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EVERY ABELIAN GROUP IS THE CLASS GROUP OF A RING OF KRULL TYPE

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Abstract. Let $Cl(A)$ denote the class group of an arbitrary integral domain A introduced by Bouvier in 1982. Then $Cl(A)$ is the ideal class (resp., divisor class) group of A if A is a Dedekind or a Prüfer (resp., Krull) domain. Let G be an abelian group. In this paper, we show that there is a ring of Krull type D such that $Cl(D) = G$ but D is not a Krull domain. We then use this ring to construct a Prüfer ring of Krull type E such that $Cl(E) = G$ but E is not a Dedekind domain. This is a generalization of Claborn's result that every abelian group is the ideal class group of a Dedekind domain.

Introduction

Let $Cl(A)$ denote the class group of a general integral domain A introduced by Bouvier in [7]. Hence, if A is a Dedekind or a Prüfer domain (resp., Krull domain), then $Cl(A)$ is the ideal class (resp., divisor class) group of A. Claborn's celebrated theorem says that given an abelian group G, there is a Dedekind domain D with ideal class group G [10, Theorem 7]. Then a subring $D + XK[\![X]\!]$ of the power series ring $K[\![X]\!]$ over the quotient field K of D is a two-dimensional non-Noetherian Prüfer domain with $Cl(D + XK[\![X]\!]) = G$ [17, Example 45.10].

For another example, let G be an abelian group, D be an integral domain with quotient field K, $X^1(D)$ be the set of height-one prime ideals of D, X be an indeterminate over D, $K[X]$ be the polynomial ring over K, and $R_1 = D + XK[X]$, i.e., $R_1 = \{f \in K[X] \mid f(0) \in D\}$. Then $Cl(R_1) = Cl(D)$ [5, Theorem 3.12] and D is a Prüfer domain if and only if R_1 is [11, Corollary 4.15]. Hence, if D is a Dedekind domain with $Cl(D) = G$, then R_1 is a Prüfer domain with $Cl(R_1) = G$. However, note that R_1 is a Dedekind domain if and only if $D = K$. Also, R_1 is a Prüfer ring of Krull type if and only if $|X^1(D)| < \infty$ [2, Corollary 2.6], and in this case, $Cl(R_1) = Cl(D) = \{0\}$. Note that

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Dedekind domain ⇒ Prüfer ring of Krull type ⇒ Prüfer domain;

hence it is natural to ask if there is a Prüfer ring of Krull type that is not a Dedekind domain and has a preassigned ideal class group. More generally, is there a ring of Krull type that is not a Krull domain and has a preassigned class group? In this paper, we prove that if G is an abelian group, there is a ring of Krull type D such that $Cl(D) = G$ but D is not a Krull domain. We then use this ring to construct a non-Noetherian Prüfer ring of Krull type with the same ideal class group.

Let Λ be a nonempty index set, $\{x_i, y_i, u_i \mid i \in \Lambda\}$ (simply, $\{x_i, y_i, u_i\}$) be an algebraically independent set over D, $v_i = y_i \cdot \frac{u_i}{x_i}$ for all $i \in \Lambda$, $\mathbb{Z}^{(\Lambda)}$ be the direct sum of Λ-copies of the additive group of integers, and $R = D[\{x_i, y_i, u_i, v_i \mid i \in \Lambda\}]$ (simply, $R = D[\{x_i, y_i, u_i, v_i\}]$). It is known that if D is a Krull domain, then R is a Krull domain with $Cl(R) = Cl(D) \oplus \mathbb{Z}^{(\Lambda)}$ [16, Proposition 14.9]. In Section 1, we first review definitions and known results related to rings of Krull type (including the t-operation and the class group of integral domains). In Section 2, we study some ring-theoretic properties of the ring R. Among other things, we show that (i) D is a PvMD (resp., a ring of Krull type, an independent ring of Krull type, a generalized Krull domain, a TV-PvMD) if and only if R is; (ii) if D is a PvMD, then $Cl(R) = Cl(D) \oplus \mathbb{Z}^{(\Lambda)}$. We also give such type of integral domains D with $Cl(D) = \{0\}$ so that $Cl(R) = \mathbb{Z}^{(\Lambda)}$.

Let H be a subgroup of $Cl(D)$ and X be an indeterminate over D. In Section 3, we show that (iii) if D is a ring of Krull type, there is a set Ω of maximal t-ideals of $D[X]$ such that $\bigcap_{Q\in\Omega} D[X]_Q$ is a ring of Krull type and $Cl(\bigcap_{Q\in\Omega} D[X]_Q) = Cl(D)/H$. Hence, by the result of Section 2, we have that (iv) if G is an abelian group, there is a ring of Krull type (resp., an independent ring of Krull type, a generalized Krull domain, a TV-PvMD) D such that $Cl(D) = G$ but D is not an independent ring of Krull type (resp., a generalized Krull domain, a Krull domain, a Krull domain). We show that (v) if D is a PvMD, there is a Prüfer domain T such that $Cl(T) = Cl(D)$ and $T \cap K = D$. Finally, we use these results to show that given an abelian group G, there is a Prüfer domain of finite character (resp., an h-local Prüfer domain, a generalized Krull domain of dimension one, a Prüfer domain in which each nonzero ideal is a v-ideal) D such that $Cl(D) = G$ but D is not an h-local Prüfer domain (resp., a generalized Krull domain of dimension one, a Dedekind domain, a Dedekind domain).

1. Rings of Krull type and the *t*-operation

Let D be an integral domain with quotient field K. An overring of D means a subring of K containing D. A valuation overring V of D is said to be *essential* for D if V is a quotient ring of D. Clearly, if M is the maximal ideal of V, then V is essential for D if and only if $V = D_{M\cap D}$.

Definition 1.1. Let $\mathfrak{V} = \{V_\alpha\}$ be a family of valuation overrings of D.

(1) $D = \bigcap_\alpha V_\alpha$.
(2) Each V_α is a rank-one discrete valuation ring (DVR).
(3) Each V_α is a rank-one valuation ring.
(4) The family $\mathfrak{V}$ has finite character, i.e., each nonzero $x \in K$ is a nonunit in only finitely many valuation rings in $\mathfrak{V}$.
(5) Each V_α is essential for D.

We say that D is a *Krull domain* (resp., *generalized Krull domain*, *ring of Krull type*) if there is a family $\mathfrak{V}$ satisfying (1), (2) and (4) (resp., (1), (3), (4) and (5); (1), (4) and (5)). A ring of Krull type D is an *independent ring of Krull type* if the valuation rings in $\mathfrak{V}$ are independent, i.e., there is no nontrivial valuation overring of D containing two distinct valuation rings in $\mathfrak{V}$.

An integral domain D is said to be of *finite character* if each nonzero nonunit of D is contained in only finitely many maximal ideals of D. We say that D is *h-local* if D is of finite character and each nonzero prime ideal of D is contained in a unique maximal ideal of D. Note that D is a Prüfer domain (i.e., each nonzero finitely generated ideal of D is invertible) if and only if D_M is a valuation domain for all maximal ideals M of D [17, Theorem 22.1]. Thus, a Prüfer domain of finite character (resp., an h-local Prüfer domain) is a ring of Krull type (resp., an independent ring of Krull type). It is well known that D is a Krull domain (resp., generalized Krull domain) if and only if $D = \bigcap_{P \in X^1(D)} D_P$, D_P is a rank-one DVR (resp., rank-one valuation ring) for all $P \in X^1(D)$, and the family $\{D_P \mid P \in X^1(D)\}$ has finite character. For this kind of characterization of rings of Krull type, we first need the notion of the t-operation on an integral domain.

