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DAGGER-SHARP TITS OCTAGONS

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Abstract. The spherical buildings associated with absolutely simple algebraic groups of relative rank 2 are all Moufang polygons. Tits polygons are a more general class of geometric structures that includes Moufang polygons as a special case. Dagger-sharp Tits n-gons exist only for $n = 3$, 4, 6 and 8. Moufang octagons were classified by Tits. We show here that there are no dagger-sharp Tits octagons that are not Moufang. As part of the proof it is shown that the same conclusion holds for a certain class of dagger-sharp Tits quadrangles.

1. Introduction

A generalized polygon is the same thing as an irreducible spherical building of rank 2. Tits observed that the spherical buildings of rank 2 that arise from absolutely simple algebraic groups all satisfy a property he called the Moufang condition. In [5], he classified Moufang octagons. He showed, in particular, that they all arise as the fixed point building of a polarity of a building of type F_4. Subsequently, the complete classification of Moufang polygons was given in [7].

The notion of a Tits polygon was introduced in [3]. A Tits polygon is a bipartite graph Γ in which for each vertex v, the set Γ_v of vertices adjacent to v is endowed with a symmetric relation we call "opposite at v" satisfying certain axioms. A Moufang polygon is the same thing as a Tits polygon all of whose local opposition relations are trivial.

Let $\mathcal{P}$ denote the set of pairs (Δ, T), where Δ is a spherical building of type M satisfying the Moufang condition and T is a Tits index of absolute type M and relative rank 2. Every pair (Δ, T) in $\mathcal{P}$ gives rise by a simple construction to a Tits polygon whose automorphism group is canonically isomorphic to the automorphism group of Δ preserving T. We call the Tits polygons that arise in this way the Tits polygons of index type. Moufang polygons are all Tits

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polygons of index type; this is the case that not just the relative rank but also the absolute rank of T is 2.

For every irreducible spherical building Δ of rank at least 2, there exist Tits indices T such that $(\Delta, T) \in \mathcal{P}$. Thus the theory of Tits polygons allows us to regard a spherical building of arbitrary rank at least 2 as a rank 2 structure to which the methods developed in [7] can be applied.

With a few exceptions, Tits polygons of index type satisfy a condition we call dagger-sharp. This is a natural condition on the action of the stabilizer of an apartment on the corresponding root groups. It is trivially satisfied by all Moufang polygons. Tits n-gons exist for every value of n (as was observed in [3, 1.2.33]), but by [3, 1.6.14], dagger-sharp Tits n-gons exist only for $n = 3$, 4, 6 and 8.

Let k be an integer at least 3. We say that a Tits polygon is k-plump if for each vertex v, the valency $|\Gamma_v|$ of v is not too small in an appropriate sense. All Tits polygons of index type corresponding to a pair (Δ, T) in $\mathcal{P}$ are k-plump if the field of definition of Δ contains at least k elements (by [3, 1.2.7]).

In [2, 5.11 and 5.12], we showed that all dagger-sharp Tits triangles are of index type (or a variation defined over a simple associative ring that is infinite dimensional over its center) and in [1, 7.7], we showed that all dagger-sharp Tits hexagons are of index type. In [4], we proved a similar (but slightly weaker) result for the Tits quadrangles of exceptional type.

The main goal of this article is to treat the case $n = 8$. We prove the following:

Theorem 1.1. *All* 9*-plump dagger-sharp Tits octagons are Moufang.*

Our proof of Theorem 1.1 is a modification of Tits' classification of Moufang octagons in [5]. It exploits the existence of a Tits subquadrangle of indifferent type. To make the proof work, we first have to prove Theorem 3.1, a classification result for this class of Tits quadrangles. As a corollary, we obtain the following:

Theorem 1.2. *All* 5*-plump dagger-sharp indifferent Tits quadrangles are Moufang.*

Our proof of Theorem 3.1 is, in turn, a modification of Tits' unpublished classification of indifferent Moufang quadrangles which eventually appeared in [6].

We conjecture that every dagger-sharp Tits polygon is of index type or a variation involving an associative ring that is infinite dimensional over its center and/or a module of infinite rank. To complete the proof, it remains only to finish the case $n = 4$.

Conventions 1.3. Let G be a group. We denote the set of non-trivial elements of G by G^*. As in [7], we set $a^b = b^{-1}ab$ and

$$[a, b] = a^{-1}b^{-1}ab$$

for all $a, b \in G$. With these definitions, we have

(i) $[ab, c] = [a, c]^b \cdot [b, c]$ and
(ii) $[a, bc] = [a, c] \cdot [a, b]^c$

for all $a, b, c \in G$.

2. Tits polygons

Tits polygons were introduced in [3]. In this section, we give the definition and assemble all the properties of Tits polygons we will need for the proofs of Theorems 1.1 and 3.1.

Definition 2.1. A *dewolla* is a triple

$$X = (\Gamma, \mathcal{A}, \{\equiv_v\}_{v \in V}),$$

where:

(i) Γ is a bipartite graph with vertex set V and $|\Gamma_v| \geq 3$ for each $v \in V$, where Γ_v denotes the set of vertices adjacent to v.
(ii) For each $v \in V$, $\equiv_v$ is an anti-reflexive symmetric relation on Γ_v. We say that vertices $u, w \in V$ are *opposite at* v if $u, w \in \Gamma_v$ and $u \equiv_v w$. A path $(w_0, w_1, \ldots, w_m)$ in Γ is called *straight* if w_{i-1} and w_{i+1} are opposite at w_i for all $i \in [1, m-1]$.
(iii) There exist $n \geq 3$ and a non-empty set $\mathcal{A}$ of circuits of length $2n$ such that every path contained in a circuit in $\mathcal{A}$ is straight.

The parameter n is called the *level* of X. The automorphism group $\mathrm{Aut}(X)$ is the subgroup of $\mathrm{Aut}(\Gamma)$ consisting of all $g \in \mathrm{Aut}(\Gamma)$ such that $\gamma^g \in \mathcal{A}$ for all $\gamma \in \mathcal{A}$ and for all $u, v, w \in V$ such that u and w are opposite at v, u^g and w^g are opposite at v^g. A *root* of X is a straight path of length n.

Definition 2.2. A *Tits n-gon* is a dewolla

$$X = (\Gamma, \mathcal{A}, \{\equiv_v\}_{v \in V})$$

of level n for some $n \geq 3$ such that Γ is connected and the following axioms hold:

(i) For all $v \in V$ and all $u, w \in \Gamma_v$, there exists $z \in \Gamma_v$ that is opposite both u and w at v.
(ii) For each straight path $\delta = (w_0, \ldots, w_k)$ of length $k \leq n-1$, δ is the unique straight path of length at most k from w_0 to w_k.
(iii) For $G = \mathrm{Aut}(X)$ and for each root $\alpha = (w_0, \ldots, w_n)$ of X, the group U_α acts transitively on the set of vertices opposite w_{n-1} at w_n, where U_α is the pointwise stabilizer of

$$\Gamma_{w_1} \cup \Gamma_{w_2} \cup \cdots \cup \Gamma_{w_{n-1}}$$

in G. The group U_α is called the *root group* associated with the root α.

A *Tits polygon* is a Tits n-gon for some $n \geq 3$. A Tits n-gon is called a *Tits triangle* if $n = 3$, a *Tits quadrangle* if $n = 4$, etc.

If $X = (\Gamma, \mathcal{A}, \{\equiv_v\}_{v\in V})$ is a Tits n-gon for some $n \geq 3$, then by [3, 1.3.12], $\mathcal{A}$ is the set of all circuits in Γ of length at most $2n$ containing only straight paths. Thus, in particular, $2n$ is, roughly speaking, the "straight girth" of Γ.

Notation 2.3. We will say that a Tits n-gon $X = (\Gamma, \mathcal{A}, \{\equiv_v\}_{v\in V})$ is *Moufang* if all the relations $\equiv_v$ are trivial, i.e., if all paths in Γ are straight. If X is Moufang, then by [3, 1.2.3], Γ is a Moufang n-gon and $\mathcal{A}$ is the set of its apartments. Conversely, if Γ is a Moufang n-gon, $\mathcal{A}$ is the set of its apartments and $\equiv_v$ is the trivial relation on Γ_v for every v in the vertex set V, then by [3, 1.2.2], $(\Gamma, \mathcal{A}, \{\equiv_v\}_{v\in V})$ is a Tits n-gon.

Notation 2.4. Let $X = (\Gamma, \mathcal{A}, \{\equiv_v\}_{v\in V})$ be a Tits n-gon for some $n \geq 3$. A *coordinate system* for X is a pair $(\gamma, i \mapsto w_i)$ where γ is an element of $\mathcal{A}$ and $i \mapsto w_i$ is a surjection from $\mathbb{Z}$ to the vertex set of γ such that w_{i-1} is adjacent to w_i for each i. For each coordinate system $(\gamma, i \mapsto w_i)$, we denote by U_i the root group associated with the root $(w_i, w_{i+1}, \ldots, w_{i+n})$ for each $i \in \mathbb{Z}$ and call the map $i \mapsto U_i$ the associated *root group labeling.* Thus $w_i = w_j$ and $U_i = U_j$ whenever i and j have the same image in $\mathbb{Z}_{2n}$. For the rest of this section, we fix a Tits n-gon $X = (\Gamma, \mathcal{A}, \{\equiv_v\}_{v\in V})$ and a coordinate system $(\gamma, i \mapsto w_i)$ of X. Let $i \mapsto U_i$ be the corresponding root group labeling and let $G = \mathrm{Aut}(X)$.

Proposition 2.5. *G acts transitively on the edge set of Γ.*

Proof. This holds by [3, 1.3.6]. □

Proposition 2.6. *Let*

$$U_{[k,m]} = \begin{cases} U_k U_{k+1} \cdots U_m & \text{if } k \leq m \text{ and} \\ 1 & \text{otherwise.} \end{cases}$$

Then the following hold:

(i) $[U_i, U_j] \subset U_{[i+1,j-1]}$ *for all i, j such that $i < j < i + n$. In particular, $[U_i, U_{i+1}] = 1$ for all i.*
(ii) *The product map $U_1 \times U_2 \times \cdots \times U_n \to U_{[1,n]}$ is bijective.*

Proof. This holds by [3, 1.3.36(ii) and (iii)]. □

Notation 2.7. For each path $(x_0, \ldots, x_m)$, we denote by $G^{(1)}_{x_1,\ldots,x_{m-1}}$ the pointwise stabilizer of $\Gamma_{x_1} \cup \cdots \cup \Gamma_{x_{m-1}}$. Thus, in particular, $U_i = G^{(1)}_{w_{i+1},\ldots,w_{i+n-1}}$ for all i and for each vertex v, $G^{(1)}_v$ is the kernel of the action of the stabilizer G_v on Γ_v.

Proposition 2.8. *$G^{(1)}_{w_{i+1},w_{i+2},\ldots w_{i+k-1}} = U_{[i+k-n,i]}$ for all i and all k such that $3 \leq k \leq n$.*

Proof. This holds by [3, 1.3.27]. □

Proposition 2.9. *Let $\alpha = (v_0, \ldots, v_n)$ be a root. Then U_α acts sharply transitively on the set of vertices that are opposite v_{n-1} at v_n.*

Proof. This holds by [3, 1.3.25]. □

Notation 2.10. Let

$$U_i^\sharp = \{a \in U_i \mid w_{i+n+1}^a \text{ is opposite } w_{i+n+1} \text{ at } w_{i+n}\}$$

for each i. By [3, 1.4.3], we have $U_i^\sharp \neq \emptyset$ and by [3, 1.4.8], we have

$$U_i^\sharp = \{a \in U_i \mid w_{i-1}^a \text{ is opposite } w_{i-1} \text{ at } w_i\}$$

for each i.

Proposition 2.11. *For each $i \in \mathbb{Z}$, there exist unique maps κ_γ and λ_γ from $U_i^\sharp$ to $U_{i+n}^\sharp$ such that for each $a \in U_i^\sharp$, the product*

$$\mu_\gamma(a) := \kappa_\gamma(a) \cdot a \cdot \lambda_\gamma(a) \tag{2.12}$$

interchanges the vertices w_{i+n-1} and w_{i+n+1}. For each $a \in U_i^\sharp$, the element $\mu_\gamma(a)$ fixes the vertices w_i and w_{i+n} and interchanges the vertices w_{i+j} and w_{i-j} for all $j \in \mathbb{Z}$ and

$$U_k^{\mu_\gamma(a)} = U_{2i+n-k} \tag{2.13}$$

for all $k \in \mathbb{Z}$.

Proof. This holds by [3, 1.4.4] and [3, 1.4.9(i)]. □

Proposition 2.14. *Let $a \in U_i^\sharp$ for some i. Then the following hold:*

(i) $a^{-1} \in U_i^\sharp$, $\mu_\gamma(a^{-1}) = \mu_\gamma(a)^{-1}$, $\kappa_\gamma(a^{-1}) = \lambda_\gamma(a)^{-1}$ *and* $\lambda_\gamma(a^{-1}) = \kappa_\gamma(a)^{-1}$.
(ii) $m = \mu_\gamma(\kappa_\gamma(a)) = \mu_\gamma(\lambda_\gamma(a))$.
(iii) $\mu_\gamma(a^g) = \mu_\gamma(a)^g$ *for all g mapping γ to itself.*
(iv) $\kappa_\gamma(\lambda_\gamma(a)) = \lambda_\gamma(\kappa_\gamma(a)) = a$.

Proof. This holds by [3, 1.4.3, 1.4.9(ii) and 1.4.13] and the third display in the proof of [3, 1.4.9]. □

Proposition 2.15. *Suppose that $U_i^* = U_i^\sharp$ for $i = 1$ and n. Then X is Moufang.*

Proof. By [3, 1.4.15], the relation $\equiv_v$ is trivial for $v = w_{n+1}$ and $v = w_{2n} = w_0$. By [3, 1.3.20], it follows that the relation $\equiv_{w_1}$ is also trivial. By Proposition 2.5, every vertex is in the same G-orbit as w_0 or w_1. Thus the relation $\equiv_v$ is trivial for all vertices v. By Notation 2.3, therefore, X is Moufang. □

Proposition 2.16. $C_H(\langle U_i, U_{i+1}\rangle) = C_H(\langle U_i, U_{i+n}\rangle) = 1$ *for all i, where H denotes the pointwise stabilizer of γ in $G = \mathrm{Aut}(X)$.*

Proof. This holds by [3, 1.4.19(ii)]. □

Proposition 2.17. $w_{i-1}^{U_iU_{i+n}} = \Gamma_{w_i} = w_{i+1}^{U_{i+n}U_i}$ *for each i.*

Proof. This holds by [3, 1.3.4]. □

Proposition 2.18. *Suppose that* $[a_1, a_n^{-1}] = a_2 \cdots a_{n-1}$ *with* $a_i \in U_i$ *for each* $i \in [1, n]$. *Then the following hold*:

(i) *If* $a_1 \in U_1^\sharp$, *then* $a_2 = a_n^{\mu_\gamma(a_1)}$ *and* $[a_2, \lambda_\gamma(a_1)^{-1}] = a_3 \cdots a_{n-1} a_n$.

(ii) *If* $a_n \in U_n^\sharp$, *then* $a_1 = a_{n-1}^{\mu_\gamma(a_n)}$ *and* $[\kappa_\gamma(a_n), a_{n-1}^{-1}] = a_1 a_2 \cdots a_{n-2}$.

Proof. This holds by [3, 1.4.16]. □

Proposition 2.19. *The following hold*:

(i) *If* $a \in U_1$ *and* $U_n^{ab} = U_2$ *for some* $b \in U_{n+1}$, *then* $a \in U_1^\sharp$ *and* $b = \lambda_\gamma(a)$

(ii) *If* $a \in U_n$ *and* $U_1^{ab} = U_{n-1}$ *for some* $b \in U_0$, *then* $a \in U_n^\sharp$ *and* $b = \lambda_\gamma(a)$.

Proof. This holds by [3, 1.4.27]. □

Remark 2.20. Both Propositions 2.18 and 2.19 remain valid if all the subscripts are shifted by a fixed amount. We have formulated both results for fixed values of the indices only for the sake of clarity.

