin{cases} x & \text{if } 0 \leq x \leq \frac{1}{2},\\ 1-x & \text{if } \frac{1}{2} \leq x \leq 1,\end{cases}$$

and

$$g(x) = 1 - 2x(1-x),\ \forall x \in [0,1].$$

Then $f, g \in \mathcal{M}([0,1])$ such that $T(f) = \{c_1\}$, $L(f) = \{I_1, I_2\}$, $T(g) = \{d_1\}$ and $L(g) = \{J_1, J_2\}$, where $c_1 = d_1 = \frac{1}{2}$, $I_1 = J_1 = [0, \frac{1}{2}]$ and $I_2 = J_2 = [\frac{1}{2}, 1]$. Since $f(I) \subseteq I_1$, we have

$$A_0(c_1+, f) = I_2 \text{ and } A_k(c_1+, f) = I_1 \text{ for } k \geq 1.$$

Also, $\epsilon_0(c_1+, f) = -1$ and $\epsilon_k(c_1+, f) = 1$ for $k \geq 1$. Therefore $\theta_0(c_1+, f) = I_2$ and $\theta_k(c_1+, f) = -I_1$ for $k \geq 1$, implying that

$$\theta(c_1+, f; t) = I_2 - I_1 t - I_1 t^2 - \cdots = (-t - t^2 - \cdots)I_1 + I_2.$$

Further, $A_0(c_1-, f) = I_1$, and since $A_k(c_1-, f) = A_k(c_1+, f)$, we get that $A_1(c_1-, f) = I_1$ for $k \geq 1$. Moreover, $\epsilon_k(c_1-, f) = 1$ for $k \geq 0$. Therefore $\theta_0(c_1-, f) = I_1$ for $k \geq 0$, implying that

$$\theta(c_1-, f; t) = I_1 + I_1 t + I_1 t^2 + \cdots = (1 + t + t^2 + \cdots)I_1.$$

Thus

$$\nu(c_1, f; t) = (I_2 - I_1) - 2I_1t - 2I_1t^2 - \cdots = (-1 - 2t - 2t^2 - \cdots)I_1 + I_2,$$

and hence

$$N(f;t) = \left[-1 - 2t - 2t^2 - \cdots,\ 1\right]_{1\times 2}.$$

Since $g(J) \subseteq J_2$, by a similar argument as above, we obtain

$$\nu(d_1, g; t) = (J_2 - J_1) + 2J_2t + 2J_2t^2 + \cdots = -J_1 + (1 + 2t + 2t^2 + \cdots)J_2.$$

Therefore

$$N(g;t) = \left[-1,\ 1 + 2t + 2t^2 + \cdots\right]_{1\times 2}.$$

Clearly, $N(g;t) = -S_1N(f;t)S_2$ and $D(g;t) = D(f;t)$. However, $[0, \frac{1}{2}]$ and $\{\frac{1}{2}, 1\}$ are precisely the set of fixed points of f and g, respectively. Therefore f and g are not topologically conjugates, because g has only two fixed points whereas f has uncountably many.

(iii) It is evident from the proof of Theorem 3.7 that $D(g;t) = D(f;t)$ whenever f and g satisfy the relation $N(g;t) = -S_mN(f;t)S_{m+1}$.

(iv) Although that of Milnor and Thurston motivated our work, we can indeed deduce their result from ours as follows. Let f be h-conjugate to g, given that h is orientation-preserving. Define $f_1 : I \to I$ by

$$f_1(x) = a + b - f(a + b - x), \quad \forall x \in I.$$

Then $f_1 \in \mathcal{M}(I)$, f is h_1-conjugate to f_1, and f_1 is $h \circ h_1$-conjugate to g, where $h_1 : I \to I$ is defined by $h_1(x) = a + b - x, \forall x \in I$. Since h_1 and $h \circ h_1$ are orientation-reversing, by Theorem 3.6, we have $N(f_1;t) = -S_mN(f;t)S_{m+1}$ and $N(g;t) = -S_mN(f_1;t)S_{m+1}$ such that $m = |T(f)|(= |T(f_1)| = |T(g)|)$. Therefore

$$N(g;t) = -S_m(-S_mN(f;t)S_{m+1})S_{m+1} = S_m^2N(f;t)S_{m+1}^2 = N(f;t),$$

proving Theorem 3.1. On the other hand, we cannot use the above approach to deduce our result from theirs, because a orientation-reversing homeomorphism can never be equal to the composition of two or more orientation-preserving homeomorphisms.