A nonzero D-submodule I of K is called a *fractional ideal* if $dI \subseteq D$ for some $0 \neq d \in D$. Let $\mathbf{F}(D)$ be the set of nonzero fractional ideals of D. It is clear that if $I \in \mathbf{F}(D)$ and $I^{-1} = \{x \in K \mid xI \subseteq D\}$, then $I^{-1} \in \mathbf{F}(D)$, and hence (i) $I_v = (I^{-1})^{-1}$ and $I_t = \bigcup\{J_v \mid J \subseteq I \text{ and } J \in \mathbf{F}(D) \text{ is finitely generated}\}$ are well-defined, (ii) $I \subseteq I_t \subseteq I_v$, (iii) $(I_t)_t = I_t$ and $(I_v)_v = (I_t)_v = (I_v)_t = I_v$, and (iv) $I_t = I_v$ if I is finitely generated. Let $* = v$ or t. An $I \in \mathbf{F}(D)$ is called a $*$-*ideal* if $I_* = I$, and a $*$-ideal is called a *maximal* $*$-*ideal* (resp., *prime* $*$-*ideal*) if it is maximal among proper integral $*$-ideals of D (resp., a prime ideal). Let $*$-Max(D) (resp., $*$-Spec(D)) denote the set of maximal (resp., prime) $*$-ideals of D. While v-Max(D) can be empty as in the case of rank-one nondiscrete valuation domains, it is well known that t-Max$(D) \neq \emptyset$ when D is not a field; a maximal t-ideal is a prime ideal; each t-ideal is contained in a maximal t-ideal; and each prime ideal minimal over a t-ideal is a t-ideal; so t-Max$(D) \subseteq t$-Spec(D) and $X^1(D) \subseteq t$-Spec(D).

An $I \in \mathbf{F}(D)$ is said to be *t-invertible* if $(II^{-1})_t = D$. A t-ideal I of D is said to be *of finite type* if $I = J_v$ for some finitely generated ideal J of D. It is known that I is t-invertible if and only if I_t is of finite type and ID_P is principal for all $P \in t$-Max(D) [24, Proposition 2.6]. We say that D is a *Prüfer v-multiplication domain* (PvMD) if every nonzero finitely generated ideal of

D is t-invertible; equivalently, the set of all fractional t-ideals of finite type forms a group under the multiplication $I * J = (IJ)_t$. It is easy to see that an invertible ideal is a t-invertible t-ideal. Thus, a Prüfer domain is a PvMD. It is also known that D is a Krull domain if and only if each nonzero (prime) ideal of D is t-invertible [25, Theorem 3.6]; hence Krull domains are PvMDs. For more on the basic properties of the v- and t-operations, see [17, Sections 32 and 34].

We now give some very useful properties of the t-operation which will be used without further comments.

Lemma 1.2. *Let D be an integral domain, S be a multiplicative set of D, and I be a nonzero fractional ideal of D.*

1. $(ID_S)_t = (I_tD_S)_t$.
2. *If $I_t = J_t$ for a finitely generated ideal J of D, then $(ID_S)^{-1} = I^{-1}D_S$.*
3. *If I is t-invertible, then $(ID_S)_v = I_vD_S = (ID_S)_t = I_tD_S$.*
4. *If D is a PvMD, then $(ID_S)_t = I_tD_S$; so if $I_t = I$, then $(ID_S)_t = ID_S$.*
5. *If ID_S is an integral t-ideal of D_S, then $ID_S \cap D$ is a t-ideal of D.*

Proof. (1) and (2). [24, Lemma 3.4]. (3) Note that both I_t and I^{-1} are of finite type. Thus, $(ID_S)_v = I_vD_S = (ID_S)_t = I_tD_S$ by (1), (2) and [8, Lemmas 2.5 and 2.6]. (4) Note that $(ID_S)_t = \bigcup\{(JD_S)_v \mid J \subseteq I$ is nonzero finitely generated$\}$ and $(JD_S)_v = J_vD_S$. Thus, $(ID_S)_t = I_tD_S$. (5) [24, Lemma 3.17]. $\square$

Let $\{X_\alpha\}$ be a nonempty set of indeterminates over K and $K[\{X_\alpha\}]$ be the polynomial ring over K. For $f \in K[\{X_\alpha\}]$, let $c(f)$ denote the fractional ideal of D generated by the coefficients of f. Dedekind-Mertens lemma states that if $f, g \in K[\{X_\alpha\}]$ are nonzero, then $c(f)^{m+1}c(g) = c(f)^m c(fg)$ for some integer $m \geq 1$ [6, Theorem 2]. Hence, if $c(f)$ is invertible (resp., t-invertible), then $c(f)c(g) = c(fg)$ (resp., $(c(f)c(g))_t = c(fg)_t$).

Lemma 1.3. (cf. [23, Theorem 1.4]) *Let Q be a prime t-ideal of $D[\{X_\alpha\}]$ such that $Q \cap D = (0)$. Then the following statements are equivalent.*

1. *Q is a maximal t-ideal.*
2. *$c(Q)_t = D$, where $c(Q) = \sum_{f\in Q} c(f)$.*
3. *Q is t-invertible.*

In this case, ht$Q = 1$.

Proof. (1) $\Rightarrow$ (2) If $c(Q)_t \subsetneq D$, then there is a maximal t-ideal P of D such that $c(Q)_t \subseteq P$. Hence, $PD[\{X_\alpha\}]$ is a maximal t-ideal [15, Lemma 2.1] such that $Q \subsetneq PD[\{X_\alpha\}]$, a contradiction.

(2) $\Rightarrow$ (3) Since $c(Q)_t = D$, there is an $f \in Q$ such that $c(f)_v = D$. If ht$Q \geq 2$, then there is a $g \in Q$ such that $gK[\{X_\alpha\}]$ is a prime ideal and $f \notin gK[\{X_\alpha\}]$. Hence, $D[\{X_\alpha\}] = (g, f)_v \subseteq Q_t = Q \subsetneq D[\{X_\alpha\}]$, a contradiction. Thus, ht$Q$=1, and hence there is an $h \in Q$ such that $Q_{D\setminus\{0\}} = hK[\{X_\alpha\}]$.

Then $Q = (f, h)_v$, so it suffices to show that Q_M is principal for all $M \in t$-Max$(D[\{X_\alpha\}])$. Let M be a maximal t-ideal of $D[\{X_\alpha\}]$. If $M \cap D \neq (0)$, then $M = (M \cap D)D[\{X_\alpha\}]$ and $M \cap D$ is a maximal t-ideal [15, Proposition 2.2]. Hence, $Q \nsubseteq M$, and thus $Q_M = D[\{X_\alpha\}]_M$. If $M \cap D = (0)$, then $c(M)_t = D$, and hence ht$M = 1$ by the previous sentence. Thus, $Q_M = hD[\{X_\alpha\}]_M$ or $Q_M = D[\{X_\alpha\}]_M$.

(3) ⇒ (1) [23, Proposition 1.3]. □

Let $S = \{f \in D[\{X_\alpha\}] \mid c(f) = D\}$ and $N_v = \{f \in D[\{X_\alpha\}] \mid c(f)_v = D\}$. Then S and N_v are saturated multiplicative sets of $D[\{X_\alpha\}]$. Clearly, $S \subseteq N_v$, and hence $D[\{X_\alpha\}]_S \subseteq D[\{X_\alpha\}]_{N_v}$. Also, $S = N_v$ if and only if $D[\{X_\alpha\}]_S = D[\{X_\alpha\}]_{N_v}$, if and only if each maximal ideal of D is a t-ideal. Let Max(A) denote the set of maximal ideals of an integral domain A. It is known that

$$\mathrm{Max}(D[\{X_\alpha\}]_{N_v}) = \{PD[\{X_\alpha\}]_{N_v} \mid P \in t\text{-}\mathrm{Max}(D)\}$$

and each maximal ideal of $D[\{X_\alpha\}]_{N_v}$ is a t-ideal [24, Propositions 2.1 and 2.2]. The ring $D[\{X_\alpha\}]_S$, denoted by $D(\{X_\alpha\})$, is called the *Nagata ring* of D. We know that Max$(D(\{X_\alpha\})) = \{MD(\{X_\alpha\}) \mid M \in \mathrm{Max}(D)\}$ [17, Proposition 33.1] and D is a Prüfer domain if and only if $D(\{X_\alpha\})$ is a Prüfer domain [17, Theorem 33.4].