Definition 2.21. Let $k \geq 3$. As in [3, 1.4.21], we call X *k-plump* if for all $v \in V$, and for every subset N of Γ_v of cardinality at most k, there exists a vertex that is opposite u at v for all $u \in N$. Thus k-plump implies $(k-1)$-plump, and "2-plump" is simply Definition 2.2(i).

Proposition 2.22. *If* X *is* 3*-plump, then for all* i, U_i *is generated by* $U_i^\sharp$.

Proof. This holds by [3, 1.4.23]. □

Notation 2.23. Let $G^\dagger$ denote the subgroup of G generated by all the root groups of X, let H be as in Proposition 2.16 and let $H^\dagger = H \cap G^\dagger$.

Proposition 2.24. *Let* $H_i = \langle mm' \mid m, m' \in \mu_\gamma(U_i^\sharp)\rangle$ *for all* i *and let* $H^\dagger$ *be as in Notation* 2.23. *Then* H_1 *and* H_n *normalize each other and if* X *is* $(n+1)$*-plump, then* $H^\dagger = H_1 H_n$.

Proof. The first claim holds by Proposition 2.14(iii) and the second claim by [3, 1.5.28]. □

Notation 2.25. Let H and $H^\dagger$ be as in Notation 2.23. The subgroup H normalizes U_i for each i. We say that X is *sharp* if for each i, every nontrivial HU_i-invariant subgroup of U_i contains elements of $U_i^\sharp$, where $U_i^\sharp$ is as in Notation 2.10. We say that X is *dagger-sharp* if for each i, every nontrivial $H^\dagger U_i$-invariant subgroup of U_i contains elements of $U_i^\sharp$. Note that dagger-sharp implies sharp. Note, too, that by [3, 1.3.13 and 1.3.40], the definitions of sharp and dagger-sharp do not depend on the choice of the coordinate system $(\gamma, i \mapsto w_i)$ in Notation 2.4.

Remark 2.26. Let H and $H^\dagger$ be as in Notation 2.23. By [7, 1.3.13], every root group of X is conjugate in G to U_1 or U_n. To show that X is sharp (respectively, dagger-sharp), it thus suffices to show that every nontrivial HU_i-invariant (respectively, $H^\dagger U_i$-invariant) subgroup of U_i contains elements of $U_i^\sharp$ for $i = 1$ and n.

Proposition 2.27. *Suppose that X is sharp and U_i is abelian for some i. Then $N_{U_{i+n}}(U_i) = 1$.*

Proof. Let $Y = N_{U_{i+n}}(U_i)$ and let H be as in Proposition 2.16. Suppose that $Y \neq 1$. The subgroup Y is normalized by H. By (2.13) with $k = i$, U_{i+n} is conjugate to U_i in G. Hence U_{i+n} is abelian. Since X is sharp, it follows that there exists $d \in Y \cap U_{i+n}^{\sharp}$. Let $m = \mu_\gamma(d)$. By (2.12), $d = emf$ for some $e, f \in U_i$. Thus

$$U_i = U_i^{f^{-1}} = (U_i^d)^{f^{-1}} = U_i^{em} = U_i^m = U_{i+8}.$$

The group U_i fixes w_{i+1}, however, but the subgroup U_{i+8} does not. With this contradiction, we conclude that $Y = 1$. □

Proposition 2.28. *Suppose that X is sharp and that $\beta = (v_0, v_1, \ldots, v_n)$ is a root such that $v_0 = w_i$, $v_n = w_{i+n}$ and $U_\beta = U_i$ for some i. Suppose, too that U_i is abelian. Then $\beta = (w_i, w_{i+1}, \ldots, w_{i+n})$.*

Proof. For each $z \in \Gamma_{w_i}$, let $\text{opp}(z)$ denote the set of vertices in Γ_{w_i} that are opposite z at w_i. By Definition 2.2(iii), U_i acts transitively on $\text{opp}(w_{i+1})$ and U_β acts transitively on $\text{opp}(v_1)$. By Definition 2.2(i), we can choose $z \in \text{opp}(w_{i+1}) \cap \text{opp}(v_1)$. Since $U_i = U_\beta$, it follows that both $\text{opp}(w_{i+1})$ and $\text{opp}(v_1)$ are equal to the U_i-orbit containing z. Hence $\text{opp}(w_{i+1}) = \text{opp}(v_1)$. In particular, $w_{i-1} \in \text{opp}(v_1)$. By Definition 2.2(iii), therefore, U_{i+n} contains an element d mapping v_1 to w_{i+1}. The subgroup U_{i+n} fixes $w_i = v_0$ and $w_{i+n} = v_n$. Thus by Definition 2.2(ii), d maps β to $(w_i, w_{i+1}, \ldots, w_{i+n})$. Hence d normalizes U_i. By Proposition 2.27, $d = 1$ and thus $\beta = (w_i, w_{i+1}, \ldots, w_{i+n})$. □

Notation 2.29. Suppose that $i < j < i+n$ and that $[a_i, a_j] = a_{i+1}a_{i+2} \cdots a_{j-1}$ with $a_k \in U_k$ for all $k \in [i, j]$. It follows from Proposition 2.6(ii) that for each $k \in [i+1, j-1]$, a_k is uniquely determined by $[a_i, a_j]$. We denote this element a_k by $[a_i, a_j]_k$.

Definition 2.30. Suppose that $n = 4$. We say that X is *indifferent* if

$$[U_1, U_3] = [U_2, U_4] = 1.$$

By [3, 1.3.13 and 1.3.40], this definition does not depend on the choice of the coordinate system $(\gamma, i \mapsto w_i)$ in Notation 2.4.

Proposition 2.31. *Suppose that $n = 4$ and that X is indifferent. Then U_i is abelian for all i.*

Proof. We first assume that $i = 2$. Let $a_2 \in U_2$. Choose $a_1 \in U_1^{\sharp}$ and let $a_4 = a_2^{\mu_\gamma(a_1)^{-1}}$. By Proposition 2.18(i), $[a_1, a_4^{-1}] = a_2 a_3$ for some $a_3 \in U_3$. Since $[U_i, U_2] = 1$ for $i = 1$, 3 and 4, it follows that $[a_2, U_2] = 1$. Thus U_2 is abelian. By Remark 2.20, in fact, U_i is abelian for all i. □

Proposition 2.32. *Suppose that $n = 4$ and that X is indifferent. Let $b_1 \in U_1$ and $b_4 \in U_4$. Then the maps $a_1 \mapsto [a_1, b_4]$ and $a_4 \mapsto [b_1, a_4]$ are homomorphisms.*

Proof. This holds by Conventions 1.3(i) and (ii). □

Notation 2.33. Suppose that $n = 8$. For each vertex z and each integer $k \geq 2$, let $G_z^{(k)}$ denote the intersection of $G^{(1)}_{v_1,\ldots,v_{k-1}}$ (as defined in Notation 2.7) for all straight k-paths $(v_0, v_1, \ldots, v_k)$ with $z = v_0$. We set

$$V_i = Z(U_{[i-4,i+4]}) \cap G^{(4)}_{w_{i+4}}$$

for all i, where $U_{[i-4,i+4]}$ is as in Notation 2.6. Thus, in particular, $V_i \subset U_i$ for all i.

Proposition 2.34. *Suppose that $n = 8$ and X is sharp as defined in Notation 2.25 and let V_i be as Notation 2.33. Then $V_i \neq 1$ for all even i or for all odd i.*

Proof. Let G be as in Notation 2.23. By [3, 1.3.7 and 1.3.13], w_i lies in the same G-orbit as w_j if $i - j$ is even and every vertex of Γ is in the same G-orbit as w_0 or w_1. The claim holds, therefore, by [3, 1.3.36(i) and 1.6.18]. □

3. Quadrangles

The main result in this section is the following:

Theorem 3.1. *Let X be a Tits quadrangle that is indifferent and 5-plump as defined in Definitions 2.21 and 2.30. Let $(\gamma, i \mapsto w_i)$ and $i \mapsto U_i$ be as in Notation 2.4, let $U_i^\sharp$ for all i be as in Proposition 2.11 and let H and $H^\dagger$ be as in Notation 2.23. Suppose that J is a subgroup of H such that $[J, H^\dagger] = 1$ and that for each i, every $JH^\dagger$-invariant subgroup of U_i contains elements of $U_i^\sharp$. Then X is Moufang.*

It follows by Notation 2.25 and Proposition 2.31 that Theorem 1.2 is the special case of Theorem 3.1 where $J = 1$. Before we begin the proof of Theorem 3.1, we prove a preliminary result which (like Theorem 3.1 itself) we will need in the proof of Theorem 1.1:

Proposition 3.2. *Let X be a 3-plump indifferent Tits quadrangle, let $(\gamma, i \mapsto w_i)$, $i \mapsto U_i$ and $U_i^\sharp$ for all i be as in Theorem 3.1. Suppose that the normalizer $N_{U_i}(U_{[i+2,i+3]})$ is trivial for all i. Then $a^2 = 1$, $\mu_\gamma(b)^2 = 1$ and $\lambda_\gamma(b) = \kappa_\gamma(b)$ for all i, all $a \in U_i$ and all $b \in U_i^\sharp$.*

Proof. Suppose X satisfies the hypotheses of Proposition 3.2. We proceed with the proof of Proposition 3.2 in a series of steps.

Proposition 3.3. *For each i, the map $a_i \mapsto \mu_\gamma(a_i)$ from $U_i^\sharp$ to G is injective, where μ_γ is as in (2.12).*

Proof. It suffices to assume that $i = 1$. Let $a_1, b_1 \in U_1^\sharp$ and suppose that $\mu_\gamma(a_1) = \mu_\gamma(b_1)$. Choose $a_4 \in U_4$. Applying the notation in Notation 2.29, we have

$$[a_1, a_4^{-1}]_2 = a_4^{\mu_\gamma(a_1)} = a_4^{\mu_\gamma(b_1)} = [b_1, a_4^{-1}]_2$$

by Proposition 2.18(i). By Proposition 2.32, therefore, $[a_1 b_1^{-1}, a_4]_2 = 1$. Since a_4 is arbitrary, it follows by Proposition 2.6(i) that $a_1 b_1^{-1} \in N_{U_1}(U_{[3,4]})$. By hypothesis, therefore, $a_1 = b_1$. □

Proposition 3.4. $\kappa_\gamma(a_i) = a_i^{\mu_\gamma(a_i)} = \lambda_\gamma(a_i)$ *for all* i *and all* $a_i \in U_i^\sharp$*, where* κ_γ *and* λ_γ *are as in* (2.12)*.*

Proof. Let $a_i \in U_i^\sharp$ for some i and let $m = \mu_\gamma(a_i)$. Then $\kappa_\gamma(a_i) \in U_{i+n}$, $\lambda_\gamma(a_i) \in U_{i+n}$ and by (2.13), also $a_i^m \in U_{i+n}$. By Proposition 2.14(ii) and (iii), we have $\mu_\gamma(a_i^m) = m^m = m = \mu_\gamma(\kappa_\gamma(a_i)) = \mu_\gamma(\lambda_\gamma(a_i))$. The claim holds, therefore, by Proposition 3.3. □

Proposition 3.5. *The elements of* $U_i^\sharp$ *are all of order* 2 *for all* i*.*

Proof. It suffices to assume that $i = 2$. Choose $a_1 \in U_1^\sharp$ and $a_2 \in U_2^\sharp$ and let $a_4 = a_2^{\mu_\gamma(a_1)^{-1}}$. Then $a_4 \in U_4^\sharp$ and $[a_1, a_4^{-1}] = a_2 a_3$ for some $a_3 \in U_3$ by Proposition 2.18(i). Hence $[a_1, a_4] = a_2^{-1} a_3^{-1}$ by Proposition 2.32. Let $a_0 = a_4^{\mu_\gamma(a_4)}$. By Proposition 3.4, $\mu_\gamma(a_4) = a_0 a_4 a_0$ and $a_0 = \kappa_\gamma(a_4)$. By Proposition 2.18(ii), therefore, $[a_0, a_3^{-1}] = a_1 a_2$. Hence $[a_0, a_3] = a_1^{-1} a_2^{-1}$ by Proposition 2.32. By Conventions 1.3 and Proposition 2.6(i), we have

$$\begin{aligned} a_1^{a_0 a_4 a_0} = a_1^{a_4 a_0} &= \big(a_1 \cdot [a_1, a_4]\big)^{a_0} \\ &= (a_1 a_2^{-1} a_3^{-1})^{a_0} = a_1 a_2^{-1} \cdot [a_0, a_3] \cdot a_3^{-1} = a_2^{-2} a_3^{-1}. \end{aligned}$$

By (2.13), we have $a_1^{\mu_\gamma(a_4)} \in U_3$. Hence by Proposition 2.6(ii), $a_2^2 = 1$. Thus the elements of $U_2^\sharp$ are all of order 2. By Proposition 2.14(i), therefore, the elements of $\mu_\gamma(U_2^\sharp)$ are all of order 2. □

Corollary 3.6. *The elements of* $\mu_\gamma(U_i^\sharp)$ *are all of order* 2 *for all* i*.*

Proof. This holds by Proposition 2.14(i) and Proposition 3.5. □

Corollary 3.7. U_i *is of exponent* 2 *for all* i*.*

Proof. This holds by Propositions 2.22, 2.31 and 3.5. □

With Proposition 3.4, Proposition 3.6 and Corollary 3.7, the proof of Proposition 3.2 is complete. □

We use the rest of this section to prove Theorem 3.1. Suppose that X satisfies the hypotheses of Theorem 3.1. Again we proceed in a series of steps.

Proposition 3.8. $N_{U_i}(U_{[i+2,i+3]}) = 1$ *for all* i *and the assertions in Proposition* 3.4 *and Corollary* 3.7 *hold.*

Proof. It suffices to assume that $i = 1$. Let $b_4 \in U_4^*$. If $c_1 \in U_1^\sharp$, then $[c_1, b_4^{-1}]_2 \neq 1$ by Proposition 2.18(i) and hence $c_1 \notin N_{U_1}(U_{[3,4]})$. Since X is sharp and the group $N_{U_1}(U_{[3,4]})$ is HU_1-invariant, it follows that $N_{U_1}(U_{[3,4]}) = 1$. By Proposition 3.2, therefore, the assertions in Proposition 3.4 and Corollary 3.7 hold. □

Proposition 3.9. *$H^\dagger$ is an abelian group.*

Proof. Let H_i for all i be as in Proposition 2.24. Then H_1 centralizes U_3 and H_4 centralizes U_2. Thus $[H_1, H_4] \subset C_H(\langle U_2, U_3\rangle)$ and hence $[H_1, H_4] = 1$ by Proposition 2.16. Now choose $m \in \mu_\gamma(U_4^\sharp)$ and $h, h' \in H_1$. We have $H_1^m = H_3$ by (2.13) and m acts trivially on U_2. Thus $[h, h']$ induces the same permutation as $[h^m, h']$ on U_2. Since $[h^m, h'] \in [H_3, H_1] = 1$, we conclude that $[h, h'] \in C_H(U_2)$. Since $h, h' \in C_H(U_3)$, if follows by Proposition 2.16 that $[h, h'] = 1$. Thus H_1 is abelian. Choosing $m \in \mu_\gamma(U_1^\sharp)$ and $h, h' \in H_4$, we conclude that $[h, h'] = 1$ by a similar argument. Thus also H_4 is abelian. Since $[H_1, H_4] = 1$, therefore, the product H_1H_4 is an abelian group. Hence by Proposition 2.24, $H^\dagger$ is abelian. □

Proposition 3.10. *Let H_i for all i be as in Proposition* 2.24*, let $h \in H_i$ and $m = \mu_\gamma(a_i)$ for some i and some $a_i \in U_i^\sharp$. Then $h^m = h^{-1}$.*

Proof. It suffices to assume that $i = 1$. We have

$$H_1 = \langle m\mu_\gamma(b_1) \mid b_1 \in U_1^\sharp\rangle.$$

By Corollary 3.6, $h^m = h^{-1}$ for $h = m\mu_\gamma(b_1)$ for all $b_1 \in U_1^\sharp$. The claim holds, therefore, by Proposition 3.9. □

Proposition 3.11. *Let $e_i \in U_i^\sharp$ and $m_i = \mu_\gamma(e_i)$ for $i = 1$ and* 4 *and let $N = \langle m_1, m_4\rangle$. Let*

$$e_{1+2i} = e_1^{(m_4m_1)^i} \quad \text{and} \quad e_{4+2i} = e_4^{(m_4m_1)^i}$$

for all i. Then N is a dihedral group of order 8 *and for all i, $e_i = e_{i+8}$, $e_i \in U_i^\sharp$, $e_i^n = e_j$ if $U_i^n = U_j$ for some $n \in N$, $\mu_\gamma(e_i) = \mu_\gamma(e_{i+4}) \in N$ and the normalizer of U_i in N centralizes U_i.*

Proof. By (2.13), we have $e_i \in U_i^\sharp$ for all i. By Proposition 2.14(iii), it follows from $m_1, m_4 \in N$ that $\mu_\gamma(e_i) \in N$ for all i. We have $m_1 \in \langle U_1, U_5\rangle$. Applying (2.13) and Proposition 2.14(iii) again, we thus have $m_1^{m_4} \in \langle U_3, U_7\rangle$. Hence $[m_1, m_1^{m_4}] = 1$. By Corollary 3.6, therefore, $(m_4m_1)^2 = (m_1m_4)^2$ and N is a dihedral group of order 8. It follows that for all i, $e_i = e_{i+8}$ and $e_i^n = e_j$ if $U_i^n = U_j$ for some $n \in N$. Thus, in particular, $e_i^{m_i} = e_{i+4}$ and hence $\mu_\gamma(e_i) = \mu_\gamma(e_i)^{m_i} = \mu_\gamma(e_i^{m_i}) = \mu_\gamma(e_{i+4})$ for all i by Proposition 2.14(iii). The normalizer of U_i in N is $\langle\mu_\gamma(e_{i+2})\rangle$ for all i. Since $[U_i, \mu_\gamma(e_{i+2})] = 1$ for all i, the last claim holds. □

Notation 3.12. Let H_i for all i be as in Proposition 2.24. For each i, let L_i denote the image of H_{i+1} in $\mathrm{Aut}(U_i)$ and let K_i denote the subring of $\mathrm{End}(U_i)$ generated by L_i. The elements of L_i are units of K_i. By Proposition 3.9, the ring K_i is commutative and by Corollary 3.7 (and Corollary 3.8), $2 = 0$ in K_i. Let $m \in \mu_\gamma(U_{i+2}^\sharp)$ for some i. Since $H_{i+1}^m = H_{i-1}$ and m centralizes U_i, L_i is also the image of H_{i-1} in $\mathrm{Aut}(U_i)$.