(v) We say that f is *topologically h-semiconjugate* to g if there exists a continuous onto map $h : I \to J$ such that $h \circ f = g \circ h$. As proved in Remark 3.16 of [4], the map $f \in \mathcal{M}([0, 1])$ defined by

$$f(x) = \begin{cases} \frac{8}{3}x & \text{if } 0 \le x \le \frac{3}{8}, \\ 2 - \frac{8}{3}x & \text{if } \frac{3}{8} \le x \le \frac{3}{4}, \\ 2x - \frac{3}{2} & \text{if } \frac{3}{4} \le x \le 1 \end{cases}$$

with $T(f) = \{\frac{3}{8}, \frac{3}{4}\}$ is h-semiconjugate to the tent map T defined in Example 2.2, where a semiconjugacy h is given in Figure 4 of [4]. However, f and T do not have the same number of turning points. Further, $h(\frac{3}{8}) \neq \frac{1}{2}$ although $\frac{3}{8}$ is a turning point of f, and $h(\frac{9}{32}) = \frac{1}{2}$ although $\frac{9}{32}$ is not a turning point of f. Therefore h does not map a turning point of f to the turning point of T and maps a point that is not a turning point of f to the turning point of T. Hence the results of Lemmas 3.2, 3.3 and 3.4, and Theorems 3.6 and 3.7 are not true in general whenever f is h-semiconjugate to g.

Example 3.9. Let $f, g : [0,1] \to [0,1]$ be the maps defined by

$$f(x) = T^2(x), \ \forall x \in [0,1]$$

and

$$g(x) = 64x^4 - 128x^3 + 80x^2 - 16x + 1, \ \forall x \in [0,1],$$

where T is the Tent map defined in Example 2.2. Then $f, g \in \mathcal{M}([0,1])$ such that

$$T(f) = \left\{c_1 = \tfrac{1}{4}, c_2 = \tfrac{1}{2}, c_3 = \tfrac{3}{4}\right\}, \ T(g) = \left\{d_1 = \tfrac{2-\sqrt{2}}{4}, d_2 = \tfrac{1}{2}, d_3 = \tfrac{2+\sqrt{2}}{4}\right\},$$

$$L(f) = \left\{I_1 = \left[0, \tfrac{1}{4}\right], I_2 = \left[\tfrac{1}{4}, \tfrac{1}{2}\right], I_3 = \left[\tfrac{1}{2}, \tfrac{3}{4}\right], I_4 = \left[\tfrac{3}{4}, 1\right]\right\}$$

and

$$L(g) = \left\{J_1 = \left[0, \tfrac{2-\sqrt{2}}{4}\right], J_2 = \left[\tfrac{2-\sqrt{2}}{4}, \tfrac{1}{2}\right], J_3 = \left[\tfrac{1}{2}, \tfrac{2+\sqrt{2}}{4}\right], J_4 = \left[\tfrac{2+\sqrt{2}}{4}, 1\right]\right\}.$$

Moreover, f is h-conjugate to g, where

$$h(x) = \frac{2}{\pi}\arcsin(\sqrt{1-x}), \ \forall x \in [0,1].$$

We have $f(c_1) = f(c_3) = 1$, $f(c_2) = 0$, $\epsilon(I_1) = \epsilon(I_3) = +1$ and $\epsilon(I_2) = \epsilon(I_4) = -1$. Since $f(0) = f(1) = 0$, we obtain

$$f^k(c_i) = \begin{cases} 0 & \text{if } i = 1,3 \text{ and } k \geq 2, \\ 0 & \text{if } i = 2 \text{ and } k \geq 1. \end{cases}$$

Now, for each $i \in \{1,3\}$, we have

$$A_0(c_i+, f) = I_{i+1}, \ A_1(c_i+, f) = I_4, \text{ and } A_k(c_i+, f) = I_1 \text{ for } k \geq 2.$$

Therefore $\epsilon_0(c_i+, f) = \epsilon_1(c_i+, f) = -1$ and $\epsilon_k(c_i+, f) = 1$ for $k \geq 2$. Hence $\theta_0(c_i+, f) = I_{i+1}$, $\theta_1(c_i+, f) = -I_4$ and $\theta_k(c_i+, f) = I_1$ for $k \geq 2$. This implies that

$$\theta(c_i+, f; t) = I_{i+1} - I_4 t + I_1 t^2 + I_1 t^3 + \cdots = (t^2 + t^3 + \cdots)I_1 + I_{i+1} - tI_4$$