Theorem 1.4. *Let D be an integral domain. Then the following statements are equivalent.*

(1) *D is a PvMD.*
(2) *D_P is a valuation domain for all $P \in t$-Max(D).*
(3) *$D[\{X_\alpha\}]$ is a PvMD.*
(4) *$D[\{X_\alpha, X_\alpha^{-1}\}]$ is a PvMD.*
(5) *$D[\{X_\alpha\}]_{N_v}$ is a Prüfer domain, where $N_v = \{f \in D[\{X_\alpha\}] \mid c(f)_v = D\}$.*
(6) *D is integrally closed and Q is t-invertible for all prime t-ideals Q of $D[\{X_\alpha\}]$ with $Q \cap D = (0)$.*
(7) *D is integrally closed and if Q is a prime ideal of $D[\{X_\alpha\}]$ such that $Q \subseteq PD[\{X_\alpha\}]$ for some $P \in t$-Max(D), then $Q = (Q \cap D)D[\{X_\alpha\}]$.*

In this case,

$$t\text{-}Spec(D[\{X_\alpha\}]) = \{PD[\{X_\alpha\}] \mid P \in t\text{-}Spec(D)\} \cup \{Q \in t\text{-}\mathrm{Max}(D[\{X_\alpha\}]) \mid Q \cap D = (0)\}.$$

Proof. See [18, Theorem 5] for (1) ⇔ (2); [24, Theorem 3.7] for (1) ⇔ (3) ⇔ (5); [26, Corollaries 2.4 and 2.6] for (1) ⇔ (4); and [24, Theorem 3.1] for (1) ⇒ (7).

(7) ⇒ (6) Let Q be a prime t-ideal of $D[\{X_\alpha\}]$ such that $Q \cap D = (0)$. Then $c(Q) \nsubseteq P$ for all $P \in t$-Max(D) by (6), and hence $c(Q)_t = D$. Thus, Q is t-invertible by Lemma 1.3.

(6) ⇒ (1) It suffices to show that every nonzero ideal of D generated by two elements is t-invertible. Let $0 \neq a, b \in D$, and let $f = aX + b$ for $X \in \{X_\alpha\}$

and $Q_f = fK[\{X_\alpha\}] \cap D[\{X_\alpha\}]$. Then Q_f is a prime t-ideal of $D[\{X_\alpha\}]$ such that $Q_f = fc(f)^{-1}[\{X_\alpha\}]$ [17, Corollary 34.9] and $Q_f \cap D = (0)$. Hence, Q_f, and so $c(f)^{-1}$, is t-invertible. Thus, $c(f) = (a, b)$ is t-invertible.

For "In this case", let I be a nonzero ideal of D. Then $(ID[\{X_\alpha\}])_t = I_t D[\{X_\alpha\}]$ ([15, Lemma 2.1] or [24, Corollary 2.3]), and hence I is a t-ideal if and only if $ID[\{X_\alpha\}]$ is a t-ideal. Thus, the result follows from (7) and Lemma 1.3. □

Corollary 1.5. *Let D be a PvMD and P be a nonzero prime ideal of D. Then the following statements are equivalent.*

(1) *P is a t-ideal.*
(2) *D_P is a valuation domain.*
(3) *$P_t \subsetneq D$.*

Proof. (1) ⇔ (2) [27, Proposition 4.1].

(1) ⇒ (3) Clear.

(3) ⇒ (2) If $P_t \subsetneq D$, then $P \subseteq Q$ for some $Q \in t\text{-Max}(D)$. Hence, $D_Q \subseteq D_P$, and since D_Q is a valuation domain by Theorem 1.4, D_P is also a valuation domain. □

An integral domain D is said to be of *finite t-character* if each nonzero nonunit of D is contained in only finitely many maximal t-ideals of D. The ring of Krull type was introduced by Griffin [19] and characterized by a PvMD of finite t-character [18, Theorem 7]. The (1)-(3) of the next theorem appears in [18], but we give the proof for easy reference of the reader.

Theorem 1.6. *Let D be an integral domain. Then the following statements are equivalent.*

(1) *D is a ring of Krull type.*
(2) *D is a PvMD of finite t-character.*
(3) *$D[\{X_\alpha\}]$ is a ring of Krull type.*
(4) *$D[\{X_\alpha\}]_{N_v}$ is a Prüfer domain of finite character.*

Proof. (1) ⇒ (2) If D is a ring of Krull type, then there is a set $\{P_\alpha \mid \alpha \in \Theta\}$ of prime ideals of D such that $\{D_{P_\alpha} \mid \alpha \in \Theta\}$ satisfies the (1) and (4) of Definition 1.1. Let P be a maximal t-ideal of D, and assume that $P \nsubseteq P_\alpha$ for all $\alpha \in \Theta$. If $0 \neq a \in P$, then there are only finitely many prime ideals in $\{P_\alpha \mid \alpha \in \Theta\}$ that contain a, say, $P_{\alpha_1}, \dots, P_{\alpha_n}$. Since $P \nsubseteq P_{\alpha_i}$ for $i = 1, \dots, n$, there is an element $b \in P \setminus \bigcup_{i=1}^n P_{\alpha_i}$. Hence, by Lemma 1.2(2),

$$\begin{aligned}(a,b)^{-1} &\subseteq \bigcap_{\alpha\in\Theta} (a,b)^{-1} D_{P_\alpha} = \bigcap_{\alpha\in\Theta} ((a,b)D_{P_\alpha})^{-1} \\ &= \bigcap_{\alpha\in\Theta} D_{P_\alpha} = D.\end{aligned}$$

Thus, $(a, b)^{-1} = D$, and hence $D = (a, b)_v \subseteq P_t = P$, a contradiction. Hence, $P \subseteq P_\alpha$ for some $\alpha \in \Theta$. Note that $P_\alpha D_{P_\alpha}$ is a t-ideal and $P_\alpha = P_\alpha D_{P_\alpha} \cap D$.

Hence, P_α is a t-ideal, and thus $P = P_\alpha$. Thus, $\{D_P \mid P \in t\text{-Max}(D)\} \subseteq \{D_{P_\alpha} \mid \alpha \in \Theta\}$, so D is a PvMD of finite t-character by Theorem 1.4.

(2) $\Rightarrow$ (1) It suffices to take $\mathfrak{V} = \{D_P \mid P \in t\text{-Max}(D)\}$ in Definition 1.1.

(2) $\Leftrightarrow$ (3) By Theorem 1.4, it suffices to prove the finite t-characterness. Let Q be a maximal t-ideal of $D[\{X_\alpha\}]$. If $Q \cap D = (0)$, then ht$Q = 1$ by Lemma 1.3, and since $K[\{X_\alpha\}]$ is a UFD, each nonzero element of $D[\{X_\alpha\}]$ is contained in only finitely many such maximal t-ideals. Thus, by Theorem 1.4, D is of finite t-character if and only if $D[\{X_\alpha\}]$ is of finite t-character.

(2) $\Leftrightarrow$ (4) Recall that $\text{Max}(D[\{X_\alpha\}]_{N_v}) = \{PD[\{X_\alpha\}]_{N_v} \mid P \in t\text{-Max}(D)\}$. Hence, D is of finite t-character if and only if $D[\{X_\alpha\}]_{N_v}$ is of finite character. Thus, the result follows directly from Theorem 1.4. □

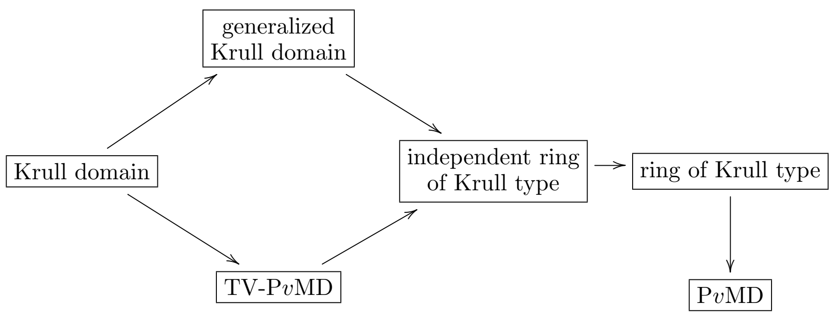
By Theorem 1.4 and [17, Theorem 22.1], a Prüfer domain is exactly the PvMD whose nonzero maximal ideals are t-ideals. Hence, by Theorem 1.6, D is a Prüfer domain of finite character if and only if D is a Prüfer ring of Krull type. We next use the PvMD to characterize generalized Krull domains and independent rings of Krull type. This result also shows that an independent Prüfer ring of Krull type is just the h-local Prüfer domain.