Proposition 3.13. *Let N be as in Proposition 3.11 and suppose that $U_i^n = U_j$ for some $n \in N$ and some i, j. Then conjugation by n induces isomorphisms from L_i to L_j and from K_i to K_j that depend on i and j but not on n.*

Proof. This holds by the last assertion in Proposition 3.11. □

Notation 3.14. By Proposition 3.13, we can use N to identify L_i with L_j and K_i with K_j whenever $i-j$ is even. We denote by φ_i the natural homomorphism from H_i to L_{i-1} for each i. By Proposition 3.13, $L_{i-1} = L_{i+1}$ and if $U_j = U_i^n$ for some $n \in N$, then

$$\varphi_j(h^n) = \varphi_i(h) \tag{3.15}$$

for all $h \in H_i$.

Notation 3.16. Let e_i be as in Proposition 3.11 for all i. For all i and all $a_i \in U_i$, let ρ_{i,a_i} denote the element of $\mathrm{Aut}(U_{i+1})$ given by

$$\rho_{i,a_i}(a_{i+1}) = [a_i, a_{i+1}^{\mu_\gamma(e_i)}]_{i+1}$$

for all $a_{i+1} \in U_{i+1}$. If $a_i \in U_i^\sharp$ for some i, then by Proposition 2.18(i),

$$\rho_{i,a_i}(a_{i+1}) = a_{i+1}^{\mu_\gamma(e_i)\mu_\gamma(a_i)}$$

for all a_{i+1} and hence

$$\rho_{i,a_i} = \varphi_i\big(\mu_\gamma(e_i)\mu_\gamma(a_i)\big) \in L_{i+1}. \tag{3.17}$$

By Proposition 2.32, we have

$$\rho_{i,a_i}(a_{i+1})\rho_{i,b_i}(a_{i+1}) = \rho_{i,a_ib_i}(a_{i+1}) \tag{3.18}$$

for all $a_i, b_i \in U_i$ and all $a_{i+1} \in U_{i+1}$. By Proposition 2.22, therefore, $\rho_{i,a_i} \in K_{i+1}$ for all $a_i \in U_i$. We denote by ψ_i (for arbitrary i) the map from U_i to the additive group of K_{i+1} given by $\psi_i(a_i) = \rho_{i,a_i}$ for all $a_i \in U_i$. The elements of $\psi_i(U_i^\sharp)$ are invertible in K_i and $\psi_i(e_i) = 1$ by (3.17), and by (3.18), ψ_i is a homomorphism.

Proposition 3.19. *Let H_i be as in Proposition 2.24 and let φ_i and ψ_i be as in Notations 3.14 and 3.16 for some i. Then the following hold:*

(i) *φ_i is an isomorphism from H_i to L_{i+1}.*
(ii) *ψ_i is an injective homomorphism from U_i to the additive group of K_{i+1}.*
(iii) *$\psi_i(a_i^h) = \varphi_i(h)^2\psi_i(a_i)$ for all $a_i \in U_i$ and all $h \in H_i$.*
(iv) *K_{i+1} is generated by the image of ψ_i.*

Proof. An element in the kernel of φ_i is contained in $C_H(\langle U_{i+1}, U_{i+2}\rangle)$. By Proposition 2.16, therefore, (i) holds. The kernel of ψ_i is $N_{U_i}(U_{[i+2,i+3]})$. By Proposition 3.8, this normalizer is trivial. Thus (ii) holds. Let $a_i \in U_i^\sharp$ and $h \in H_i$. Then

$$\begin{aligned}
\psi_i(a_i^h) &= \varphi_i\big(\mu_\gamma(a_i^h)\mu_\gamma(e_i)\big) && \text{by (3.17)}\\
&= \varphi_i\big(\mu_\gamma(a_i)^h\mu_\gamma(e_i)\big) && \text{by Proposition 2.14(iii)}\\
&= \varphi_i\big(h^2\mu_\gamma(a_i)\mu_\gamma(e_i)\big) && \text{by Proposition 3.10}\\
&= \varphi_i(h)^2 \cdot \psi_i(a_i)
\end{aligned}$$

Hence by Proposition 2.22, (iii) holds. By (3.17), L_{i+1} is contained in the subring of K_{i+1} generated by $\psi_i(U_i)$. Since K_{i+1} is generated by L_{i+1}, (iv) holds. □

Notation 3.20. Let $\varepsilon = 1$ or -1 and let $m_{i+\varepsilon} \in \mu_\gamma(U_{i+\varepsilon}^\sharp)$ for some i. We set $\alpha_i^\varepsilon(h) = [m_{i+\varepsilon}, h]$ for all $h \in H_i$. We also set $\alpha_i^+ = \alpha_i^\varepsilon$ if $\varepsilon = 1$ and $\alpha_i^- = \alpha_i^\varepsilon$ if $\varepsilon = -1$.

Proposition 3.21. *Then for all i, the following hold*:

(i) α_i^ε *is a homomorphism from H_i to $H_{i+\varepsilon}$ for $\varepsilon = 1$ and -1.*
(ii) α_i^ε *is independent of the choice of $m_{i+\varepsilon}$ in Notation 3.20 for $\varepsilon = 1$ and -1.*
(iii) $\alpha_i^+\big(\alpha_{i+1}^-(h)\big) = h^2$ *for all $h \in H_{i+1}$.*

Proof. Choose i and let $j = i + \varepsilon$ for $\varepsilon = 1$ or -1. If $h \in H_i$ and $a_j \in U_j^\sharp$, then $[\mu_\gamma(a_j), h] = \mu_\gamma(a_j)\mu_\gamma(a_j^h) \in H_j$ by Proposition 2.14(iii). By Conventions 1.3(ii) and Proposition 3.9, it follows that α_i^ε is a homomorphism. Thus (i) holds.

Choose $h \in H_i$ and let $m, m' \in \mu_\gamma(U_{i+\varepsilon}^\sharp)$. Then $[mm', h] = 1$ by Proposition 3.9 and $[m, h]^{m'} = [m, h]^{-1}$ by (i) and Proposition 3.10. By Conventions 1.3(i), therefore, $[m, h] = [m', h]$. Thus (ii) holds.

Let $h \in H_{i+1}$, $m \in \mu_\gamma(U_{i+1}^\sharp)$ and $m' \in \mu_\gamma(U_i^\sharp)$. Then $m^{m'}$ is contained in $\langle U_{i-1}, U_{i+3}\rangle$ and hence commutes with H_{i+1}. By Proposition 3.10, $h^m = h^{-1}$. Thus

$$\begin{aligned}
[m, [m', h]] &= m \cdot h^{-1}m'hm' \cdot m \cdot m'h^{-1}m'h\\
&= mh^{-1}m' \cdot hm^{m'}h^{-1} \cdot m'h\\
&= mh^{-1}m' \cdot m^{m'} \cdot m'h = mh^{-1}m \cdot h = h^2.
\end{aligned}$$

Thus (iii) holds. □

Proposition 3.22. *For each i and each $a_i \in U_i^\sharp$, let ξ_i be the map from K_i to K_{i+1} given by*

$$\xi_i(s) = \psi_i(a_i)^{-1} \cdot \psi_i(sa_i) \tag{3.23}$$

for all $s \in K_i$. Then the following hold:

(i) ξ_i *is an injective homomorphism of rings from* K_i *to* K_{i+1} *mapping the identity* 1 *of* K_i *to the identity* 1 *of* K_{i+1} *that does not depend on the choice of* a_i.

(ii) $\xi_{i+1}\big(\xi_i(s)\big) = s^2$ *for all* $s \in K_i$.

(iii) *The map* $s \mapsto s^2$ *is an injective endomorphism of* K_i.

Proof. Choose i and $a_i \in U_i^\sharp$. By Proposition 3.19(ii), ψ_i is injective. Hence ξ_i is injective. Let $j = i + \varepsilon$ for $\varepsilon = 1$ or -1, let $h \in H_j$ and let $s = \varphi_j(h)$. Then

$$\begin{aligned} \psi_i(a_i)^{-1} \cdot \psi_i\big(sa_i\big) &= \psi_i(a_i)^{-1} \cdot \psi_i(a_i^h) \\ &= \varphi_i\big(\mu_\gamma(a_i)\mu_\gamma(e_i)\big) \cdot \varphi_i\big(\mu_\gamma(e_i)\mu_\gamma(a_i^h)\big) \\ &= \varphi_i\big(\mu_\gamma(a_i)\mu_\gamma(a_i^h)\big) \\ &= \varphi_i\big([\mu_\gamma(a_i), h]\big) = \varphi_i\big(\alpha_j^{-\varepsilon}(h)\big). \end{aligned} \tag{3.24}$$

Thus by Proposition 3.21(ii), the restriction of ξ_i to $\varphi_j(H_j)$ is independent of the choice of a_i and, by Proposition 3.21(i), this restriction is multiplicative. Since K_i is generated by L_i additively, $\varphi_j(H_j) = L_i$ and ξ_i is additive, it follows that ξ_i is a homomorphism of rings that is independent of the choice of a_i. Thus (i) holds.

By (3.24), we have $\xi_i \circ \varphi_{i+1} = \varphi_i \circ \alpha_{i+1}^-$ and $\xi_i \circ \varphi_{i-1} = \varphi_i \circ \alpha_{i-1}^+$ (composing from right to left). Replacing i by $i + 1$ in the second equation, we obtain $\xi_{i+1} \circ \varphi_i = \varphi_{i+1} \circ \alpha_i^+$. Thus

$$\xi_{i+1} \circ \xi_i \circ \varphi_{i+1} = \xi_{i+1} \circ \varphi_i \circ \alpha_{i+1}^- = \varphi_{i+1} \circ \alpha_i^+ \circ \alpha_{i+1}^-.$$

By Proposition 3.21(iii), therefore,

$$\xi_{i+1}\big(\xi_i(s)\big) = s^2$$

for all s in the subset $\varphi_{i+1}(H_{i+1}) = L_i$ of K_i. This subset generates K_i additively and, as was observed in Notation 3.12, $2 = 0$ in K_i. Thus (ii) holds. Since ξ_i and ξ_{i+1} are both injective homomorphisms, it follows that (iii) holds. ☐

Corollary 3.25. *Let σ be an automorphism of K_i for some i and suppose that $\sigma(s^2) = s^2$ for all $s \in K_i$. Then σ is the identity.*

Proof. This follows from Proposition 3.22(iii). ☐

Proposition 3.26. *Let N be as in Proposition* 3.11 *and suppose that $U_i^n = U_j$ for some $n \in N$. Then $\psi_j(a_i^n) = \psi_i(a_i)$ for all $a_i \in U_i$.*

Proof. Let $a_i \in U_i^\sharp$. Then

$$\begin{aligned} \psi_j(a_i^n) &= \varphi_j\big(\mu_\gamma(e_j)\mu_\gamma(a_i^n)\big) && \text{by (3.17)} \\ &= \varphi_j\big(\mu_\gamma(e_j)\mu_\gamma(a_i)^n\big) && \text{by Proposition 2.14(iii)} \\ &= \varphi_j\big((\mu_\gamma(e_i)\mu_\gamma(a_i))^n\big) && \text{by Proposition 3.11} \end{aligned}$$

$$= \varphi_i\big(\mu_\gamma(a_i)\mu_\gamma(e_i)\big) \qquad \text{by (3.15)}$$
$$= \psi_i(a_i) \qquad \text{by (3.17)}.$$

By Proposition 2.22, therefore, the claim holds. □

Proposition 3.27. *Let $b \in U^\sharp_{i+3\varepsilon}$ for some i and for $\varepsilon = 1$ or -1. Then*

$$\psi_{i+2\varepsilon}(a_i^{\mu_\gamma(b)}) = \psi_i(a_i) \cdot \xi_{i+2\varepsilon}\big(\psi_{i+3\varepsilon}(b)\big)$$

for all $a_i \in U_i$, where ξ_i is as in Proposition 3.22.

Proof. It suffices to assume that $i = 1$ and $\varepsilon = 1$. Let $m_4 = \mu_\gamma(e_4)$ (as in Proposition 3.11), let $m' = \mu_\gamma(b)$ and choose $a_1 \in U_1$. Then $\psi_4(b) = \varphi_4(m_4m')$ by (3.18) and thus

$$\begin{aligned}\psi_3(a_1^{m'}) &= \psi_3\big(a_1^{m_4\cdot m_4m'}\big)\\ &= \psi_3\big(\varphi_4(m_4m')a_1^{m_4}\big)\\ &= \psi_3\big(\psi_4(b)a_1^{m_4}\big)\\ &= \psi_3(a_1^{m_4}) \cdot \xi_3\big(\psi_4(b)\big)\end{aligned}$$

by (3.23). By Proposition 3.26, we have $\psi_3(a_1^{m_4}) = \psi_1(a_1)$. □

Notation 3.28. Let $K = K_4$, let $F = \xi_3(K_3)$, let $\tilde{K} = \psi_3(U_3)$ and let $\tilde{F} = \xi_3(\psi_2(U_2))$. By Proposition 3.26, we have $\tilde{K} = \psi_i(U_i)$ for all odd i and

$$\tilde{F} = \xi_{i+1}(\psi_i(U_i)) = \xi_{i+1}(\psi_{i+2}(U_{i+2}))$$

for all even i. By (3.17), we have $\psi_3(e_3) = \xi_3(\psi_2(e_2)) = 1$, so both $\tilde{K}$ and $\tilde{F}$ contain 1. By Proposition 3.19(ii) and (iv), $\tilde{K}$ is an additive subgroup of K that generates K as a ring and (since ξ_3 is a homomorphism of rings) $\tilde{F}$ is an additive subgroup of F that generates F as a ring. The group U_2 is generated by $U_2^\sharp$ (by Proposition 2.22), $\psi_1(U_1) = \psi_3(U_3)$ and $\psi_2(U_2) = \psi_4(U_4)$. By Proposition 3.27, therefore, $\tilde{K}\tilde{F} \subset \tilde{K}$. By Proposition 3.22(ii), $\tilde{K}^2 = \psi_1(U_1)^2 = \xi_3\big(\xi_2(\psi_1(U_1))\big)$ and by Proposition 3.27, $\xi_2(\psi_1(U_1)) \subset \psi_2(U_2)$. Therefore $\tilde{K}^2\tilde{F} \subset \xi_3(\psi_2(U_2)) = \tilde{F}$. We conclude that $(K, \tilde{K}, \tilde{F})$ satisfies all the properties of an indifferent set as defined in [7, 10.1] except that we do not know that K is a field.