for $i = 1,3$. Also, $A_0(c_i-, f) = I_i$. Further, since $A_k(c_i-, f) = A_k(c_i+, f)$, we have $A_1(c_i-, f) = I_4$ and $A_k(c_i-, f) = I_1$ for $k \geq 2$. Therefore $\epsilon_0(c_i-, f) = 1$, $\epsilon_1(c_i-, f) = -1$ and $\epsilon_k(c_i-, f) = 1$ for $k \geq 2$. Hence $\theta_0(c_i-, f) = I_i$, $\theta_1(c_i-, f) = I_4$ and $\theta_k(c_i-, f) = -I_1$ for $k \geq 2$. This implies that

$$\theta(c_i-, f; t) = I_i + I_4 t - I_1 t^2 - I_1 t^3 - \cdots = (-t^2 - t^3 - \cdots)I_1 + I_i + tI_4$$

for $i = 1,3$. Therefore

$$\begin{aligned} \nu(c_i, f; t) &= (I_{i+1} - I_i) - 2I_4 t + 2I_1 t^2 + 2I_1 t^3 + \cdots \\ &= (2t^2 + 2t^3 + \cdots)I_1 - I_i + I_{i+1} - 2tI_4 \end{aligned}$$

for $i = 1,3$. By a similar argument as above, we obtain

$$\nu(c_2, f; t) = (2t + 2t^2 + \cdots)I_1 - I_2 + I_3.$$

Thus

$$N(f;t) = \begin{bmatrix} -1+2t^2+2t^3+\cdots & 1 & 0 & -2t \\ 2t+2t^2+\cdots & -1 & 1 & 0 \\ 2t^2+2t^3+\cdots & 0 & -1 & 1-2t \end{bmatrix}_{3\times 4},$$

and hence $D(f;t) = (-1)^{1+1}(1-t)^{-1}(1-4t) = 1-3t-3t^2-\cdots$. Also, by a similar argument, we have

$$N(g;t) = \begin{bmatrix} -1+2t & 1 & 0 & -2t^2-2t^3-\cdots \\ 0 & -1 & 1 & -2t-2t^2-\cdots \\ 2t & 0 & -1 & 1-2t^2-2t^3-\cdots \end{bmatrix}_{3\times 4},$$

and therefore $D(g;t) = (-1)^{4+1}(1-t)^{-1}(-1+4t) = 1-3t-3t^2-\cdots$. Clearly, $D(f;t) = D(g;t)$, and an easy computation shows that $N(g;t) = -S_3N(f;t)S_4$.

4. Consequences of Theorem 3.6

4.1. Reduction of computational complexity

The kneading matrix of a piecewise monotone map f captures much crucial dynamical information of all the iterates of f. On the other hand, if f is a piecewise monotone map, then from (2) it follows that

$$|T(f)| \leq |T(f^2)| \leq |T(f^3)| \leq \cdots,$$

and therefore the 'complexity' of the behaviour of the iterates of f increases with the increase in order of iteration. So, in general, the process of finding the kneading matrix of a piecewise monotone map involves tedious computations. However, Theorem 3.6 is very effective in reducing this computational complexity to a reasonable extent. More precisely, if

$$\mathcal{N}_{\nearrow} = \{N(f;t) : |T(f)| \text{ is odd and } f \text{ is strictly increasing on its first lap}\}$$

and

$$\mathcal{N}_{\searrow} = \{N(g;t) : |T(g)| \text{ is odd and } g \text{ is strictly decreasing on its first lap}\},$$

then from Theorem 3.6 and the following corollary it is evident that any one of the two sets $\mathcal{N}_{\nearrow}$ and $\mathcal{N}_{\searrow}$ determines the other.

Corollary 4.1. *If $f \in \mathcal{M}(I)$ such that $|T(f)|$ is odd and $f \nearrow I_1$, then there exist $g \in \mathcal{M}(J)$ and an orientation-reversing homeomorphism $h : I \to J$ such that $g \searrow J_1$ and f is h-conjugate to g.*

Proof. Let $|T(f)| = m$ for some positive odd integer m. Define $g : J \to J$ by $g = h \circ f \circ h^{-1}$, where $h : I \to J$ is the orientation-reversing homeomorphism defined by

$$h(x) = \frac{c-d}{b-a}x + \frac{bd-ac}{b-a}, \ \forall x \in I.$$

Then $g \in \mathcal{M}(J)$ and f is h-conjugate to g. Since $f \nearrow I_1$, we have $\epsilon(I_1) = +1$, implying by result (i) of Lemma 3.4 that $\epsilon(J_{m+1}) = +1$. Then $\epsilon(J_j) = +1$ for every even $j \in \{1, 2, \ldots, m+1\}$, because $m+1$ is even. This implies $\epsilon(J_1) = -1$, and hence $g \searrow J_1$. □

4.2. Classification up to topological conjugacy

As observed in Section 3, Theorems 3.1 and 3.6 together show that any two piecewise monotone maps f and g with $|T(f)| = |T(g)| = m$ are not topologically conjugates whenever neither of the conditions $N(f;t) = N(g;t)$ and $N(f;t) = -S_m N(g;t) S_{m+1}$ are satisfied. This gives a combinatorial approach to prove the nonexistence of topological conjugacy, and thereby helps to classify the dynamical systems up to topological conjugacy. As an illustration, we have the following.