Corollary 1.7. (1) *D is an independent ring of Krull type if and only if D is a PvMD of finite t-character in which no two distinct maximal t-ideals contain a nonzero prime ideal.*

(2) *D is a* generalized Krull domain *if and only if D is a PvMD of finite t-character in which each prime t-ideal is a maximal t-ideal.*

Proof. This is an immediate consequence of Theorem 1.6. □

Following [22], we say that D is a TV-PvMD if D is a PvMD on which $t = v$, i.e., $I_t = I_v$ for all nonzero fractional ideals I of D. It is known that D is a TV-PvMD if and only if D is an independent ring of Krull type whose maximal t-ideals are t-invertible [22, Theorem 3.1]. Obviously, a Krull domain is a TV-PvMD. Hence, by Definition 1.1 and Theorem 1.6, we have the following implications:



However, none of the implications is reversible. For example, the ring $\mathbb{Z} + X\mathbb{Q}[X]$ is a PvMD but not a ring of Krull type, and see Example 2.7 or Corollary 3.4 for the other implications.

The next result is already known (see [2, Corollary 2.9], [22, Proposition 4.6], and [17, Theorem 43.11] for the case of a single indeterminate).

Corollary 1.8. *D is an independent ring of Krull type* (*resp., a TV-PvMD, a generalized Krull domain, a Krull domain*) *if and only if $D[\{X_\alpha\}]$ is.*

Proof. By Theorem 1.6, D and $D[\{X_\alpha\}]$ are rings of Krull type. Let Q be a prime t-ideal of $D[\{X_\alpha\}]$. Then either $Q \cap D = (0)$ or $Q = PD[\{X_\alpha\}]$ for some prime t-ideal P of D by Theorem 1.4. If $Q \cap D = (0)$, then Q is a maximal t-ideal, and hence Q is t-invertible and $D[\{X_\alpha\}]_Q$ is a rank-one DVR by Lemma 1.3. Furthermore, if P is a prime t-ideal of D, then $PD[\{X_\alpha\}]$ is a prime t-ideal, $D[\{X_\alpha\}]_{PD[\{X_\alpha\}]} = D_P(\{X_\alpha\})$ is a valuation domain such that $\mathrm{ht}P = \dim(D_P) = \dim(D[\{X_\alpha\}]_{PD[\{X_\alpha\}]}) = \mathrm{ht}(PD[\{X_\alpha\}])$, and P is t-invertible if and only if $PD[\{X_\alpha\}]$ is t-invertible. Thus, the results follow from these observations and the definitions. □

Let $A \subseteq B$ be an extension of integral domains. We say that B is *t-linked over A* if $I^{-1} = A$ for a nonzero finitely generated ideal I of A implies $(IB)^{-1} = B$; equivalently, if Q is a prime t-ideal of B, then either $Q \cap A = (0)$ or $Q \cap A \neq (0)$ and $(Q \cap A)_t \subsetneq A$ [4, Proposition 2.1]. The notion of t-linkedness was introduced in [13] in order to study the PvMD analogue of [12, Theorem 1] that D is a Prüfer domain if and only if each overring of D is integrally closed. It is clear that if S is a multiplicative set of A, then A_S is t-linked over A. Also, if A and B are Krull domains, then B is t-linked over A if and only if $\mathrm{ht}(Q \cap A) \leq 1$ for all maximal t-ideals Q of B, i.e., condition (PDE) is satisfied (cf. [16, Theorem 6.2]).

Let Λ be a set of prime t-ideals of an integral domain D. Then $\bigcap_{P \in \Lambda} D_P$ is called a subintersection of D. It is known that if D is a PvMD, then an overring of D is t-linked over D if and only if it is a subintersection of D [24, Theorem 3.8]. Hence, every t-linked overring of a ring of Krull type is a ring of Krull type [27, Corollary 7.2]. The following lemma presents a complete picture from our perspective.

Lemma 1.9. *Let D be a ring of Krull type* (*resp., an independent ring of Krull type, a generalized Krull domain, a Krull domain*)*. If R is a t-linked overring of D, then R is also a ring of Krull type* (*resp., an independent ring of Krull type, a generalized Krull domain, a Krull domain*)*.*

Proof. Let Q be a maximal t-ideal of R, and let $P = Q \cap D$. Then P is a prime t-ideal of D, and hence D_P is a valuation domain by Theorem 1.4. Hence, $D_P = R_Q$ and R_Q is a valuation domain. Thus, R is a PvMD. Next, note that two incomparable prime t-ideals of D are not contained in the same maximal t-ideal. Thus, R is a ring (resp., an independent ring) of Krull type when D is a ring (resp., an independent ring) of Krull type. Finally, if $\mathrm{ht}P = 1$ (resp., D_P is a rank-one DVR), then $\mathrm{ht}Q = 1$ (resp., R_Q is a rank-one DVR). Thus, if D is a generalized Krull domain (resp., Krull domain), then R is also a generalized Krull domain (resp., Krull domain). □

Let $T(D)$ be the set of t-invertible fractional t-ideals of an integral domain D and $Prin(D)$ be the set of nonzero principal fractional ideals of D. Then $T(D)$ is an abelian group under the t-multiplication $I * J = (IJ)_t$ [7, Lemme 1] and $Prin(D)$ is a subgroup of $T(D)$. Let $Cl(D) = T(D)/Prin(D)$ be the factor group of $T(D)$ modulo $Prin(D)$. For $I \in T(D)$, let $cl(I) \in Cl(D)$ denote the equivalence class of $T(D)$ containing I. Hence, $cl(I) = cl(J)$ if and only if $I = xJ$ for some $0 \neq x \in K$, and $cl(I) + cl(J) = cl((IJ)_t)$ in $Cl(D)$ for all $I, J \in T(D)$. We say that $Cl(D)$ is the *class group* of D. The notion of class groups was introduced by Bouvier in [7]. Let $Inv(D)$ be the set of invertible fractional ideals of D. It is easy to see that $Inv(D)$ is a subgroup of $T(D)$ containing $Prin(D)$, and thus $Pic(D) = Inv(D)/Prin(D)$ is a subgroup of $Cl(D)$ and called the *Picard group* or the *ideal class group* of D.

Clearly, if D is a Krull domain, then $Cl(D)$ is the usual divisor class group of D (see Remark 1.10), and if D is a Dedekind domain or a Prüfer domain, then $Cl(D)$ is the ideal class group of D, i.e., $Cl(D) = Pic(D)$ [7, Lemme 3]. The notion of the class group of D is very useful when we study the factorization properties of D. For example, a UFD (resp., GCD-domain, Bezout domain) is just a Krull domain (resp., PvMD, Prüfer domain) with $Cl(D) = \{0\}$ [16, Proposition 6.1] (resp., [8, Corollary 1.5]). In fact, $Cl(D)$ measures how far from a UFD (resp., GCD-domain) a Krull domain (resp., PvMD) is.

Remark 1.10. (1) Let D be an integral domain and $\mathcal{D}(D)$ be the set of divisor classes of D, i.e., $\mathcal{D}(D) = \{A \mid A \in \mathbf{F}(D) \text{ and } A_v = A\}$. Clearly, $\mathcal{D}(D)$ is a commutative semigroup under $A \oplus B = (AB)_v$ for all $A, B \in \mathcal{D}(D)$, and $Prin(D)$ is a subgroup of $\mathcal{D}(D)$. Moreover, $\mathcal{D}(D)$ is a group if and only if D is completely integrally closed (c.i.c.) [17, Theorem 34.3]. The divisor class group of a c.i.c. domain D is defined by the factor group $\mathcal{D}(D)/Prin(D)$ of $\mathcal{D}(D)$ modulo $Prin(D)$. It is well known that (i) a Krull domain is completely integrally closed and (ii) D is a Krull domain if and only if every nonzero ideal of D is t-invertible, and in this case, $t = v$, i.e., $I_v = I_t$ for all $I \in \mathbf{F}(D)$. Thus, if D is a Krull domain, then $Cl(D) = \mathcal{D}(D)/Prin(D)$.