Notation 3.29. Let $\tilde{K}$ and $\tilde{F}$ be as in Notation 3.28. By Proposition 3.19(ii) and Proposition 3.22(i), ψ_i is an isomorphism from U_i to the additive group of $\tilde{K}$ for i odd and $\xi_{i-1}\circ\psi_i$ is an isomorphism from U_i to the additive group of $\tilde{F}$ for i even. We set $x_i(s) = \psi_i^{-1}(s)$ for all $s \in \tilde{K}$ if i is odd and $x_i(t) = (\xi_{i-1}\circ\psi_i)^{-1}(t)$ for all $t \in \tilde{F}$ if i is even. Note that by (3.17), $x_i(1) = e_i$ for all i.

Proposition 3.30. $[x_1(s), x_4(t)] = x_2(s^2t)x_3(st)$ *for all $s \in \tilde{K}$ and all $t \in \tilde{F}$.*

Proof. Let $a_1 = x_1(s)$ for some $s \in \tilde{K}$ and $a_4 = x_4(t)$ for some $t \in \tilde{F}$. By Proposition 2.22 and Proposition 2.32, it suffices to assume that $a_i \in U_i^\sharp$ for $i = 1$ and 4. Let $n_i = \mu_\gamma(a_i)$ for $i = 1$ and 4. Then

$$[a_1, a_4] = a_4^{n_1} a_1^{n_4}$$

by Proposition 2.18,

$$\psi_3(a_1^{n_4}) = \psi_1(a_1)\xi_3(\psi_4(a_4)) = st$$

by Proposition 3.27 and

$$\xi_3\big(\psi_2(a_4^{n_1})\big) = \xi_3\big(\psi_4(a_4)\big) \cdot \xi_3\big(\xi_2(\psi_1(a_1))\big) = s^2 t$$

by Proposition 3.22(ii) and Proposition 3.27. □

Proposition 3.31. *$F \subset \tilde{K}F \subset \tilde{K}$ and $K^2 \subset K^2\tilde{F} \subset \tilde{F} \subset F$.*

Proof. By Notation 3.28, $\tilde{K}\tilde{F} \subset \tilde{K}$, F is generated by $\tilde{F}$ as a ring and $1 \in \tilde{K}$. It follows that $F \subset \tilde{K}F \subset \tilde{K}$. Similarly, we know that $\tilde{K}^2\tilde{F} \subset \tilde{F}$, K is generated by $\tilde{K}$ as a ring and $1 \in \tilde{F}$ and hence $K^2 \subset K^2\tilde{F} \subset \tilde{F}$. □

Proposition 3.32. *Let $K^\times$ denote the group of invertible elements of K and suppose that $r \in \tilde{K}^\times := \tilde{K} \cap K^\times$ and $u \in \tilde{F}^\times := \tilde{F} \cap K^\times$. Then $r^{-1} \in \tilde{K}$ and $u^{-1} \in \tilde{F}$.*

Proof. By Proposition 3.31, $r^{-2} \subset \tilde{F}$ and $\tilde{F}K^2 \subset \tilde{F}$. Hence $r^{-1} = r \cdot r^{-2} \in \tilde{K}\tilde{F} \subset \tilde{K}$ and $u^{-1} = u \cdot u^{-2} \subset \tilde{F}K^2 \subset \tilde{F}$. □

Notation 3.33. Let $x_0(t) = x_4(t)^{m_4}$ and $x_5(t) = x_1(t)^{m_1}$, where m_1 and m_4 are as in Proposition 3.11 and thus $m_1 = \mu_\gamma(x_1(1))$ and $m_4 = \mu_\gamma(x_4(1))$ by Notation 3.29. By Proposition 2.18 and Proposition 3.30, we have $x_4(t)^{m_1} = x_2(t)$ and $x_3(s)^{m_4} = x_1(s)$ for all $s \in \tilde{K}$ and all $t \in \tilde{F}$. Conjugating the relation in Proposition 3.30 by m_4 and by m_1, we thus obtain

$$[x_0(t), x_3(s)] = x_1(st)x_2(s^2t) \tag{3.34}$$

and

$$[x_2(t), x_5(s)] = x_3(st)x_4(s^2t) \tag{3.35}$$

for all $s \in \tilde{K}$ and all $t \in \tilde{F}$.

Proposition 3.36. *Let $s \in \tilde{K}$ and $t \in \tilde{F}$. Then $x_1(s) \in U_1^\sharp$ if and only if $s \in \tilde{K}^\times$ and $x_4(t) \in U_4^\times$ if and only if $t \in \tilde{F}^\times$.*

Proof. Suppose that $x_1(s) \in U_1^\sharp$ for some $s \in \tilde{K}$ and $x_4(t) \in U_4^\sharp$ for some $t \in \tilde{F}$. Then $\lambda_\gamma(x_1(s)) = x_5(r)$ for some $r \in \tilde{K}$ and $\kappa_\gamma(x_4(t)) = x_0(u)$ for some $u \in \tilde{F}$. By Proposition 2.18(i) applied to $[x_1(s), x_4(1)] = x_2(s^2)x_3(s)$, we obtain $[x_2(s^2), x_5(r)]_4 = x_4(1)$. By (3.35), it follows that $(sr)^2 = 1$. By Proposition 3.22(iii), therefore, $sr = 1$ and hence $s \in \tilde{K}^\times$. By Proposition 2.18(ii) applied to $[x_1(1), x_4(t)] = x_2(t)x_3(t)$, we have $[x_0(u), x_3(t)]_1 = x_1(1)$. By (3.34), it follows that $tu = 1$. Hence $t \in \tilde{F}^\times$.

Suppose, conversely, that $s \in \tilde{K}^\times$ and $t \in \tilde{F}^\times$. By Proposition 3.30, (3.34) and (3.35) and bit of calculation, we obtain

$$U_4^{x_1(s)x_5(s^{-1})} = U_2 \quad \text{and} \quad U_3^{x_0(t^{-1})x_4(t)} = U_1.$$

Hence $x_1(s) \in U_1^\sharp$ and $x_4(t) \in U_4^\sharp$ by Proposition 2.19. □

Proposition 3.37. *$x_1(s)^{\mu_\gamma(x_1(1))\mu_\gamma(x_1(r))} = x_1(r^2 s)$ for all $r \in \tilde{K}^\times$ and all $s \in \tilde{K}$.*

Proof. Let $\alpha_r = \mu_\gamma(x_1(1))\mu_\gamma(x_1(r))$ for all $r \in \tilde{K}^\times$. By Proposition 2.18 and Proposition 3.30, we have $x_4(t)^{\mu_\gamma(x_1(r))} = x_2(r^2 t)$ and hence $x_4(t)^{\alpha_r} = x_4(r^{-2}t)$ for all $r \in \tilde{K}^\times$ and all $t \in \tilde{F}$. We have $[\mu_\gamma(U_1^\sharp), U_3] = 1$. Conjugating the identity $[x_1(s), x_4(1)]_3 = x_3(s)$ by α_r and then applying Proposition 3.30, we conclude that $x_1(s)^{\alpha_r} = x_1(r^2 s)$ for all $r \in \tilde{K}^\times$ and all $s \in \tilde{K}$. □

Proposition 3.38. *Let σ be an automorphism of K, let S denote the subgroup $\{s \mapsto r^2 s \mid r \in \tilde{K}^\times\}$ of the automorphism group of the additive group of K and suppose that $[\sigma, S] = 1$. Then σ is the identity.*

Proof. Since $[\sigma, S] = 1$, we have $\sigma(r^2) = r^2$ for all $r \in \tilde{K}^\times$. By Proposition 2.22 and Proposition 3.36, $\tilde{K}$ is generated additively by $\tilde{K}^\times$ and as was observed in Proposition 3.28, K is generated as a ring by $\tilde{K}$. Therefore K is generated as a ring by $\tilde{K}^\times$. Hence $\sigma(s^2) = s^2$ for all $s \in K$. The claim holds, therefore, by Corollary 3.25. □

Proposition 3.39. *Let $h \in H$, where H is as in Notation 2.23. Then there exist $\rho \in \tilde{K}^\times$ and $\sigma \in \mathrm{Aut}(K)$ such that $x_1(s)^h = x_1(\rho s^\sigma)$ for all $s \in \tilde{K}$.*

Proof. There exist $\rho \in \tilde{K}$ and $\eta \in \tilde{F}$ such that

$$x_1(1)^h = x_1(\rho) \quad \text{and} \quad x_4(1)^h = x_4(\eta). \tag{3.40}$$

By Proposition 3.36, $x_i(1) \in U_i^\sharp$ for $i = 1$ and 4 and thus $\rho \in \tilde{K}^\times$ and $\eta \in \tilde{F}^\times$.

By Notation 3.28 and Proposition 3.32, $\eta\tilde{K} = \tilde{K}$ and $\rho^2\tilde{F} = \tilde{F}$. We can thus set $\hat{x}_1(s) = x_1(\rho s)$ and $\hat{x}_3(s) = x_3(\rho\eta s)$ for all $s \in \rho^{-1}\tilde{K}$ and $\hat{x}_2(t) = x_2(\rho^2\eta t)$ and $\hat{x}_4(t) = x_4(\eta t)$ for all $t \in \eta^{-1}\tilde{F}$. By Proposition 3.30, we have

$$[\hat{x}_1(s), \hat{x}_4(t)] = \hat{x}_2(s^2 t)\hat{x}_3(st) \tag{3.41}$$

for all $s \in \rho^{-1}\tilde{K}$ and all $t \in \eta^{-1}\tilde{F}$.

Next we let β_i be the map from $\tilde{K}$ to $\rho^{-1}\tilde{K}$ such that $x_i(s)^h = \hat{x}_i(\beta_i(s))$ for $i = 1$ and 3 and all $s \in \tilde{K}$ and let β_i be the map from $\tilde{F}$ to $\eta^{-1}\tilde{F}$ such that $x_i(s)^h = \hat{x}_i(\beta_i(t))$ for $i = 2$ and 4 and all $t \in \tilde{F}$. The maps β_i are all additive. By (3.40), we have $\beta_1(1) = 1$ and $\beta_4(1) = 1$. Conjugating the identity $[x_1(s), x_4(1)]_3 = x_3(s)$ by h, we thus obtain $\hat{x}_3(\beta_3(s)) = [\hat{x}_1(\beta_1(s)), \hat{x}_4(1)]_3$ for all $s \in \tilde{K}$ and hence $\beta_1 = \beta_3$ by (3.41). Conjugating the identity $[x_1(1), x_4(t)]_2 = x_2(t)x_3(t)$ by h, we obtain $[\hat{x}_1(1), \hat{x}_4(\beta_4(t))] = \hat{x}_2(\beta_2(t))\hat{x}_3(\beta_3(t))$ for all $t \in \tilde{F}$. By (3.41), it follows that $\beta_2 = \beta_4$ and that β_4 is the restriction of β_3 to $\tilde{F}$. Let $\beta = \beta_1$. Conjugating the identity $[x_1(s), x_4(t)]_2 = x_2(s^2t)$ by h, we obtain $[\hat{x}_1(\beta(s)), \hat{x}_4(\beta(t))]_2 = \hat{x}_2(\beta(s^2t))$ and hence

$$\beta(s)^2\beta(t) = \beta(s^2t)$$

for all $s \in \tilde{K}$ and all $t \in \tilde{F}$ by one more application of (3.41). Setting $t = 1$, it follows that $\beta(s)^2 = \beta(s^2)$ for all $s \in \tilde{K}$ and since $K^2 \subset \tilde{F} \subset \tilde{K}$ by Proposition 3.31, we thus obtain

$$\beta(s^2)\beta(u^2) = \beta(s^2u^2) \tag{3.42}$$

for all $s \in \tilde{K}$ and all $u \in K$. Since $\tilde{K}$ generates K, it follows that (3.42) holds for all $s, u \in K$. In other words, β restricts to an automorphism of K^2. By Proposition 3.22(iii), every element of K^2 has a unique square root in K. This implies that the map β has a unique extension to an automorphism σ if K. Hence $x_1(s)^h = \hat{x}_1(s^\sigma) = x_1(\rho s^\sigma)$ for all $s \in \tilde{K}$. □

Proposition 3.43. *Suppose that $[H^\dagger, h] = 1$ for some $h \in H$. Then there exists $\rho \in \tilde{K}^\times$ such that $x_1(s)^h = x_1(\rho s)$ for all $s \in \tilde{K}$.*

Proof. By Proposition 3.37, the subgroup of $\mathrm{Aut}(U_1)$ induced by $H^\dagger$ contains the group

$$\{x_1(s) \mapsto x_1(r^2s) \mid r \in \tilde{K}^\times\}.$$

The claim holds, therefore, by Proposition 3.38 and Proposition 3.39. □

Proposition 3.44. *K and F are fields and X is Moufang.*

Proof. Suppose $s \in \tilde{K}$ is a non-zero element that does not lie in $K^\times$ and let I denote the principal ideal of K generated by s. Then $x_1(I \cap \tilde{K})$ is a non-trivial subgroup of U_1. By Proposition 3.36, either $x_1(I \cap \tilde{K}) \cap U_1^\sharp = \emptyset$ or $I = K$. By hypothesis, the subgroup J in Theorem 3.1 centralizes $H^\dagger$. By Proposition 3.43, therefore, the subgroup $x_1(I \cap \tilde{K})$ is J-invariant. Again by hypothesis, this implies that $x_1(I \cap \tilde{K}) \cap U_1^\sharp \neq \emptyset$. Hence $I = K$. We conclude that every non-zero element of $\tilde{K}$ lies in $K^\times$ and thus $U_1^* = U_1^\sharp$. By Proposition 3.31, $K^2 \subset \tilde{K}$ and $K^2 \subset F \subset K$. It follows from the first containment that K is a field and hence the second containment implies that also F is a field. By Proposition 3.36 again, it follows that $U_4^* = U_4^\sharp$. By Proposition 2.15, therefore, X is Moufang. □

This concludes the proof of Theorem 3.1. Note that Notation 3.28 and Proposition 3.44, we now know that $(K, \tilde{K}, \tilde{F})$ is an indifferent set as defined in [7, 10.1]. Thus by [7, 7.5] and Proposition 3.30, Γ is isomorphic to the Moufang polygon described in [7, 16.4] with $(K, \tilde{K}, \tilde{F})$ in place of (K, K_0, F_0).

4. Octagons

Our goal in this section is to prove Theorem 1.1. Suppose that X satisfies the hypotheses of Theorem 1.1, let $(\gamma, i \mapsto w_i)$ and $i \mapsto U_i$ be as in Notation 2.4, let $U_i^\sharp$ for all i be as in Proposition 2.11 and let H be as in Notation 2.23.

Let V_i for all i be as in Notation 2.33. We have $V_i \subset U_i$ and $[V_i, U_j] = 1$ whenever $|i-j| \leq 4$. By Proposition 2.34, we can assume that the map $i \mapsto w_i$ has been chosen so that $V_i \neq 1$ for all even i. Since X is sharp, it follows that

$$V_i \cap U_i^\sharp \neq \emptyset \tag{4.1}$$

for all even i. By (2.13), we have

$$V_j^{\mu_\gamma(a_i)} = V_{2i+8-j} \tag{4.2}$$

for all i, j and all $a_i \in U_i^\sharp$.

Remark 4.3. Let $a_i \in V_i^\sharp$ for some even i and let $(v_0, \ldots, v_4, v_5)$ be a straight 5-path with $v_0 = w_{i+4}$. By Definition 2.2(iii) and Proposition 2.17, $U_{[i-4,i+4]}$ acts transitively on the set of straight 5-paths that start at w_{i+4}. Since $w_{i+9}^{a_i}$ is opposite w_{i+9} at w_{i+8}, it follows that $v_5^{a_i}$ is opposite v_5 at v_4.