Example 4.2. Consider the maps $f, g : [0,1] \to [0,1]$ defined by

$$f(x) = \begin{cases} 2.8x + 0.3 & \text{if} \quad 0 \le x \le 0.25, \\ -2.8x + 1.7 & \text{if} \quad 0.25 \le x \le 0.5, \\ 7x - 3.2 & \text{if} \quad 0.5 \le x \le 0.6, \\ -1.125x + 1.675 & \text{if} \quad 0.6 \le x \le 1, \end{cases}$$

and

$$g(x) = \begin{cases} -6.5x + 0.65 & \text{if} \quad 0 \le x \le 0.1, \\ 1.5x - 0.15 & \text{if} \quad 0.1 \le x \le 0.6, \\ -2.5x + 2.25 & \text{if} \quad 0.6 \le x \le 0.9, \\ 4x - 3.6 & \text{if} \quad 0.9 \le x \le 1. \end{cases}$$

Then $f, g \in \mathcal{M}([0,1])$ (see Figure 1) with

$$T(f) = \{c_1 = 0.25, c_2 = 0.5, c_3 = 0.6\},\ T(g) = \{d_1 = 0.1, d_2 = 0.6, d_3 = 0.9\},$$

$$L(f) = \{I_1 = [0, 0.25], I_2 = [0.25, 0.5], I_3 = [0.5, 0.6], I_4 = [0.6, 1]\}$$

and

$$L(g) = \{J_1 = [0, 0.1], J_2 = [0.1, 0.6], J_3 = [0.6, 0.9], J_4 = [0.9, 1]\}.$$

Claim: $N(g;t) \ne N(f;t)$ and $N(g;t) \ne -S_3 N(f;t) S_4$.

Suppose that $N(g;t) = N(f;t)$ or $N(g;t) = -S_3 N(f;t) S_4$. Then an easy computation shows that $N^2_{24}(f;t) = N^2_{24}(g;t)$ or $N^2_{24}(f;t) = -N^2_{21}(g;t)$. Since $f^2(c_2) = f^2(0.5) = 0.86 \in I_4 \setminus T(f)$, we have $A_2(c_2+, f) = A_2(c_2-, f) = I_4$, implying that $N^2_{24}(f;t) \ne 0$. Now $g^2(d_2) = g^2(0.6) = 0.375 \in J_2 \setminus T(g)$, and therefore $A_2(d_2+, g) = A_2(d_2-, g) = J_2$. This implies that $N^2_{22}(g;t) \ne 0$ and hence $N^2_{21}(g;t) = N^2_{24}(g;t) = 0$, because $N^2_{2j}(g;t) \ne 0$ for one and only one $j \in \{1, 2, 3, 4\}$. Thus, neither $N^2_{24}(f;t) = N^2_{24}(g;t)$ nor $N^2_{24}(f;t) = -N^2_{21}(g;t)$, which is a contradiction. This proves the claim, implying by Theorems 3.1 and 3.6 that f and g are not conjugates.

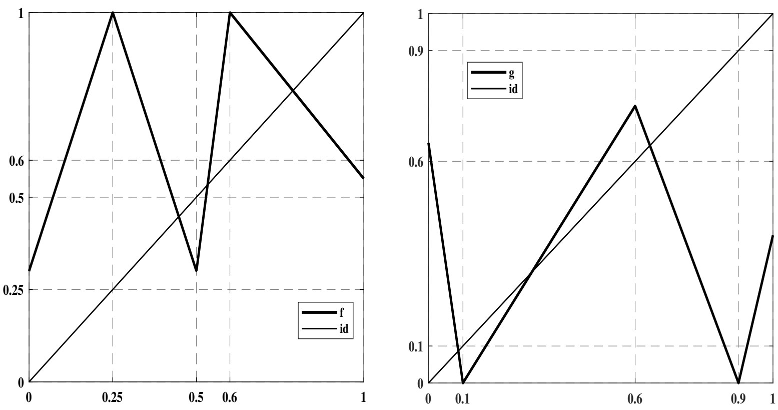

FIGURE 1. Maps f and g

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