(2) Let $V(D)$ be the set of v-invertible fractional v-ideals of D. Then $V(D)$ is an abelian group under the v-multiplication $I * J = (IJ)_v$ and $Prin(D)$ is a subgroup of $V(D)$. Hence, the factor group $Cl_v(D) = V(D)/Prin(D)$ of $V(D)$ modulo $Prin(D)$ is an abelian group. In particular, if D is c.i.c. (e.g., a Krull domain), then $Cl_v(D)$ is the divisor class group of D. Furthermore, note that a t-invertible t-ideal is a v-invertible v-ideal, so $Cl(D)$ is a subgroup of $Cl_v(D)$. However, $Cl_v(D) \neq Cl(D)$ in general. For example, if D is a rank-one nondiscrete valuation domain with value group $G \subsetneq \mathbb{R}$, then $Cl(D) =$

$\{0\} \subsetneq \mathbb{R}/G = Cl_v(D)$ [3, Theorem 2.7]. Thus, the divisor class group of Krull domains can be generalized to arbitrary integral domains in at least two ways.

(3) There is another symbol used for the class group of integral domains in order to distinguish the divisor class group of c.i.c. domains and the class group of general integral domains. It is $Cl_t(D)$ and called the t-class group of D.

(4) Let A and B be integral domains. We mean by $Cl(A) = Cl(B)$ that there is a group isomorphism from $Cl(A)$ onto $Cl(B)$. It is well known that $Cl(D[\{X_\alpha\}]) = Cl(D)$ if and only if D is integrally closed [14, Corollary 2.13].

2. The ring $D[\{x_i, y_i, u_i, v_i\}]$ with $x_i v_i = y_i u_i$

Throughout D denotes an integral domain with quotient field K, Λ is a nonempty index set, and $\mathbb{Z}^{(\Lambda)}$ is the direct sum of Λ-copies of the additive group of integers. Let $\{X_i, Y_i, U_i, V_i \mid i \in \Lambda\}$ (simply, $\{X_i, Y_i, U_i, V_i\}$) be a set of indeterminates over D, $D[\{X_i, Y_i, U_i, V_i\}]$ be the polynomial ring over D, $(\{X_iV_i - Y_iU_i\})$ be the prime ideal of $D[\{X_i, Y_i, U_i, V_i\}]$ generated by $\{X_iV_i - Y_iU_i \mid i \in \Lambda\}$, and $R = D[\{X_i, Y_i, U_i, V_i\}]/(\{X_iV_i - Y_iU_i\})$. Hence, if we let x_i, y_i, u_i, v_i be the images of X_i, Y_i, U_i, V_i in R, respectively, then

$$R = D[\{x_i, y_i, u_i, v_i\}] \text{ with } x_i v_i = y_i u_i \text{ for all } i \in \Lambda$$

and $R_{D\setminus\{0\}} = K[\{x_i, y_i, u_i, v_i\}]$. Let S (resp., T) be the multiplicative set of R generated by $\{x_i \mid i \in \Lambda\}$ (resp., $\{v_i \mid i \in \Lambda\}$). Clearly, $\{x_i, y_i, u_i\}$, $\{x_i, y_i, \frac{u_i}{x_i}\}$, $\{v_i, y_i, u_i\}$, and $\{v_i, y_i, \frac{u_i}{v_i}\}$ are algebraically independent sets over D, respectively,

- $R_S = D[\{x_i, y_i, \frac{u_i}{x_i}\}]_S = D[\{x_i, y_i, u_i\}]_S$,
- $R_T = D[\{v_i, y_i, \frac{u_i}{v_i}\}]_T = D[\{v_i, y_i, u_i\}]_T$, and

$D[\{x_i, y_i, u_i\}] \cup D[\{v_i, y_i, u_i\}] \subseteq R \subseteq D[\{x_i, y_i, \frac{u_i}{x_i}\}] \cap D[\{v_i, y_i, \frac{u_i}{v_i}\}]$.

Let $\{a_\alpha\}$ be a subset of an integral domain A. We denote by $\langle\{a_\alpha\}\rangle$ the multiplicative set of A generated by $\{a_\alpha\}$. In this section, we study some ring-theoretic properties of the ring R.

Lemma 2.1. *Let $R = D[\{x_i, y_i, u_i, v_i\}]$ and I a nonzero fractional ideal of D.*

(1) *$(IR)^{-1} = I^{-1}R$, and hence $(IR)_v = I_vR$.*
(2) *$(IR)_t = I_tR$.*
(3) *I is t-invertible if and only if IR is t-invertible.*
(4) *I is a prime t-ideal of D if and only if IR is a prime t-ideal of R.*
(5) *If I is a prime ideal, then $R_{IR} = D_I(\{x_i, y_i, u_i\})$.*
(6) *If I is a t-invertible height-one prime ideal, then $ht(IR) = 1$.*

Proof. (1) Clearly, $I^{-1}R \subseteq (IR)^{-1}$. For the reverse containment, let $h \in (IR)^{-1}$. Then $hI \subseteq R \subseteq D[\{x_i, y_i, \frac{u_i}{x_i}\}] \cap K[\{x_i, y_i, u_i, v_i\}]$. Since $\{x_i, y_i, \frac{u_i}{x_i}\}$ are algebraically independent over D, $h \in (ID[\{x_i, y_i, \frac{u_i}{x_i}\}])^{-1} = I^{-1}D[\{x_i, y_i, \frac{u_i}{x_i}\}]$

[20, Lemma 4.1]. Also, $h \in K[\{x_i, y_i, u_i, v_i\}]$. Note that $u_i = x_i \cdot \frac{u_i}{x_i}$ and $v_i = y_i \cdot \frac{u_i}{x_i}$; so

$$h(\{x_i, y_i, u_i, v_i\}) = h(\{x_i, y_i, x_i\frac{u_i}{x_i}, y_i\frac{u_i}{x_i}\}) \in I^{-1}D[\{x_i, y_i, \frac{u_i}{x_i}\}],$$

and since $\{x_i, y_i, \frac{u_i}{x_i}\}$ is a set of indeterminates over D, the coefficients of h must be in I^{-1}. Thus, $h \in I^{-1}R$.

(2) If A is a nonzero finitely generated subideal of IR, there is a nonzero finitely generated subideal J of I such that $A \subseteq JR$. Hence, by (1), $A_v \subseteq (JR)_v = J_vR \subseteq I_tR$, and thus $(IR)_t \subseteq I_tR$. For the reverse containment, let $0 \neq a \in I_t$. Then $a \in H_v$ for some nonzero finitely generated subideal H of I, and hence $a \in H_vR = (HR)_v \subseteq (IR)_t$. Thus, $I_t \subseteq (IR)_t$, and so $I_tR \subseteq (IR)_t$.

(3) By (1) and (2), $((IR)(IR)^{-1})_t = ((IR)(I^{-1}R))_t = (II^{-1})_tR$. Also, it is clear that $(II^{-1})_tR \cap K = (II^{-1})_t$. Thus, $(II^{-1})_t = D$ if and only if $((IR)(IR)^{-1})_t = R$.

(4) Let $S = \langle\{x_i\}\rangle$. It is clear that if $I \subseteq D$, then $IR_S \cap R = IR$. Hence, I is a prime ideal of D if and only if $ID[\{x_i, y_i, u_i\}]_S = IR_S$ is a prime ideal, if and only if IR is a prime ideal. Thus, the result follows from (2).

(5) By the proof of (4), IR is a prime ideal of R. Hence, if $S = \langle\{x_i\}\rangle$, then

$$\begin{aligned} R_{IR} &= D[\{x_i, y_i, u_i, v_i\}]_{ID[\{x_i,y_i,u_i,v_i\}]} = (D[\{x_i, y_i, u_i\}]_S)_{ID[\{x_i,y_i,u_i\}]_S} \\ &= D[\{x_i, y_i, u_i\}]_{ID[\{x_i,y_i,u_i\}]} = D_I(\{x_i, y_i, u_i\}). \end{aligned}$$

(6) By (5), $R_{IR} = D_I(\{x_i, y_i, u_i\})$, and since D_I is a rank-one DVR, R_{IR} is also a rank-one DVR [17, Proposition 18.7]. Thus, $\mathrm{ht}(IR) = 1$. □

Let S be a multiplicative set of D and I be a nonzero fractional ideal of D. It is known that if ID_S is a t-ideal of D_S, then $ID_S \cap D$ is a t-ideal of D (Lemma 1.2(5)). Thus, if I is a maximal t-ideal of D, then ID_S is a t-ideal of D_S if and only if ID_S is a maximal t-ideal.