Notation 4.4. Let u be a vertex at even distance from w_4. By Proposition 2.5, we can choose an element $g \in G$ such that $u = w_4^g$. Let $M_u = (V_0^\sharp)^g$. By Remark 4.3, the set $V_0^\sharp$ is normalized by the stabilizer G_{w_4}. Hence the set M_u is independent of the choice of g. In particular, $M_{w_i} = V_{w_{i-4}}^\sharp$ for all even i.

Proposition 4.5. *Let v be a vertex at odd distance from w_0 and let $u, z \in \Gamma_v$ be distinct. Then $M_u \cap M_z = \emptyset$, where M_u and M_z are as in Notation 4.4.*

Proof. By Definition 2.2(i), we can choose a vertex y in Γ_v that is opposite both u and z. Let $\alpha = (v_0, \ldots, v_8)$ be a root with $v_7 = y$ and $v_8 = v$. By Definition 2.2(iii), there exists $g \in U_\alpha$ mapping u to z. Suppose that $a \in M_u \cap M_z$ and let $v_4' = v_4^a$. By Remark 4.3 and Notation 4.4, v_4' and v_4 are opposite at v_5. By Proposition 2.9, a is the unique element of M_z mapping v_4 to v_4'. Since g acts trivially on Γ_{v_5}, the element $a^g \in (M_u)^g = M_z$ maps v_4 to v_4'. It follows that $[a, g] = 1$. Thus $g \in G^{(1)}_{v_1,v_2,v_3,v_4,v_5,v_4',v_3',v_2'}$, where $v_3' = v_3^a$ and $v_2' = v_2^a$. Let $\beta = (v_0, v_1, v_2, v_3, v_4, v_5, v_4', v_3', v_2')$. Then β is a root (because v_4' and v_4 are opposite at v_5) and g is an element of U_β acting trivially on $\Gamma_{v_2'}$. By Proposition 2.9, it follows that $g = 1$. This contradicts the assumption that $u \neq z$. $\square$

Proposition 4.6. $[a_1, a_6^{-1}] = a_6^{\mu_\gamma(a_1)} \in V_4$ *for all $a_1 \in U_1^\sharp$ and all $a_6 \in V_6$.*

Proof. Let $a_1 \in U_1^\sharp$, $a_6 \in V_6$, $u_9 = \kappa_\gamma(a_1)$, $v_9 = \lambda_\gamma(a_1)$ and $m = \mu_\gamma(a_1)$. Thus $m = u_9 a_1 v_9$. By the choice of a_1,

$$(w_{10}, w_9, w_{10}', w_{11}')$$

is a straight 3-path, where

$$w_i' = w_i^{a_1^{-1}}$$

for $i = 10$ and 11. By Notation 2.33,

$$a_6 \in G^{(1)}_{w_{10},w_9,w'_{10},w'_{11}}.$$

It follows that

$$[a_1, a_6^{-1}] \subset G^{(1)}_{w_6,\ldots,w_{11}}.$$

By Proposition 2.8, therefore, $[a_1, a_6^{-1}] \in U_{[4,5]}$. Let $a_k = [a_1, a_6^{-1}]_k$ for $k = 4$ and 5. Since $[a_6, u_9] \in [V_6, U_9] = 1$, we have

$$\begin{aligned} a_4a_5a_6 &= [a_1, a_6^{-1}] \cdot a_6 = [u_9^{-1}mv_9^{-1}, a_6^{-1}] \cdot a_6 \\ &= [mv_9^{-1}, a_6^{-1}] \cdot a_6 = v_9m^{-1}a_6mv_9^{-1} \\ &= a_6^m \cdot [a_6^m, v_9^{-1}] \end{aligned} \tag{4.7}$$

by Conventions 1.3(i). By (4.2), we have $a_6^m \in V_6^m = V_4$. Thus by Proposition 2.6(i), $[a_6^m, v_9^{-1}] \in U_{[5,8]}$. By Proposition 2.6(ii) and (4.7), it follows that $a_4 = a_6^m \in V_4$. Hence

$$[a_1, a_6^{-1}] = a_6^m a_5. \tag{4.8}$$

The element $a_6 \in V_6$ centralizes $U_{[2,8]}$. By Proposition 2.6(i), a_1 normalizes $U_{[2,8]}$. It follows that $a_4a_5 = [a_1, a_6^{-1}]$ centralizes $U_{[2,8]}$. Since $a_4 \in V_4$ centralizes $U_{[2,8]}$, we conclude that

$$[a_5, U_{[2,8]}] = 1. \tag{4.9}$$

Choose $a_{10} \in V_{10}^\sharp$ and let $u = w_9^{a_{10}^{-1}}$ and $v = w_8^{a_{10}^{-1}}$. Then (w_9, w_{10}, u, v) is a straight 3-path. Hence there exists $b \in U_{[2,3]}$ such that $u^b = w_{11}$ and $v^b = w_{12}$. By (4.9), $[a_5, b] = 1$. Since $a_5 \in G^{(1)}_{w_{11},w_{12}}$, it follows that $a_5 \in G^{(1)}_{u,v}$. Therefore

$$[a_5, a_{10}] \in G^{(1)}_{w_8,w_9,\ldots,w_{12}}. \tag{4.10}$$

The element a_{10} centralizes $U_{[6,12]}$ and by Proposition 2.6(i), a_5 normalizes $U_{[6,12]}$. It follows that $[a_5, a_{10}]$ centralizes $U_{[6,12]}$. Choose $a_{12} \in U_{12}^\sharp$. By (4.10), therefore,

$$[a_5, a_{10}] \in G^{(1)}_{w_8,w_9,\ldots,w_{11},w_{12},w'_{11},\ldots,w'_9,w'_8}, \tag{4.11}$$

where

$$w'_i = w_i^{a_{12}}$$

for all $i \in [8, 11]$. By the choice of a_{12}, the sequence

$$(w_8, w_9, \ldots, w_{11}, w_{12}, w'_{11}, \ldots, w'_9, w'_8)$$

is a straight 8-path. By Proposition 2.9 and (4.11), it follows that

$$[a_5, a_{10}] = 1. \tag{4.12}$$

By the choice of a_{10},

$$(w_6, w_7, w_8, w_9, w_{10}, w''_9, w''_8, w''_7, w''_6)$$

is a straight 8-path, where $w_i'' = w_i^{a_{10}}$ for all $i \in [6,9]$, and by (4.12),

$$a_5 \in G^{(1)}_{w_6,w_7,w_8,w_9,w_{10},w_9'',w_8'',w_7'',w_6''}.$$

By another application of Proposition 2.9, we conclude that $a_5 = 1$. By (4.8), therefore, the claim holds. □

Corollary 4.13. *$[U_1, \langle V_6^\sharp\rangle] \subset \langle V_4^\sharp\rangle$, $[U_1, V_6] \subset V_4$, U_1 is abelian and for each $a_6 \in V_6^\sharp$, the map $a_1 \mapsto [a_1, a_6]$ from U_1 to V_4 is a faithful homomorphism.*

Proof. By Conventions 1.3(ii) and Proposition 4.6, we have $[U_1^\sharp, \langle V_6^\sharp\rangle] \subset \langle V_4^\sharp\rangle$ and $[U_1^\sharp, V_6] \subset V_4$. By Conventions 1.3(i) and Proposition 2.22, therefore, we have $[U_1, \langle V_6^\sharp\rangle] \subset \langle V_4^\sharp\rangle$ and $[U_1, V_6] \subset V_4$. Choose $a_6 \in V_6^\sharp$. By Conventions 1.3(i), the map $a_1 \mapsto [a_1, a_6]$ from U_1 to V_4 is a homomorphism. Choose a_1 in the kernel of this map and let $u = w_{10}^{a_1}$. Since $[a_1, a_6] = 1$ and $a_6 \in M_{w_{10}}$, we have $a_6 \in M_{w_{10}} \cap M_u$ and hence $u = w_{10}$ by Proposition 4.5. By Proposition 2.9, therefore, $a_1 = 1$. Thus the map $a_1 \mapsto [a_1, a_6]$ is injective. Since V_4 is abelian, it follows that U_1 is too. □

Remark 4.14. Let D be the dihedral group generated by the permutations $i \mapsto 8-i$ and $i \mapsto 10-i$ of $\mathbb{Z}_{16}$. By (2.13), Proposition 2.14(iii) and Proposition 4.6, we have

$$[a_i, a_j^{-1}] = a_j^{\mu_\gamma(a_i)} \in V_k$$

for all $a_i \in U_i^\sharp$ and $a_j \in V_j$ whenever $(i,j) \in (1,6)^D$. We will use this observation implicitly whenever we refer to Proposition 4.6. A similar comment applies to all the identities and assertions that follow. Thus, for example, it follows from Corollary 4.13 that $[U_i, V_j] \subset V_{j-2}$ whenever $(i,j) \in (1,6)^D$ and that U_i is abelian for all odd i.

Proposition 4.15. *For each $a_0 \in V_0^\sharp$ and each $a_5 \in U_5^\sharp$,*

(i) $[a_2, v_8] = a_3a_5a_6$ *and*
(ii) $[a_5, v_8] = a_6$,

where $v_8 = \lambda_\gamma(a_0)$, $a_2 = a_0^{\mu_\gamma(a_5)}$, $a_3 = (a_5^{-1})^{\mu_\gamma(a_0)}$ and $a_6 = a_2^{\mu_\gamma(a_0)}$.

Proof. Choose $a_0 \in V_0^\sharp$ and $a_5 \in U_5^\sharp$. Let $u_8 = \kappa_\gamma(a_0)$, $v_8 = \lambda_\gamma(a_0)$, $m = \mu_\gamma(a_0)$, $a_2 = a_0^{\mu_\gamma(a_5)}$, $a_3 = (a_5^{-1})^m$, $a_6 = a_2^m$ and $w_0 = u_8^m$. Then $m = u_8a_0v_8$, $a_k \in U_k^\sharp$ for $k = 2$, 3 and 6 and $w_0 \in U_0^\sharp$ by (2.13). By Proposition 2.6(i), $a_2^{mw_0^{-1}} \in U_{[1,5]}a_6$. Since $[a_0, a_2] \in [V_0, U_2] = 1$, we have $a_2^{a_0v_8} = a_2^{v_8} \in a_2U_{[3,7]}$ by Proposition 2.6(i). Since $mw_0^{-1} = u_8^{-1}m = a_0v_8$, it follows that

$$a_2^{v_8} \in a_2U_{[3,7]} \cap U_{[1,5]}a_6.$$

Thus $a_2^{v_8} \in a_2U_{[3,5]}a_6$ by Proposition 2.6(ii).

By Proposition 4.6 and Remark 4.14, $[a_5, a_0^{-1}] = [a_0, a_5] = a_2$, so

$$a_2^{v_8} = ((a_5^{-1})^{a_0}a_5)^{v_8} = (a_5^{-1})^{mw_0^{-1}}a_5^{v_8}.$$

We have $(a_5^{-1})^{mw_0^{-1}} = a_3^{w_0^{-1}} \in U_{[1,2]}a_3$ and $a_5^{v_8} \in a_5U_{[6,7]}$ by Proposition 2.6(i). Thus

$$a_2^{v_8} \in U_{[1,2]}a_3 \cdot a_5U_{[6,7]}.$$

By Proposition 2.6(ii) and the conclusion of the previous paragraph, therefore, $a_2^{v_8} = a_2a_3a_5a_6$ and $a_5^{v_8} = a_5a_6$. □

Corollary 4.16. $[U_5, \lambda_\gamma(V_0^\sharp)] \subset \langle V_6^\sharp\rangle$.

Proof. By Proposition 4.15(ii), $[U_5^\sharp, \lambda_\gamma(V_0^\sharp)] \subset V_6^\sharp$. The claim follows by Conventions 1.3(i) and Proposition 2.22 since $[U_5, V_6] = 1$. □

Proposition 4.17. $[U_5, U_7] = [U_3, U_7] = 1$.

Proof. Choose $a_0 \in V_0^\sharp$ and let $v_8 = \lambda_\gamma(a_0)$ and $m = \mu_\gamma(a_0)$. Choose $a_3 \in U_3^\sharp$ and $a_7 \in U_7^\sharp$ and let $a_5 = (a_3^{-1})^{m^{-1}}$. By Proposition 4.15(i), $[a_2, v_8] = a_3a_5a_6$ for $a_2 = a_0^{\mu_\gamma(a_5)}$ and $a_6 = a_2^m$. By Proposition 4.6, $[a_7, a_2^{-1}] = a_4$ and therefore $a_2^{a_7} = a_4a_2$ for $a_4 = a_2^{\mu_\gamma(a_7)} \in V_4$. Thus $[a_2, v_8]^{a_7} = [a_4a_2, v_8]$ since $[U_7, U_8] = 1$. Since $[a_4, v_8] \in [V_4, U_8] = 1$, we have $[a_4a_2, v_8] = [a_2, v_8]$ by Conventions 1.3(i). Thus

$$[a_3a_5a_6, a_7] = [[a_2, v_8], a_7] = 1. \tag{4.18}$$

We have $[a_6, U_{[3,7]}] \in [V_6, U_{[3,7]}] = 1$ and thus $[a_3a_5a_6, a_7] = [a_3a_5, a_7]$. By Proposition 2.6(i) and Corollary 4.13, we have $[a_5, [a_3, a_7]] \in [a_5, U_{[4,6]}] = 1$. By Conventions 1.3(i), therefore, $[a_3a_5, a_7] = [a_3, a_7] \cdot [a_5, a_7]$. Hence $[a_3, a_7] = [a_5, a_7]^{-1}$ by (4.18). We conclude that $[U_3^\sharp, U_7^\sharp] = [U_5^\sharp, U_7^\sharp]$. By Proposition 2.6(i), $[U_5, U_7] \subset U_6$ and thus $[U_3^\sharp, U_7^\sharp] \subset U_6$. By Remark 4.14, $[U_3^\sharp, U_7^\sharp] \subset U_6$ implies that $[U_3^\sharp, U_7^\sharp] \subset U_4$. By Proposition 2.6(ii), $U_4 \cap U_6 = 1$. It follows that $[U_5^\sharp, U_7^\sharp] = [U_3^\sharp, U_7^\sharp] = 1$. By Proposition 2.22, therefore, the claim holds. □

Proposition 4.19. $[U_1, U_7] \subset U_3U_5$.

Proof. Let i be odd. By Proposition 4.17, $[U_i, U_{i+2}] = 1$. By Definition 2.2(iii), it follows that $U_i \subset G_u^{(1)}$ for all u opposite w_{i+1} at w_{i+2} and $U_{i+2} \subset G_v^{(1)}$ for all v opposite w_{i+7} at w_{i+8}. Thus

$$[U_1^\sharp, U_7^\sharp] \subset G_{w_6,\ldots,w_{10}}^{(1)}.$$

By 2.8, therefore, $[U_1^\sharp, U_7^\sharp] \subset U_{[3,5]}$.

Now choose $a_1 \in U_1^\sharp$ and let $u_9 = \kappa_\gamma(a_1)$, $v_9 = \lambda_\gamma(a_1)$ and $m = \mu_\gamma(a_1)$, so $m = u_9a_1v_9$. Let $a_7 \in U_7^\sharp$. Then $a_7^{mv_9^{-1}} = a_7^{u_9a_1} = a_7^{a_1} \in U_{[3,5]}a_7$ by the conclusion of the previous paragraph. Since $a_7^m \in U_3$ by (2.13), we also have $a_7^{mv_9^{-1}} \in a_7^mU_{[5,7]}$ by Remark 4.14 and the conclusion of the previous paragraph. Thus

$$a_7^{a_1} \in a_7^mU_{[5,7]} \cap U_{[3,5]}a_7 \subset a_7^mU_5a_7$$

by Proposition 2.6(ii). Hence $[U_1^\sharp, U_7^\sharp] \subset U_3U_5$. The claim follows now by Conventions 1.3(i)–(ii), Proposition 2.22 and Proposition 4.17. □

Proposition 4.20. *Let*

$$\hat{G} = H \cdot \langle U_i \mid i \text{ odd}\rangle,$$

where H is as in Notation 2.23. Then there exist an indifferent Tits quadrangle

$$\hat{X} = (\hat{\Gamma}, \hat{\mathcal{A}}, \{\hat{\equiv}_v\}_{v\in\hat{V}}),$$

a coordinate system $(\hat{\gamma}, i \mapsto \hat{w}_i)$ of $\hat{X}$ with root group labeling $i \mapsto \hat{U}_i$ and a homomorphism φ from $\hat{G}$ to $\mathrm{Aut}(\hat{X})$ such that $\varphi(H)$ is the pointwise stabilizer of $\hat{\gamma}$ in $\varphi(\hat{G})$ and the restriction of φ to U_i is an isomorphism from U_i to $\hat{U}_{(i+1)/2}$ for all odd i.