Lemma 2.2. *Let $R = D[\{x_i, y_i, u_i, v_i\}]$, $S = \langle\{x_i\}\rangle$, and $T = \langle\{v_i\}\rangle$.*

(1) *$(x_i, v_j)_v = R$ for all $i, j \in \Lambda$.*
(2) *If A is a nonzero fractional t-ideal of R, then $A = AR_S \cap AR_T$.*
(3) *$R = R_S \cap R_T$.*
(4) *If Q is a maximal t-ideal of R, either Q_S or Q_T is a maximal t-ideal.*
(5) *(x_k, y_k) is a t-invertible height-one prime ideal of R for all $k \in \Lambda$.*

Proof. (1) Let $k \in \Lambda$. Clearly, $x_kD[x_k, y_k, u_k, v_k] = (X_k, Y_kU_k)/(X_kV_k - Y_kU_k)$; $v_kD[x_k, y_k, u_k, v_k] = (V_k, Y_kU_k)/(X_kV_k - Y_kU_k)$; and $(X_k, Y_kU_k) \cap (V_k, Y_kU_k) = (X_kV_k, Y_kU_k)$ in $D[X_k, Y_k, U_k, V_k]$ because X_k, V_k are algebraically independent over $D[Y_k, U_k]$. Thus,

$$x_kD[x_k, y_k, u_k, v_k] \cap v_kD[x_k, y_k, u_k, v_k] = x_kv_kD[x_k, y_k, u_k, v_k],$$

and hence $((x_k, v_k)D[x_k, y_k, u_k, v_k])_t = D[x_k, y_k, u_k, v_k]$. Hence, $(x_k, v_k)_v = R$ by Lemma 2.1(2) because $R = D[x_k, y_k, u_k, v_k][\{x_i, y_i, u_i, v_i \mid i \neq k\}]$.

Also, note that if $i \neq j$, then v_j is transcendental over $D[x_i, y_i, u_i, v_i]$. Thus, $(x_i, v_j)_v = R$.

(2) Clearly, $A \subseteq AR_S \cap AR_T$. For the reverse containment, let $0 \neq h \in AR_S \cap AR_T$. Then $h = \frac{f}{s} = \frac{g}{z}$ for some $s \in S$, $z \in T$ and $f, g \in A \Rightarrow zf = sg \in sR \cap zR = szR$ (because $(s, z)_v = R$ by (1)) $\Rightarrow f = sf_1$ for some $f_1 \in R \Rightarrow f_1 z = g \in A$. Thus, $h = f_1 \in (f_1 s, f_1 z)_v = (f, g)_v \subseteq A_t = A$.

(3) This follows directly from (2) above.

(4) Since Q is a maximal t-ideal of R, it suffices to show that $(QR_S)_t \subsetneq R_S$ or $(QR_T)_t \subsetneq R_T$. Assume to the contrary that $(QR_S)_t = R_S$ and $(QR_T)_t = R_T$. Then there is a nonzero finitely generated ideal A of R such that $A \subseteq Q$ and $R_S = (AR_S)^{-1} = A^{-1}R_S$ and $R_T = A^{-1}R_T$. Hence, $A^{-1} \subseteq R_S \cap R_T = R$, and thus $R = A_v \subseteq Q_t \subseteq R$. Thus, $Q_t = R$, a contradiction.

(5) Let $Q = (x_k, y_k)$ be the ideal of R generated by x_k, y_k. Then $\frac{v_k}{y_k} = \frac{u_k}{x_k} \in Q^{-1}$, and hence $(x_k, v_k) \subseteq QQ^{-1}$. Hence, by (1), $R = (x_k, v_k)_v \subseteq (QQ^{-1})_t \subseteq R$, and thus $(QQ^{-1})_t = R$. Next, if $P = (x_k, y_k)D[x_k, y_k, u_k, v_k]$, then

$$P = (X_k, Y_k)/(X_k V_k - Y_k U_k) \subsetneq D[X_k, Y_k, U_k, V_k]/(X_k V_k - Y_k U_k),$$

and since $\mathrm{ht}(X_k, Y_k) = 2$ as a prime ideal of $D[X_k, Y_k, U_k, V_k]$, P is a height-one prime ideal. Note that

$$Q = PD[x_k, y_k, u_k, v_k][\{x_i, y_i, u_i, v_i \mid i \neq k\}];$$

so P is t-invertible by Lemma 2.1(3). Thus, Q is a height-one prime ideal of R by Lemma 2.1(6). □

We next give the structure of prime t-ideals of $D[\{x_i, y_i, u_i, v_i\}]$ when D is a PvMD. This result is very useful when we study the (independent) rings of Krull type property of $D[\{x_i, y_i, u_i, v_i\}]$.

Proposition 2.3. *Let D be a PvMD and $R = D[\{x_i, y_i, u_i, v_i\}]$.*

1. *R is a PvMD.*
2. *If A is a t-ideal of R such that $A \subseteq R$ and $A \cap D \neq (0)$, then $A \cap D$ is a t-ideal of D and $A = (A \cap D)R$.*
3. *If $Q \in t\text{-Max}(R)$ with $Q \cap D = (0)$, then $htQ = 1$ and Q is t-invertible.*
4. *t-Spec$(R) = \{PR \mid P \in t\text{-Spec}(D)\} \cup \{Q \in t\text{-Max}(R) \mid Q \cap D = (0)\}$.*
5. *$t\text{-Max}(R) = \{PR \mid P \in t\text{-Max}(D)\} \cup \{Q \in t\text{-Max}(R) \mid Q \cap D = (0)\}$.*
6. *If D is a field, then R is a Krull domain.*

Proof. Let $S = \langle \{x_i\} \rangle$ and $T = \langle \{v_i\} \rangle$. And recall that $R_S = D[\{x_i, y_i, u_i\}]_S$ and $R_T = D[\{v_i, y_i, u_i\}]_T$.

(1) Since D is a PvMD and $\{x_i, y_i, u_i\}$ are algebraically independent over D, by Theorem 1.4, both $D[\{x_i, y_i, u_i\}]$ and $D[\{v_i, y_i, u_i\}]$ are PvMDs. Hence, both R_S and R_T are PvMDs. Let Q be a maximal t-ideal of R. By Lemma 2.2(4), we may assume that Q_S is a maximal t-ideal of R_S. Thus, $R_Q = (R_S)_{Q_S}$ is a valuation domain. Therefore, by Theorem 1.4, R is a PvMD.

(2) Since R is a PvMD, both A_S and A_T are t-ideals. Note that $A_S \cap D[\{x_i, y_i, u_i\}]$ is a t-ideal and $(A_S \cap D[\{x_i, y_i, u_i\}]) \cap D \neq (0)$. Note also that

if $0 \neq a \in D$ and $f \in D[\{x_i, y_i, u_i\}]$, then $(a, f)_v = (aD + c(f))_v D[\{x_i, y_i, u_i\}]$; hence

$$A_S \cap D[\{x_i, y_i, u_i\}] = (A \cap D)D[\{x_i, y_i, u_i\}]$$

and $A \cap D$ is a t-ideal. Thus, $(A \cap D)R$ is a t-ideal by Lemma 2.1(2) and $A_S = (A \cap D)R_S$. Similarly, $A_T = (A \cap D)R_T$. Thus, $A = A_S \cap A_T = (A \cap D)R$ by Lemma 2.2(2).

(3) By Lemma 2.2(4), we may assume that Q_S is a maximal t-ideal of R_S, and hence $Q_0 := Q_S \cap D[\{x_i, y_i, u_i\}]$ is a prime t-ideal of $D[\{x_i, y_i, u_i\}]$ such that $Q_0 \cap D = (0)$. Hence, by Lemma 1.3 and Theorem 1.4, $\mathrm{ht}Q_0 = 1$ and Q_0 is t-invertible. Thus, $\mathrm{ht}Q = \mathrm{ht}Q_S = \mathrm{ht}(Q_0)_S = 1$ and $Q_S = (Q_0)_S$ is t-invertible. Similarly, $Q_T = R_T$ or Q_T is t-invertible. Hence, there is a nonzero finitely generated ideal A of R such that $Q_S = (AR_S)_t$ and $Q_T = (AR_T)_t$. Since R is a PvMD, A is t-invertible, whence by Lemma 1.2(3), $(AR_S)_t = A_t R_S$ and $(AR_T)_t = A_t R_T$. Thus, $Q = A_t$ by Lemma 2.2(2), and hence Q is t-invertible.

(4) Let Q be a prime t-ideal of R, and let M be a maximal t-ideal of R such that $Q \subseteq M$. If $M \cap D = (0)$, then $\mathrm{ht}M = 1$ by (3), and hence $Q = M$. Next, assume that $M \cap D \neq (0)$. Then $M_S = (M \cap D)R_S$ by (2), and hence $Q_S = (Q \cap D)R_S$ (cf. Theorem 1.4(6)). By symmetry, $Q_T = (Q \cap D)R_T$. Thus, $Q \subseteq (Q \cap D)R_S \cap (Q \cap D)R_T = (Q \cap D)R$ by Lemma 2.2(2), and hence $Q = (Q \cap D)R$. The reverse containment follows directly from Lemma 2.1(4).

(5) This follows directly from (4) above.

(6) Let Q be a prime t-ideal of R. If Q' is a maximal t-ideal of R containing Q, then $Q' \cap D = (0)$ by assumption, and hence Q' is a t-invertible height-one prime ideal by (3). Thus, $Q = Q'$, whence Q is t-invertible. Therefore, R is a Krull domain [25, Theorem 3.6]. □

Corollary 2.4. *Let $R = D[\{x_i, y_i, u_i, v_i\}]$.*

(1) *D is a PvMD if and only if R is a PvMD.*
(2) *D is a ring (resp., an independent ring) of Krull type if and only if R is a ring (resp., an independent ring) of Krull type.*
(3) *D is a generalized Krull domain if and only if R is a generalized Krull domain.*
(4) [16, Corollary 14.7] *D is a Krull domain if and only if R is a Krull domain.*
(5) *D is a TV-PvMD if and only if R is a TV-PvMD.*

Proof. (1) If D is a PvMD, then R is a PvMD by Proposition 2.3(1). Conversely, assume that R is a PvMD. Then R_S is a PvMD, where $S = \langle \{x_i\} \rangle$. Note that $\{x_i, y_i, u_i\}$ are algebraically independent over D and

$$R_S = D[\{x_i, y_i, u_i\}]_S = D[\{y_i, u_i\}][\{x_i, x_i^{-1}\}].$$

Thus, $D[\{y_i, u_i\}]$, and hence D, is a PvMD by Theorem 1.4.

(2) This follows directly from (1) and Proposition 2.3.

(3) By Proposition 2.3(3)-(4), t-$\dim(D) = 1$, i.e., each prime t-ideal of D is a maximal t-ideal, if and only if t-$\dim(R) = 1$. Thus, by (2), the result follows.

(4) By Proposition 2.3, t-Spec$(R) = \{PR \mid P \in t\text{-Spec}(D)\} \cup \{Q \in t$-Max$(R) \mid Q \cap D = (0)\}$ and Q is t-invertible for all $Q \in t$-Max(R) with $Q \cap D = (0)$. Hence, by Lemma 2.1(3), every prime t-ideal of D is t-invertible if and only if every prime t-ideal of R is t-invertible. Thus, D is a Krull domain if and only if R is a Krull domain [25, Theorem 3.6].

(5) This follows directly from (2), Lemma 2.1(3), and Proposition 2.3. □

Let $A \subseteq B$ be an extension of integral domains such that B is t-linked over A. It is known that if I and J are t-invertible t-ideals of A, then

$$((IJ)_tB)_t = ((IJ)B)_t = ((IB)(JB))_t = ((IB)_t(JB)_t)_t$$

by [4, Proposition 2.1]. Hence, the map $\varphi : Cl(A) \to Cl(B)$ given by $\varphi(cl(I)) = cl((IB)_t)$ is a group homomorphism [4, Theorem 2.2].

Lemma 2.5. *Let $R = D[\{x_i, y_i, u_i, v_i\}]$.*

(1) *R is t-linked over D.*
(2) *The map $\varphi : Cl(D) \to Cl(R)$ given by $\varphi(cl(I)) = cl((IR)_t)$ is a group monomorphism.*

Proof. (1) If I is a nonzero finitely generated ideal of D such that $I^{-1} = D$, then $(IR)^{-1} = I^{-1}R = R$ by Lemma 2.1(1). Thus, R is t-linked over D.

(2) By (1), R is t-linked over D, and thus φ is a group homomorphism. Next, let I be a nonzero t-invertible t-ideal of D such that $(IR)_t = fR$ for some $f \in R$ and $S = \langle\{x_i\}\rangle$. Then $(IR)_t = IR$ by Lemma 2.1(2), whence $fD[\{x_i, y_i, u_i\}]_S = fR_S = IR_S = ID[\{x_i, y_i, u_i\}]_S$. Note that

$$fD[\{x_i, y_i, u_i\}]_S = gD[\{x_i, y_i, u_i\}]_S$$

for some $g \in D[\{x_i, y_i, u_i\}]$ with $x_i \nmid g$ in $D[\{x_i, y_i, u_i\}]$ for all $i \in \Lambda$; hence the previous equality shows that $g \in D$. Thus, $I = ID[\{x_i, y_i, u_i\}]_S \cap D = gD[\{x_i, y_i, u_i\}]_S \cap D = gD$. Hence, φ is injective. □

We next give the PvMD analogue of [16, Proposition 14.9] that if D is a Krull domain, then $R = D[\{x_i, y_i, u_i, v_i\}]$ is a Krull domain with $Cl(R) = Cl(D) \oplus \mathbb{Z}^{(\Lambda)}$.

Theorem 2.6. *If D is a PvMD, $Cl(D[\{x_i, y_i, u_i, v_i\}]) = Cl(D) \oplus \mathbb{Z}^{(\Lambda)}$.*

Proof. Let $R = D[\{x_i, y_i, u_i, v_i\}]$. Then, by Lemma 2.5, the map $\varphi : Cl(D) \to Cl(R)$ given by $\varphi(cl(I)) = cl((IR)_t)$ is a group monomorphism.

Now, let $D^* = D \setminus \{0\}$. Then $R_{D^*} = K[\{x_i, y_i, u_i, v_i\}]$, and hence R_{D^*} is a Krull domain with $Cl(R_{D^*}) = \mathbb{Z}^{(\Lambda)}$ [16, Proposition 14.8]. Let $\psi : Cl(R) \to Cl(R_{D^*})$ be defined by $\psi(cl(A)) = cl(A_{D^*})$, then ψ is a group homomorphism, and since R is a PvMD by Proposition 2.3(1), ψ is surjective. Note that if I is a nonzero t-invertible t-ideal of D, then $((IR)_t)R_{D^*} = (IR_{D^*})_t = R_{D^*}$; hence $\psi \circ \varphi = 0$. Let A be a t-invertible t-ideal of R such that $A_{D^*} = fR_{D^*}$ for some $0 \neq f \in R$. Then $\frac{1}{f}A_{D^*} = R_{D^*}$, and since A is of finite type, there is an $s \in D^*$ with $s\frac{1}{f}A \subseteq R$. Note that $s\frac{1}{f}A \cap D \neq (0)$ and $s\frac{1}{f}A$ is a t-ideal; hence by

Proposition 2.3(2), $s\frac{1}{f}A = JR$ for some t-ideal J of D. Since JR is t-invertible, J is t-invertible by Lemma 2.1(3). Thus, $cl(A) = cl((JR)_t) = \varphi(cl(J))$, and therefore we have an exact sequence

$$0 \to Cl(D) \to Cl(R) \to Cl(R_{D^*}) \to 0.$$

Note that (x_i, y_i) is a t-invertible prime t-ideal of R by Lemma 2.2(5) and $Cl(R_{D^*})$ is generated by $\{cl((x_i, y_i)R_{D^*}) \mid i \in \Lambda\}$ [16, Proof of Proposition 14.8]; so if we define $\theta : Cl(R_{D^*}) \to Cl(R)$ by

$$\theta(\sum k_i cl(((x_i, y_i)R_{D^*}))) = \sum k_i cl((x_i, y_i)),$$

then θ is a well-defined group homomorphism. Clearly, $\psi \circ \theta$ is the identity function of $Cl(R_{D^*})$, and hence the exact sequence above is split. Thus, $Cl(R) = Cl(D) \oplus Cl(R_{D^*}) = Cl(D) \oplus \mathbb{Z}^{(\Lambda)}$. □

Let D be a PvMD (resp., a ring of Krull type, an independent ring of Krull type, a generalized Krull domain, a Krull domain, a TV-PvMD) with $Cl(D) = \{0\}$, and let $R = D[\{x_i, y_i, u_i, v_i\}]$. Then R is a PvMD (resp., a ring of Krull type, an independent ring of Krull type, a generalized Krull domain, a Krull domain, a TV-PvMD) with $Cl(R) = \mathbb{Z}^{(\Lambda)}$ by Corollary 2.4 and Theorem 2.6. We end this section with some examples of such rings.