Proof. Let Φ_8 be as in [1, 2.1] and let α_i denote the root $(w_i, w_{i+1}, \ldots, w_{i+n})$ for each i. We identify Φ_8 with $\{\alpha_i \mid i \in \mathbb{Z}\}$ as described in [1, 4.7]. By [1, 5.1], the map $\alpha_i \mapsto U_i$ is a stable Φ_8-grading of G with torus H as defined in [1, 2.3]. By Proposition 2.11, we can assume that the set M_{α_i} that appears in [1, 2.3(iii)] equals $\mu_\gamma(U_i^\sharp)$. After identifying $\{\alpha_i \mid i \text{ odd}\}$ with Φ_4, we observe that the restriction of the map $\alpha_i \mapsto U_i$ to $\{\alpha_i \mid i \text{ odd}\}$ is a stable Φ_4-grading of $\hat{G}$ with torus H (and with the same sets M_{α_i}). Let $\hat{X}$ be the Tits quadrangle obtained by applying [1, 5.2 and 5.3] to this Φ_4-grading, let $(\hat{\gamma}, i \mapsto \hat{w}_i)$ be the coordinate system of $\hat{X}$ described in [1, 5.7] and let $i \mapsto \hat{U}_i$ be the corresponding root group labeling. Let φ be the homomorphism from $\hat{G}$ to $\mathrm{Aut}(\hat{X})$ corresponding to the action of $\hat{G}$ on $\hat{X}$ by right multiplication. Then by [1, 5.3], $\varphi(H)$ is the pointwise stabilizer of $\hat{\gamma}$ in $\varphi(\hat{G})$. By [1, 5.19], the restriction of φ to U_i is an isomorphism from U_i to $\hat{U}_{(i+1)/2}$ for all odd i. By Definition 2.30, Remark 4.14 and Proposition 4.17, $\hat{X}$ is indifferent. □

Proposition 4.21. *$\varphi(U_i^\sharp) = \hat{U}_{(i+1)/2}^\sharp$ for all odd i, $\hat{\lambda}_{\hat{\gamma}} \circ \varphi = \varphi \circ \lambda_\gamma$ and $\hat{\kappa}_{\hat{\gamma}} \circ \varphi = \varphi \circ \kappa_\gamma$, where $\hat{U}_{(i+1)/2}$ and φ are as in Proposition 4.20 and $\hat{\lambda}_{\hat{\gamma}}$ and $\hat{\kappa}_{\hat{\gamma}}$ are as in Proposition 2.11 applied to $\hat{X}$.*

Proof. To prove the first claim, it suffices to assume that $i = 1$. Let $\hat{a}_1 = \varphi(a_1)$ for some $a_1 \in U_1$. Suppose first that $\hat{a}_1 \in \hat{U}_1^\sharp$, let $\hat{c}_9 = \hat{\lambda}_{\hat{\gamma}}(a_1)$ and let c_9 be the unique element of U_9 such that $\varphi(c_9) = \hat{c}_9$. By Proposition 2.11, we have $U_9^{a_1c_9} = U_1$. By Proposition 2.28, therefore, a_1c_9 maps the root $(w_1, w_0, w_{15}, \ldots, w_9)$ to the root $(w_1, w_2, w_3, \ldots, w_9)$. Since $U_8 = G^{(1)}_{w_9,w_{10},\ldots,w_{15}}$ and $U_2 = G^{(1)}_{w_3,w_4,\ldots,w_9}$, it follows that $U_8^{a_1c_9} = U_2$. By Proposition 2.19(i), therefore, $a_1 \in U_1^\sharp$ and $c_9 = \lambda_\gamma(a_1)$. Suppose, conversely, that $a_1 \in U_1^\sharp$, let $c_9 = \lambda_\gamma(a_1)$ and let $\hat{c}_9 = \varphi(c_9)$. By Proposition 4.17 and Proposition 2.11 applied to X, we have $\hat{U}_4^{\hat{a}_1\hat{c}_9} = \hat{U}_2$. By Proposition 2.19(i) again, it follows that $\hat{a}_1 \in \hat{U}_1^\sharp$ and $\hat{c}_9 = \hat{\lambda}_{\hat{\gamma}}(\hat{a}_1)$. Thus $\varphi(U_i^\sharp) = \hat{U}_{(i+1)/2}^\sharp$ and $\hat{\lambda}_{\hat{\gamma}} \circ \varphi = \varphi \circ \lambda_\gamma$. By Proposition 2.14(i), it follows that $\hat{\kappa}_{\hat{\gamma}} \circ \varphi = \varphi \circ \kappa_\gamma$. □

Corollary 4.22. *If $\hat{X}$ is Moufang, then $U_i^\sharp = U_i^*$ for all odd i.*

Proof. If $\hat{X}$ is Moufang, then by Notations 2.3 and 2.10, $\hat{U}_i^\sharp = \hat{U}_i^*$ for all i. The claim holds, therefore, by Proposition 4.21. ☐

Corollary 4.23. *$\hat{X}$ is sharp.*

Proof. Let $H^\dagger$ be as in Notation 2.23. Since X is dagger-sharp and U_i is abelian, every non-trivial $H^\dagger$-invariant subgroup of U_i for i odd contains elements of $U_i^\sharp$. Every non-trivial $\varphi(H^\dagger)$-invariant subgroup of $\hat{U}_j$ is the image under φ of a non-trivial $H^\dagger$-invariant subgroup of U_i. By Proposition 4.21, it follows that for all j, every non-trivial $\varphi(H^\dagger)$-invariant subgroup of $\hat{U}_j$ contains elements of $\hat{U}_j^\sharp$. Since $\hat{U}_j$ is abelian for all j, it follows that $\hat{X}$ is sharp. ☐

Proposition 4.24. *Let i be odd and let $j = (i+1)/2$. Then there exists a bijection π_i from Γ_{w_i} to $\hat{\Gamma}_{\hat{w}_i}$ mapping $\equiv_{w_i}$ to $\hat{\equiv}_{\hat{w}_j}$ and $w_{i+2\varepsilon}$ to $\hat{w}_{j+\varepsilon}$ for $\varepsilon = 1$ and -1 such that $\pi_i(u^g) = \pi_i(u)^{\varphi(g)}$ for all $g \in \langle U_i, H, U_{i+8}\rangle$.*

Proof. Let $Q_i = \langle U_i, H, U_{i+8}\rangle$ and let

$$S_i = \bigcap_{g\in\langle U_i, U_{i+8}\rangle} H^g.$$

By [1, 5.1], we can identify X with the Tits octagon that arises as in [1, 5.2–5.3] starting with $i \mapsto U_i$ and H. By [1, 5.2(a)], the group Q_i acts transitively on Γ_{w_i} and hence by [1, 5.4(i)], S_i is the kernel of this action. Let $\hat{H} = \varphi(H)$. The homomorphism φ maps Q_i to $\hat{Q}_j := \langle \hat{U}_j, \hat{H}, \hat{U}_{j+4}\rangle$ and S_i to

$$\hat{S}_j := \bigcap_{g\in\langle \hat{U}_j, \hat{U}_{j+4}\rangle} \hat{H}^g.$$

Suppose that $\varphi(g) \in \hat{H}$ for some $g \in Q_i$. Let $j = i$ or $i+8$. Then $\varphi(U_j) = \hat{U}_j^{\varphi(g)} = \varphi(U_j^g)$. By [1, (2.4) and 5.1], we have $Q_j = U_jU_{j+8}U_jH$. Thus $g = abch$ with $a, c \in U_j$, $b \in U_{j+8}$ and $h \in H$. Thus

$$\varphi(U_j^g) = \varphi(U_j^b)^{\varphi(c)\varphi(h)}.$$

Since $\varphi(c)$ and $\varphi(h)$ normalize $\varphi(U_j)$, it follows that $\varphi(U_j^b) = \varphi(U_j)$. By Proposition 2.27 and Corollary 4.23, it follows that $b = 1$. Hence $g = ach$. Thus g normalizes both U_i and U_{i+8}. The group Q_i stabilizes both w_i and w_{i+8}. By Proposition 2.28, it follows that $g \in H$. We conclude that $\varphi^{-1}(\hat{H}) = H$. Hence $\varphi^{-1}(S_j) = S_i$. Therefore φ induces an isomorphism from Q_i/S_i to $\hat{Q}_j/\hat{S}_j$.

It follows that φ induces a bijection from the set of right cosets of $B_i := U_iH$ in Q_i to the set of right cosets of $\hat{B}_j := \hat{U}_j\hat{H}$ in $\hat{Q}_j$. By [1, 5.4(i)], therefore, there exists a bijection π_i from Γ_{w_i} to $\hat{\Gamma}_{\hat{w}_j}$ mapping w_{i+2} to $\hat{w}_{j+1}$ such that $\pi_i(u^g) = \pi_i(u)^{\varphi(u)}$ for all $u \in Q_i$. Choose $a \in U_i^\sharp$ and let $\hat{a} = \varphi(a)$. By Proposition 4.21, $\hat{a} \in \hat{U}_i^\sharp$ and φ maps $m := \mu_\gamma(a)$ to $\hat{m} := \mu_{\hat{\gamma}}(\hat{a})$. Thus π_i maps $w_{i-2} = w_{i+2}^m$ to $\hat{w}_{j-1} = \hat{w}_{j+1}^{\hat{m}}$ and φ maps the double coset B_imB_i to the double coset $\hat{B}_j\hat{m}\hat{B}_j$. Thus by [1, 5.2(c)], vertices $u, v \in \Gamma_{w_i}$ are opposite at w_i if and only if $\pi_i(u)$ and $\pi_i(v)$ are opposite at $\hat{w}_j$. In other words, π_i maps $\equiv_{w_i}$ to $\hat{\equiv}_{\hat{w}_j}$. □

Corollary 4.25. *$\hat{X}$ is 5-plump.*

Proof. By hypothesis, X is 9-plump. By Proposition 4.24, therefore, $\hat{X}$ is also 9-plump "at $\hat{w}_j$" for all j, so by Proposition 2.5, $\hat{X}$ is 9-plump. Thus, in particular, $\hat{X}$ is 5-plump. □

Proposition 4.26. *The normalizer $N_{\hat{U}_1}(\hat{U}_3\hat{U}_4)$ is trivial.*

Proof. By Proposition 2.18(i), we have $N_{\hat{U}_1^\sharp}(\hat{U}_3\hat{U}_4) = \emptyset$. By Proposition 4.21, therefore, $N_{U_1^\sharp}(U_5U_7) = \emptyset$. Since X is dagger-sharp, it follows that $N_{U_1}(U_5U_7) = 1$. Hence $N_{\hat{U}_1}(\hat{U}_3\hat{U}_4) = 1$. □

Proposition 4.27. *The following hold:*

(i) $\exp(U_i) = \exp(V_{i+1}) = 2$ *for all odd i.*
(ii) $\mu_\gamma(a_0)^2 = \mu_\gamma(a_1)^2 = 1$ *for all $a_0 \in V_0^\sharp$ and $a_1 \in U_1^\sharp$.*
(iii) $\kappa_\gamma(a_0) = \lambda_\gamma(a_0)^{-1}$ *and* $\kappa_\gamma(a_1) = \lambda_\gamma(a_1)$ *for all $a_0 \in V_0^\sharp$ and $a_1 \in U_1^\sharp$.*

Proof. By Propositions 3.2, 4.21 and 4.26, we have $\exp(U_i) = 2$ and $\kappa_\gamma(a_i) = \lambda_\gamma(a_i)$ for all odd i and all $a_i \in U_i^\sharp$. Choose $a_1 \in U_1^\sharp$ and $a_4 \in V_4$. By Proposition 4.6, there exists $a_6 \in V_6$ such that $[a_1, a_6] = a_4$. Then $a_4^2 = [a_1^2, a_6] = 1$ since $[a_1, a_4] \in [U_1, V_4] = 1$. Thus $\exp(V_4) = 2$ and hence $\exp(V_i) = 2$ for all even i. Thus (i) holds. By Proposition 2.14(i), it follows that (ii) and the first claim in (iii) hold. □

Proposition 4.28. $[U_4, \kappa_\gamma(a_0)] = [U_4, \lambda_\gamma(a_0)] = 1$ *for all $a_0 \in V_0^\sharp$.*

Proof. Choose $a_0 \in V_0^\sharp$ and let $u_8 = \kappa_\gamma(a_0)$, $v_8 = \lambda_\gamma(a_0)$, $m = \mu_\gamma(a_0)$ and

$$w_0 = v_8^{m^{-1}}. \tag{4.29}$$

Then $v_8m^{-1} \cdot u_8a_0 = 1$ and hence $m = w_0u_8a_0$. Let $a_4 \in U_4$. By (2.13), $a_4^m \in U_4$, so $[a_0, a_4^m] \in [V_0, U_4] = 1$. Thus $a_4^{w_0} = a_4^{ma_0^{-1}u_8^{-1}} = a_4^{mu_8^{-1}} = a_4^m \cdot [a_4^m, u_8^{-1}] \in a_4^m U_{[5,7]}$ by Proposition 2.6(i). On the other hand, $a_4^{w_0} = [w_0, a_4^{-1}] \cdot a_4 \in U_{[1,3]}a_4$ by Proposition 2.6(i). Thus by Proposition 2.6(ii), a_4 commutes with m, u_8 and w_0. By (4.29), a_4 commutes with v_8 as well. □

Proposition 4.30. $[a_2, a_8]_6 = a_2^{\mu_\gamma(a_8)} \in V_6$ *and* $[a_2, a_8]_7 = 1$ *for each $a_2 \in V_2$ and $a_8 \in U_8^\sharp$.*

Proof. Choose $a_2 \in V_2$ and $a_8 \in U_8^\sharp$. Let $u_0 = \kappa_\gamma(a_8)$, $v_0 = \lambda_\gamma(a_8)$ and $m = \mu_\gamma(a_8)$, so $m = u_0a_8v_0$. Then $a_2^m = a_2^{u_0a_8v_0} = a_2^{a_8v_0} = a_2 \cdot [a_2, a_8]^{v_0}$ since $[U_0, a_2] \subset [U_0, V_2] = 1$. By Proposition 2.6(i), $a_2 \cdot [a_2, a_8]^{v_0} \in U_{[1,6]}a_7$, where $a_7 = [a_2, a_8]_7$. By (2.13), $a_2^m \in V_6$. By Proposition 2.6(ii), therefore, $a_7 = 1$.

Thus $a_2\cdot[a_2,a_8]^{v_0}\in U_{[1,5]}a_6$, where $a_6=[a_2,a_8]_6$. By Proposition 2.6(ii) again, we conclude that $a_6=a_2^m$. ☐

Corollary 4.31. *$[a_2,a_8]\in U_{[3,5]}\cdot\langle V_6^\sharp\rangle$ for all $a_2\in\langle V_2^\sharp\rangle$ and all $a_8\in U_8^\sharp$.*

Proof. This holds by Proposition 2.6(i) and Proposition 4.30. ☐

Corollary 4.32. *$[\langle V_2^\sharp\rangle,\langle V_8^\sharp\rangle]\subset\langle V_4^\sharp\rangle U_5\langle V_6^\sharp\rangle$.*

Proof. By Proposition 2.6(i) and Proposition 4.30, $[V_2^\sharp,V_8^\sharp]\subset U_{[3,5]}V_6^\sharp$. By Remark 4.14, therefore, $[V_2^\sharp,V_8^\sharp]\subset V_4^\sharp U_{[5,7]}$. Hence

$$[V_2^\sharp,V_8^\sharp]\subset V_4^\sharp U_{[5,7]}\cap U_{[3,5]}V_6^\sharp=V_4^\sharp U_5V_6^\sharp$$

by Proposition 2.6(ii). The claim follows now by Conventions 1.3(i)–(ii). ☐

Proposition 4.33. *Let $a_0\in V_0^\sharp$ and $a_3\in U_3^\sharp$. Then $[a_3,v_8]=a_4a_5$ and $[a_3,v_8^{-1}]=a_4a_5a_6$, where $v_8=\lambda_\gamma(a_0)$, $a_5=a_3^{\mu_\gamma(a_0)}$, $a_6=a_0^{\mu_\gamma(a_5)\mu_\gamma(a_0)}$ and $a_4=a_0^{\mu_\gamma(a_6)}$.*

Proof. Let $u_8=\kappa_\gamma(a_0)$, $v_8=\lambda_\gamma(a_0)$, $m=\mu_\gamma(a_0)$, $a_5=a_3^m$ and $w_0=u_8^m$. Then $m=u_8a_0v_8$. By (2.13), $a_5\in U_5^\sharp$ and $w_0\in U_0$. By Proposition 2.6(i), therefore, $a_3^{mw_0^{-1}}\in U_{[1,4]}a_5$ and, since $[a_0,a_3]\in[V_0,U_3]=1$, $a_3^{a_0v_8}=a_3^{v_8}\in a_3U_{[4,7]}$. Since $m=a_0v_8w_0$, it follows that

$$a_3^{v_8}\in a_3U_{[4,7]}\cap U_{[1,4]}a_5.$$

Therefore $a_3^{v_8}\in a_3U_4a_5$ by Proposition 2.6(ii). Thus $[a_3,v_8]=a_4a_5$ for some $a_4\in U_4$. By Proposition 4.27(iii), $u_8=v_8^{-1}$. By Conventions 1.3(ii), therefore,

$$1=[a_3,v_8u_8]=[a_3,u_8]\cdot(a_4a_5)^{u_8}.$$

By Proposition 4.28, $[a_4,u_8]=1$. By Proposition 4.15(ii), $[a_5,v_8]=a_6$, where $a_6=a_0^{\mu_\gamma(a_5)m}\in V_6^\sharp$. Since $[a_6,U_8]\subset[V_6,U_8]=1$, it follows by Conventions 1.3(ii) that $[a_5,v_8^{-1}]=a_6^{-1}$. Hence

$$[a_5,u_8]=[a_5,v_8^{-1}]=a_6$$

by Proposition 4.27(i). We conclude that $[a_3,u_8]=(a_4a_5a_6)^{-1}$. By Proposition 4.27(i), $(a_4a_5a_6)^{-1}=a_4^{-1}a_5a_6$ since $[a_4,a_6]\in[U_4,V_6]=1$. It remains to show only that $a_4=a_0^{\mu_\gamma(a_6)}$, since then $a_4\in V_4$ by (2.13) and thus $a_4=a_4^{-1}$ by Proposition 4.27(i).

Since $[a_0,a_3a_4^{-1}]\in[V_0,U_{[3,4]}]=1$, we have

$$a_3^{u_8a_0}=(a_3\cdot[a_3,u_8])^{a_0}=(a_3a_4^{-1}a_5a_6)^{a_0}=a_3a_4^{-1}a_5^{a_0}a_6^{a_0}.$$

By Proposition 4.6, $a_5^{a_0}\in V_2a_5$. By Corollary 4.32, $[V_0^\sharp,V_6^\sharp]\subset V_2U_3V_4$ and hence

$$[a_0,a_6]\in V_2U_3a_0^{\mu_\gamma(a_6)}$$

by Proposition 4.30. Thus by Proposition 4.17,

$$a_3^{u_8a_0} = a_3a_4^{-1}a_5^{a_0}a_6^{a_0} \in V_2a_3a_4^{-1}a_5a_6^{a_0} \subset V_2U_3a_4^{-1}a_0^{\mu_\gamma(a_6)}a_5a_6$$

since $[V_2, U_{[3,5]}] = 1$. On the other hand, $a_3^{u_8a_0} = a_3^{mu_8} = a_5^{u_8} = a_5a_6$ since $u_8 = v_8^{-1}$. Thus $a_4 = a_0^{\mu_\gamma(a_6)}$ by Proposition 2.6(ii). □

By Proposition 4.27(i), $\exp(U_i) = \exp(V_{i+1}) = 2$ for all odd i. From now on, we will use this fact without explicitly referring to Proposition 4.27(i).

Proposition 4.34. $N_{V_2}(U_{[4,8]}) = 1$.

Proof. Let $a_2 \in V_2^\sharp$ and $a_5 \in U_5^\sharp$. By (2.13), we have $\lambda_\gamma(a_2^{\mu_\gamma(a_5)}) \in U_8$ and by Proposition 4.15(i),

$$[a_2, \lambda_\gamma(a_2^{\mu_\gamma(a_5)})]_3 \neq 1.$$

Thus a_2 does not normalize $U_{[4,8]}$. Since X is sharp, the claim follows. □

Proposition 4.35. *Suppose that* $[a_2, a_8]_5 = 1$ *for some* $a_2 \in \langle V_2^\sharp\rangle$ *and some* $a_8 \in V_8^\sharp$. *Then* $a_2 = 1$.

Proof. By Corollary 4.32, we have $[a_2, a_8] \in V_4V_6$. Thus $[[a_2, a_8], U_8] = 1$. Since $[a_8, U_8] \in [V_8, U_8] = 1$, it follows that $[[a_2, U_8], a_8] = 1$ by [7, 2.3]. Hence

$$[a_2, U_8] \subset U_{[3,7]} \cap C_G(a_8) = U_{[4,7]}$$

by Proposition 2.6(i) and Proposition 4.13. Thus a_2 normalizes $U_{[4,8]}$. By Proposition 4.34, it follows that $a_2 = 1$. □

Proposition 4.36. *For each* $a_6 \in \langle V_6^\sharp\rangle$ *and* $a_8 \in V_8^\sharp$, *there exists* $a_3 \in U_3$ *such that* $[a_3, a_8] = a_6$.

Proof. Choose $a_6 \in \langle V_6^\sharp\rangle$ and $a_8 \in V_8^\sharp$ and let $u_0 = \kappa_\gamma(a_8)$, $v_0 = \lambda_\gamma(a_8)$ and $m = \mu_\gamma(a_8)$. Let $a_2 = a_6^m$. Then $a_2 \in \langle V_2^\sharp\rangle$ by (2.13) and $m = m^{-1}$ by Proposition 4.27(ii). By Proposition 4.30 and Corollary 4.32, therefore, $[a_2, a_8] \in V_4U_5a_6$. Let $a_5 = [a_2, a_8]_5$, $a_3 = a_5^m$ and $b_2 = [v_0^{-1}, a_3]$. By (2.13), $a_3 \in U_3$ and thus

$$[a_3, a_8] \in V_6 \tag{4.37}$$

by Corollary 4.13. By Corollary 4.16, we have $[\lambda_\gamma(V_8^\sharp), U_3] \subset \langle V_2^\sharp\rangle$. Thus $[v_0, a_3] \in \langle V_2^\sharp\rangle$. Since $[U_0, V_2] = 1$, it follows that $b_2 = [v_0, a_3]^{-1} \in \langle V_2^\sharp\rangle$. Hence

$$a_3^{v_0^{-1}} = [v_0^{-1}, a_3] \cdot a_3 = b_2a_3$$

by Proposition 4.27(i). Thus by Corollary 4.32, we have

$$\begin{aligned} a_5^{mv_0^{-1}a_8} &= a_3^{v_0^{-1}a_8} = (b_2a_3)^{a_8} \\ &= b_2 \cdot [b_2, a_8] \cdot a_3 \cdot [a_3, a_8] \\ &\in U_{[2,4]} \cdot [b_2, a_8]_5 \cdot [b_2, a_8]_6 \cdot [a_3, a_8] \end{aligned} \tag{4.38}$$

since $[a_3,[b_2,a_8]_6] \in [U_3,[V_2,V_8]_6] \subset [U_3,V_6] = 1$. We have $mv_0^{-1}a_8 = u_0$. Since $a_5^{u_0} \in U_{[1,4]}a_5$, we conclude that

$$[b_2,a_8]_5 = a_5 \text{ and } [b_2,a_8]_6 = [a_3,a_8]$$

by Proposition 2.6(ii), (4.37) and (4.38). Since $[b_2,[a_2,a_8]] \in [V_2,U_{[4,6]}] = 1$, the first of these equations implies that $[a_2b_2,a_8]_5 = a_5^2 = 1$, so $a_2 = b_2$ by Proposition 4.35. Thus $[a_3,a_8] = [b_2,a_8]_6 = [a_2,a_8]_6 = a_6$. □

Proposition 4.39. *Let $a_3 \in U_1$, $a_6 \in V_6^\sharp$ and $a_8 \in V_8^\sharp$ and suppose that $[a_3,a_8] = a_6$. Then $a_3 \in U_3^\sharp$.*

Proof. Let $u = w_{12}^{a_3}$ and let $b = a_8^{a_3}$. Then $a_8 \in M_{w_{12}}$ and $b \in M_u$, where $M_{w_{12}}$ and M_u are as in Notation 4.4. Since a_3 fixes w_{10} and w_{11}, u is opposite w_{10} at w_{11}. Since $[a_3,a_8] = a_6$, we $a_6a_8 = b \in M_u$. By Proposition 4.5, u is the unique vertex in $\Gamma_{w_{11}}$ such that a_6a_8 is contained in M_u. By Proposition 4.36, it follows that for *all* $a_6 \in V_6^\sharp$ and $a_8 \in V_8^\sharp$, there exists a unique vertex u in $\Gamma_{w_{11}}$ such that $a_6a_8 \in M_u$ and u is opposite w_{10} at w_{11}. By symmetry, the vertex u is also opposite w_{12} at w_{11}. Thus $a_1 \in U_1^\sharp$ by Notation 2.10. □

Proposition 4.40. $[U_2,U_5] \subset \langle V_4^\sharp \rangle$.

Proof. Choose $a_0 \in V_0^\sharp$, $b_2 \in U_2$ and $a_5 \in U_5^\sharp$ and let $v_8 = \lambda_\gamma(a_0)$ and $a_2 = a_0^{\mu_\gamma(a_5)}$. By (2.13), $a_2 \in V_2^\sharp$, so $[a_2,b_2] = 1$ and by Conventions 1.3(ii), Proposition 2.6(i) and Corollary 4.13,

$$[a_2,[b_2,v_8]] \in [a_2,U_{[3,7]}] = [a_2,U_7] \subset \langle V_4^\sharp \rangle$$

since $[a_2,U_{[3,6]}] \subset [V_2,U_{[3,6]}] = 1$. It follows that $[b_2,[a_2,v_8]] \in \langle V_4^\sharp \rangle$ by [7, 2.3] applied to the quotient group $U_{[2,8]}/\langle V_4^\sharp \rangle$. By Proposition 4.15(i),

$$[a_2,v_8] \in U_3a_5V_6 = U_3V_6a_5,$$

so $[b_2,a_5] = [b_2,[a_2,v_8]] \in \langle V_4^\sharp \rangle$ since $[U_2,U_3V_6] = 1$. Thus $[U_2,U_5^\sharp] \subset \langle V_4^\sharp \rangle$. The claim holds, therefore, by Proposition 2.22. □

Proposition 4.41. *Let $v_8 = a_8w_8$ for some $a_8 \in V_8$ and some $w_8 \in \lambda_\gamma(U_0^\sharp)$ and suppose that $v_8 \in U_8^\sharp$. Then $a_8 \in \langle V_8^\sharp \rangle$.*

Proof. Let $a_0 = \kappa_\gamma(v_8)$, so $a_0 \in U_0^\sharp$ and by Proposition 2.14(iv), $v_8 = \lambda_\gamma(a_0)$. Let $u_8 = \kappa_\gamma(a_0)$, $m = \mu_\gamma(a_0)$ and $w_0 = u_8^m$. Choose $a_3 \in U_3^\sharp$. Then $m = u_8a_0v_8$ and by (2.13), $a_3^m \in U_5^\sharp$ and $w_0 \in U_0^\sharp$. Hence $a_3^{mw_0^{-1}} \in U_{[1,5]}$ by Proposition 2.6(i). Let $a_2 = [a_0,a_3^{-1}]$. By Proposition 4.40, $a_2 \in \langle V_2^\sharp \rangle$. We have $a_3^{a_0v_8} = (a_2a_3)^{v_8} \in U_{[2,7]}$ by Proposition 2.6(i). Since $m = a_0v_8w_0$, it follows that

$$(a_2a_3)^{v_8} \in U_{[2,7]} \cap U_{[1,5]}.$$

Therefore

$$(a_2a_3)^{v_8} \in U_{[2,5]} \tag{4.42}$$

by Proposition 2.6(ii). By Corollary 4.31, $[a_2, v_8] \in U_{[3,5]} \cdot \langle V_6^\sharp \rangle$. Thus

$$a_2 \cdot [a_2, v_8] \cdot a_3 \in U_{[2,5]} \langle V_6^\sharp \rangle.$$

Since

$$(a_2 a_3)^{v_8} = a_2 \cdot [a_2, v_8] \cdot a_3 \cdot [a_3, v_8],$$

it follows by Proposition 2.6(ii) and (4.42) that $[a_3, v_8] \in U_{[4,5]} \cdot \langle V_6^\sharp \rangle$. Since $[V_8, U_{[4,8]}] = 1$, we have $v_8 = w_8 a_8$ and $[[a_3, w_8], a_8] = 1$. Thus

$$[a_3, v_8] = [a_3, w_8 a_8] = [a_3, a_8] \cdot [a_3, w_8]^{a_8} = [a_3, a_8] \cdot [a_3, w_8]$$

by Conventions 1.3(ii). We have $[a_3, w_8] \in U_{[4,5]}$ by Proposition 4.33. Since $[V_6, U_{[4,5]}] = 1$, it follows that $[a_3, a_8] \in U_{[4,5]} \cdot \langle V_6^\sharp \rangle$. By Proposition 4.6, therefore, $a_8 \in \langle V_8^\sharp \rangle$. □

Proposition 4.43. *Let* $a_0 \in U_0^\sharp$. *If* $[a_0, a_5] \in U_{[1,2]}$ *for some* $a_5 \in U_5$, *then* $[a_0, a_5] \in \langle V_2^\sharp \rangle$.

Proof. Suppose that $[a_0, a_5] = a_1 a_2$ with $a_0 \in U_0^\sharp$ and $a_i \in U_i$ for $i = 1$, 2 and 5. Let $u_8 = \kappa_\gamma(a_0)$, $v_8 = \lambda_\gamma(a_0)$ and $m = \mu_\gamma(a_0)$. Then $a_5^m \in U_3$ by (2.13), so

$$a_5^{u_8 a_0} = a_5^{m v_8^{-1}} \in U_{[3,7]} \tag{4.44}$$

by Proposition 2.6(i). By Proposition 4.40, $a_5^{u_8} = a_5 a_6$ for some $a_6 \in \langle V_6^\sharp \rangle$, so

$$a_5^{u_8 a_0} = (a_5 a_6)^{a_0} = [a_0, a_5] \cdot a_5 \cdot [a_0, a_6] \cdot a_6 = a_1 a_2 a_5 \cdot [a_0, a_6] \cdot a_6.$$

By Corollary 4.31, $[a_0, a_6] \in \langle V_2^\sharp \rangle U_{[3,5]}$. Thus $a_5^{u_8 a_0} \in a_1 a_2 \langle V_2^\sharp \rangle U_{[3,6]}$. Hence

$$a_5^{u_8 a_0} \in U_{[3,7]} \cap a_1 a_2 \langle V_2^\sharp \rangle U_{[3,6]}$$

by (4.44). By Proposition 2.6(ii), therefore, $a_1 = 1$ and $a_2 \in \langle V_2^\sharp \rangle$. □

Proposition 4.45. *Let* $a_4 \in U_4$ *and suppose that* $[a_1, a_4] = 1$ *for some* $a_1 \in U_1^\sharp$, *Then* $[a_4, v_9] \in U_5 a_4^m$, *where* $v_9 = \lambda_\gamma(a_1)$ *and* $m = \mu_\gamma(a_1)$.

Proof. Let $a_6 = a_4^m$. Then $a_6 \in U_6$ by (2.13), $m = \mu_\gamma(v_9)$ by Proposition 2.14(ii), $a_1 = \kappa_\gamma(v_9)$ by Proposition 2.14(iv) and $\kappa_\gamma(v_9) = \lambda_\gamma(v_9)$ by Proposition 4.27(iii). Thus $m = a_1 v_9 a_1$. We have $a_4^{m a_1} \in U_{[2,5]} a_6$ and $a_4^{a_1 v_9} = a_4^{v_9} \in a_4 U_{[5,8]}$ by Proposition 2.6(i). Since $m a_1 = a_1 v_9$, it follows that $a_4^{v_9} \in a_4 U_{[5,8]} \cap U_{[2,5]} a_6$. By Proposition 2.6(ii), therefore, $a_4^{v_9} \in a_4 U_5 a_6$. Thus $[a_4, v_9] \in U_5 a_6 = U_5 a_4^m$. □

Proposition 4.46. *Let* $a_4 \in U_4^\sharp$ *and suppose that* $[a_1, a_4] = 1$ *for some* $a_1 \in U_1^\sharp$. *Then* $a_4 \in V_4$.

Proof. Let $v_9 = \lambda_\gamma(a_1)$ and $m = \mu_\gamma(a_1)$. By Proposition 4.45, $[a_4, v_9] \in U_5 a_4^m$. We have $a_4^m \in U_6$. By Proposition 2.6(ii) and Proposition 4.43, therefore, $a_4^m \in V_6$. Hence $a_4 \in V_4$. □

Proposition 4.47. *Suppose that* $[a_0, a_5] = a_1a_2$ *and* $a_i \in U_i$ *for* $i = 0, 1, 2$ *and* 5. *Then* $a_1 = 1$.

Proof. The subgroup V_4 is normal in $U_{[0,5]}$. By Conventions 1.3(i), Proposition 2.6(i), Proposition 4.17 and Proposition 4.40, we have

$$[[U_0, U_4], U_5] \subset [U_{[1,3]}, U_5] \subset V_4.$$

Since $[U_4, U_5] = 1$, it follows by [7, 2.3] applied to the quotient group $U_{[0,5]}/V_4$ that $[[U_0, U_5], U_4] \subset V_4$. Thus $[a_1a_2, U_4] \subset V_4$. Choose $b_4 \in U_4$. By Conventions 1.3(i), we have $[a_1a_2, b_4] = [a_1, b_4]^{a_2} \cdot [a_2, b_4]$. By Proposition 4.40, $[a_1, b_4]^{a_2} \in V_2$ and by Proposition 2.6(i), $[a_2, b_4] \in U_3$. By Proposition 2.6(ii), therefore, $[a_1, b_4] = 1$. Since b_4 is arbitrary, it follows that $a_1 \in C_{U_1}(U_4)$. By Proposition 4.15(ii), on the other hand, $U_4^\sharp \not\subset V_4$, so by Proposition 4.46, $C_{U_1^\sharp}(U_4) = \emptyset$. Since X is sharp, it follows that $C_{U_1}(U_4) = 1$. Thus $a_1 = 1$. □

Proposition 4.48. *Let* $a_0 \in U_0$. *If* $[a_0, a_5] \in U_{[1,2]}$ *for some* $a_5 \in U_5$, *then* $[a_0, a_5] \in \langle V_2^\sharp \rangle$.

Proof. Suppose that $[a_0, a_5] = a_1a_2$ with $a_i \in U_i$ for $i = 0, 1, 2$ and 5. By Proposition 4.47, we have $a_1 = 1$. Choose $b_7 \in U_7^\sharp$. By Proposition 2.6(i), a_0 normalizes $U_{[1,6]}$ and hence $a_0^{b_7} = fa_0$ for some $f \in U_{[1,6]}$. Again by Proposition 2.6(i), U_2 normalizes $U_{[3,6]}$ and hence $f = eb_2$ for some $b_2 \in U_2$ and some $e \in U_1U_{[3,6]}$. By Corollary 4.13, U_5 is abelian. By Proposition 2.6(i) and Proposition 4.17, therefore, $[e, a_5] = 1$ and thus

$$a_2^{b_7} = [a_0, a_5]^{b_7} = [a_0^{b_7}, a_5^{b_7}] = [eb_2a_0, a_5] = [b_2a_0, a_5] \tag{4.49}$$

by Conventions 1.3(i). By Conventions 1.3(i) and Proposition 4.40, we have $[b_2a_0, a_5] = d_4 \cdot [a_0, a_5] = d_4a_2 = a_2d_4$ for some d_4 in $\langle V_4^\sharp \rangle$. By (4.49), therefore, we have $[a_2, b_7] = d_4$. Let $d_2 = d_4^{\mu_\gamma(b_7)}$. By (2.13), $d_2 \in \langle V_2^\sharp \rangle$ and by 4.6, $[d_2, b_7] = d_4$. Thus $[a_2d_2, b_7] = 1$ by Conventions 1.3(i) and Proposition 4.27(i). Therefore

$$a_2b_2 \in U_2 \cap U_2^{b_7} \subset G^{(1)}_{w_3,w_4,w_5,w_6,w_7,w_6',w_5',w_4'},$$

where $w_i' = w_i^{b_7}$ for all i. The path $(w_2, w_3, w_4, w_5, w_6, w_7, w_6', w_5', w_4')$ is straight and of length 8. Thus $\alpha := (w_2, w_3, w_4, w_5, w_6, w_7, w_6', w_5', w_4')$ is a root and

$$U_\alpha = G^{(1)}_{w_3,w_4,w_5,w_6,w_7,w_6',w_5'}.$$

By Proposition 2.9, therefore, $a_2d_2 = 1$. Hence $a_2 \in \langle V_2^\sharp \rangle$. □

Proposition 4.50. *Let* $a_4 \in U_4$. *If* $[a_1, a_4] = 1$ *for some* $a_1 \in U_1^\sharp$, *then* $a_4 \in V_4$.

Proof. Let $v_9 = \lambda_\gamma(a_1)$ and $m = \mu_\gamma(a_1)$. By Proposition 4.45, $[a_4, v_9] \in U_5a_4^m$. We have $a_4^m \in U_6$. By Proposition 2.6(ii) and Proposition 4.48, it follows that $a_4^m \in V_6$. Hence $a_4 \in V_4$. □

Proposition 4.51. *Let $e_1 \in U_1^\sharp$ and $a_6 \in V_6^\sharp$. Then*

$$V_6^\sharp = \{a_6^{\mu_\gamma(e_1)\mu_\gamma(a_1)} \mid a_1 \in U_1^\sharp\}.$$

Proof. Let $a_4 = a_6^{\mu_\gamma(e_1)}$ and choose $b_6 \in V_6^\sharp$. By Proposition 4.36, there exists $a_1 \in U_1^*$ such that $[a_1, b_6] = a_4$. By Proposition 4.39, $a_1 \in U_1^\sharp$. Thus $a_6^{\mu_\gamma(e_1)\mu_\gamma(a_1)} = b_6$ by Proposition 4.6. □

Let $W_i = \lambda_\gamma(V_{i-8}^\sharp)$ for all even i.

Proposition 4.52. *$W_i \subset U_i^\sharp$ for all even i.*

Proof. This holds by Proposition 2.11. □

Proposition 4.53. *$U_8 = V_8 \cdot \langle W_8 \rangle$.*

Proof. Choose $a_5 \in U_5^\sharp$ and $a_8 \in U_8$. By Proposition 4.40, $[a_5, a_8] \in \langle V_6^\sharp \rangle$. By Proposition 4.15(ii), $[a_5, W_8]$ contains elements of $V_6^\sharp$. The product $\mu_\gamma(e_1)$ $\mu_\gamma(a_1)$ for $e_1, a_1 \in U_1^\sharp$ normalizes W_8 and by Proposition 4.17, it centralizes U_5. By Proposition 4.51, therefore, $V_6^\sharp \subset [a_5, W_8]$. Therefore $\langle V_6^\sharp \rangle \subset [a_5, \langle W_8 \rangle]$. Thus there exists $b \in \langle W_8 \rangle$ such that $[a_5, a_8] = [a_5, b]$. Hence $[a_5, a_8 b^{-1}] = 1$. By 4.50, we conclude that $a_8 b^{-1} \in V_8$. □

Proposition 4.54. *$[U_4, U_8] = 1$.*

Proof. This holds by Proposition 4.28 and Proposition 4.53. □

Proposition 4.55. *$[H_1 H_7, H_8] = 1$, where H_i for all i is as in Proposition* 2.24.

Proof. By Proposition 4.17, H_1 centralizes U_5 and H_7 centralizes U_3. By Proposition 4.54, H_8 centralizes U_4. Thus $[H_1, H_8] \subset C_H(\langle U_4, U_5 \rangle)$ and $[H_7, H_8] \subset C_H(\langle U_3, U_4 \rangle)$. Thus $[H_1, H_8] = [H_7, H_8] = 1$ by Proposition 2.16. □

Proposition 4.56. *Let $\hat{X}$ be as in Proposition* 4.20*. Then $\hat{X}$ is Moufang and $U_i^\sharp = U_i^*$ for all odd i.*

Proof. Let $H^\dagger$ be as in Proposition 2.23. We have $H_1 H_7 \subset H^\dagger$ and by Proposition 2.24, $H^\dagger = H_1 H_8$. By Proposition 2.24 and Proposition 4.21, we have $\varphi(\hat{U}_i^\sharp) = \hat{U}_{(i+1)/2}^\sharp$ for all odd i and $\hat{H}^\dagger = \varphi(H_1 H_7)$, where φ is as in Proposition 4.20 and $\hat{H}^\dagger$ is as in Proposition 2.23 applied to $\hat{X}$. Since X is dagger-sharp, every non-trivial $H^\dagger$-invariant subgroup of U_i for i odd contains elements of $U_i^\sharp$. Hence every non-trivial $\varphi(H^\dagger)$-invariant subgroup of $\hat{U}_i$ for arbitrary i contains elements of $\hat{U}_i^\sharp$. By Proposition 4.25 and Proposition 4.55, therefore, we can apply Theorem 3.1 with $J = \varphi(H_8)$. Thus $\hat{X}$ is Moufang. The second claim holds, therefore, by Proposition 4.22. □

Proposition 4.57. *Let $e_1 \in U_1^\sharp$ and $a_6 \in V_6^\sharp$. Then*

$$\langle V_6^\sharp \rangle^* = \{a_6^{\mu_\gamma(a_1)\mu_\gamma(e_1)} \mid a_1 \in U_1^\sharp\}.$$

Proof. Choose $b_6 \in \langle V_6^\sharp \rangle^*$ and let $b_4 = b_6^{\mu_\gamma(e_1)}$. By (2.13), $b_4 \in \langle V_4^\sharp \rangle^*$ and by 4.36, there exists $a_1 \in U_1^*$ such that $[a_1, a_6] = b_4$. By Proposition 4.56, $a_1 \in U_1^\sharp$. Thus $b_6^{\mu_\gamma(e_1)\mu_\gamma(a_1)} = a_6$ by Proposition 4.6. □

Corollary 4.58. *$\langle V_i^\sharp \rangle^* = V_i^\sharp$ for all even i.*

Proof. This holds by Proposition 4.57. □

Proposition 4.59. *$U_8 = V_8 \cup V_8 W_8$.*

Proof. Choose $a_5 \in U_5^\sharp$ and $a_8 \in U_8$. By Proposition 4.40, $[a_5, a_8] \in \langle V_6^\sharp \rangle$. By Proposition 4.15(ii), $[a_5, W_8]$ contains elements of $V_6^\sharp$. The product $\mu_\gamma(a_1)$ $\mu_\gamma(e_1)$ for $e_1, a_1 \in U_1^\sharp$ normalizes W_8 and by Proposition 4.17, it centralizes U_5. By Proposition 4.57 and Corollary 4.58, therefore, $\langle V_6^\sharp \rangle^* \subset [a_5, W_8]$. Thus there exists $b \in W_8 \cup \{1\}$ such that $[a_5, a_8] = [a_5, b]$. Hence $[a_5, a_8 b^{-1}] = 1$. By Proposition 4.50, we conclude that $a_8 b^{-1} \in V_8$. □

Proposition 4.60. *$\langle W_8 \rangle \subset \langle V_8^\sharp \rangle \cup \langle V_8^\sharp \rangle \cdot W_8$.*

Proof. Choose $a_3 \in U_3^\sharp$ and $b_8 \in \langle W_8 \rangle$. By Proposition 4.59, there exists $a_8 \in V_8$ and $w_8 \in W_8 \cup \{1\}$ such that $b_8 = a_8 w_8$. We have

$$[a_3, W_8] \subset U_{[4,5]} \tag{4.61}$$

and $[a_3, W_8^{-1}] \subset U_{[4,5]} V_6^\sharp$ by Proposition 4.33. By Conventions 1.3(ii), Proposition 4.40 and 4.54, it follows that

$$[a_3, b_8]_6 \in [a_3, \langle W_8 \rangle]_6 \subset \langle V_6^\sharp \rangle.$$

By Conventions 1.3(ii), Proposition 4.6 and (4.61), on the other hand, we have

$$[a_3, b_8] = [a_3, a_8 w_8] = [a_3, w_8] \cdot [a_3, a_8]^{w_8} \in U_{[4,5]} a_8^{\mu_\gamma(a_3)}.$$

Hence $a_8 \in \langle V_8^\sharp \rangle$. □

Corollary 4.62. *$\hat{U}_8 := \langle V_8^\sharp \rangle \cup \langle V_8^\sharp \rangle \cdot W_8$ is a subgroup of U_8.*

Proof. Since $V_8 \subset Z(U_8)$, the product $\langle V_8^\sharp \rangle \cdot \langle W_8 \rangle$ is a subgroup. This subgroup contains $\hat{U}_8$. By Proposition 4.60, on the other hand, $\langle V_8^\sharp \rangle \cdot \langle W_8 \rangle \subset \hat{U}_8$. □

Proposition 4.63. *$V_8 \cap \hat{U}_8 = \langle V_8^\sharp \rangle$, where $\hat{U}_8$ is as in Proposition 4.62.*

Proof. Let $a_3 \in U_3^\sharp$, $a_8 \in \langle V_8^\sharp \rangle$ and $w_8 \in W_8$. By Conventions 1.3(ii) and Proposition 4.13,

$$[a_3, a_8 w_8] = [a_3, w_8] \cdot [a_3, a_8]^{w_8} \in [a_3, w_8] V_6.$$

By Proposition 4.33, therefore, $[a_3, a_8 w_8]_4 \neq 1$. Hence $a_8 w_8 \notin V_8$ by another application of Corollary 4.13. □

Proposition 4.64. *$V_8 = \langle V_8^\sharp \rangle$.*

Proof. Let $\hat{U}_8$ be as in Corollary 4.62. By Proposition 4.41 and Proposition 4.59, $U_8^\sharp \subset \hat{U}_8$. By Proposition 2.22 and Corollary 4.62, it follows that $U_8 = \hat{U}_8$. Hence $V_8 = V_8 \cap \hat{U}_8 = \langle V_8^\sharp \rangle$ by Proposition 4.63. □

Corollary 4.65. $V_i^\sharp = V_i^*$.

Proof. This holds by Proposition 4.58 and Proposition 4.64. □

We observe now that we can continue to follow the proof of [7, 17.7] given in [7, 31.1–31.34] verbatim, starting with [7, 31.22]. The arguments from this point on require only Proposition 4.52, Proposition 4.56, and Corollary 4.65; the equality $U_i^\sharp = U_i^*$ for i even is never required. The results [7, 31.22–31.34] yield the conclusion that there exist an octagonal set (K, σ), isomorphisms x_i from the additive group of K to U_i for all odd i, isomorphisms x_i from the additive group of K to the center of U_i for all even i and and injections y_i from the set K to U_i for all even i such that $U_i = y_i(K)x_i(K)$ and

$$y_i(s)y_i(t) = y_i(s+t)x_i(s^\sigma t) \tag{4.66}$$

for all $s, t \in K$ and for all even i and all the commutator relations in [7, 16.9] hold.

It is now a lengthy but straightforward calculation to show using (4.66) and the commutator relations in [7, 16.9] that

$$U_7^{x_0((u+v^\sigma)/R^\sigma)y_0(u/R)x_8(t)y_8(u)} = U_1$$

for all $s, t \in K$ not both zero, where

$$R = v^{\sigma+2} + uv + u^\sigma$$

(cf. [7, 10.14 and 32.13]). By Proposition 2.19(ii), therefore, $U_8^* = U_8^\sharp$. By Proposition 4.56, it follows that $U_i^* = U_i^\sharp$ for all i. Hence by Proposition 2.15, X is Moufang. This concludes the proof of Theorem 1.1.

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