Example 2.7. (1) Let D be a non-discrete valuation domain that is not a field. Then D is an independent ring of Krull type such that $Cl(D) = \{0\}$ but D is neither a TV-PvMD nor a Krull domain, and D is a generalized Krull domain if and only if $\dim(D) = 1$, i.e., each nonzero prime ideal of D is a maximal ideal.

(2) Let V be a discrete valuation domain of (Krull) dimension ≥ 2. Then V is a TV-PvMD with $Cl(V) = \{0\}$ but not a Krull domain.

(3) Let D be a Prüfer domain with $1 < |\mathrm{Max}(D)| < \infty$, K be the quotient field of D, X be an indeterminate over D, and $R_1 = D + XK[X]$. Then R_1 is a ring of Krull type, $Cl(R_1) = Cl(D) = \{0\}$, but R_1 is not an independent ring of Krull type (cf. [11, Section 4] for the proof).

(4) Let $\{X_\alpha\}$ be a nonempty set of indeterminates over D, and let $N_v = \{f \in D[\{X_\alpha\}] \mid c(f)_v = D\}$. Then $Cl(D[\{X_\alpha\}]_{N_v}) = \{0\}$ [24, Theorem 2.14], and D is a PvMD (resp., a ring of Krull type, an independent ring of Krull type, a generalized Krull domain, a Krull domain, a TV-PvMD) if and only if $D[\{X_\alpha\}]_{N_v}$ is a Prüfer domain (resp., a Prüfer domain of finite character, an h-local Prüfer domain, a generalized Krull domain of dimension one, a principal ideal domain, a Prüfer domain whose nonzero ideals are v-ideals) (cf. Theorem 1.6 and [24]).

3. The class group of rings of Krull type

Let D be an integral domain with quotient field K, X be an indeterminate over D, and $D[X]$ be the polynomial ring over D. For $0 \neq f \in D[X]$ that is irreducible in $K[X]$, let $Q_f = fK[X] \cap D[X]$. Hence, Q_f is a height-one prime ideal (so a t-ideal) of $D[X]$ such that $Q_f \cap D = (0)$.

The next result is already well known for Krull domains ([10, Proposition 4] or [16, Theorem 14.3]).

Lemma 3.1. *Let D be a ring of Krull type and I be a t-invertible t-ideal of D.*

(1) *$I^{-1} = (a,b)_v$ for some $0 \neq a, b \in K$.*
(2) *For $f = aX + b \in K[X]$ with $I = (a,b)^{-1}$, let $Q_f = fK[X] \cap D[X]$. Then Q_f is a t-invertible t-ideal of $D[X]$ such that $cl(ID[X]) = cl(Q_f)$.*
(3) *Every class of $Cl(D[X])$ contains a prime ideal of the form Q_f for some $f = aX + b \in D[X]$.*

Proof. (1) If $0 \neq d \in I$, then $dI^{-1} \subseteq D$. Let $0 \neq c \in dI^{-1}$. Then, since D is of finite t-character, there are only finitely many maximal t-ideals of D, say, $P_1, \dots, P_n$ that contain c. Let $S = D \setminus \bigcup_{i=1}^n P_i$. Since dI^{-1} is t-invertible, $dI^{-1}D_S$ is invertible, and hence $dI^{-1}D_S = eD_S$ for some $0 \neq e \in dI^{-1}$. Thus, $dI^{-1} = (c,e)_v$ or $I^{-1} = (\frac{c}{d}, \frac{e}{d})_v$.

(2) Note that $Q_f = fK[X] \cap D[X] = fc(f)^{-1}[X] = fID[X]$. Since D is a PvMD, I is t-invertible. Thus, Q_f is t-invertible and $cl(Q_f) = cl(ID[X])$.

(3) Let A be a t-invertible t-ideal of $D[X]$. Then we may assume that $A \subseteq D[X]$. If $A \cap D \neq (0)$, then $A = (A \cap D)D[X]$ [22, Lemma 4.5] and $A \cap D$ is a t-invertible t-ideal. Next, if $A \cap D = (0)$, then there are $0 \neq h \in D[X]$ and a fractional t-ideal J of D such that $A = hJD[X]$ [22, Lemma 4.5]. Since A is t-invertible, J is also t-invertible. Thus, the result follows directly from (1) and (2). □

Nagata theorem states that if D is a Krull domain and if Δ is a set of height-one prime ideals of D, then $R = \bigcap_{P \in \Delta} D_P$ is a Krull domain with $Cl(R) = Cl(D)/H$, where H is the subgroup of $Cl(D)$ generated by $\{cl(P) \mid P \in X^1(D) \setminus \Delta\}$ [16, Theorem 7.1]. The next result is a partial analogue of rings of Krull type (cf. [10, The proof of Proposition 5] for Krull domains).

Theorem 3.2. *Let D be a ring of Krull type, H be a subgroup of $Cl(D)$, U be the set of all linear polynomials $f \in D[X]$ such that $cl(Q_f) \in H$, $\Omega =$ t-$\mathrm{Max}(D[X]) \setminus \{Q_f \mid f \in U\}$, and $R = \bigcap_{Q \in \Omega} D[X]_Q$.*

(1) *R is t-linked over $D[X]$.*
(2) *$R = D[X]_{N_v} \cap K[X]_{\langle U \rangle}$, where $N_v = \{f \in D[X] \mid c(f)_v = D\}$ and $\langle U \rangle$ is the multiplicative set of $D[X]$ generated by U.*
(3) *t-$\mathrm{Max}(R) = \{PD[X]_{N_v} \cap R \mid P \in t\text{-}\mathrm{Max}(D)\} \cup \{fK[X]_{\langle U \rangle} \cap R \mid f$ is irreducible in $K[X]$ but $f \notin U\}$.*
(4) *R is a ring of Krull type and $Cl(R) = Cl(D)/H$.*

Proof. (1) Since D is a PvMD, $D[X]$ is a PvMD. Thus, R is t-linked over $D[X]$ [24, Theorem 3.8].

(2) Let $\Delta = \{f \in D[X] \mid fK[X]$ is a prime ideal and $f \notin U\}$, and note that t-Max$(D[X]) = \{P[X] \mid P \in t\text{-Max}(D)\} \cup \{Q_f \mid f \in \Delta \cup U\}$. Then

$$R = (\bigcap_{P \in t\text{-Max}(D)} D[X]_{P[X]}) \cap (\bigcap_{f \in \Delta} D[X]_{Q_f}) = D[X]_{N_v} \cap K[X]_{\langle U \rangle}.$$

(3) Note that $R = D[X]_{N_v} \cap K[X]_{\langle U \rangle}$ by (2); so $R_{PD[X]_{N_v} \cap R} = D[X]_{P[X]}$ for all $P \in t$-Max(D) and $R_{fK[X]_{\langle U \rangle} \cap R} = D[X]_{Q_f}$ for all $f \in \Delta$. Hence, the intersection $R = \bigcap_{Q \in \Omega} D[X]_Q$ is locally finite. Thus, the result follows (cf. the proof of Theorem 1.6).

(4) R is t-linked over $D[X]$ by (1), and D is a ring of Krull type if and only if $D[X]$ is a ring of Krull type by Theorem 1.6. Thus, if D is a ring of Krull type, then R is a ring of Krull type by Lemma 1.9. Hence, it suffices to show that $Cl(R) = Cl(D)/H$.

Since R is t-linked over $D[X]$, the map $\varphi : Cl(D[X]) \to Cl(R)$ given by $\varphi(cl(A)) = cl((AR)_t)$ is a group homomorphism. We first show that φ is surjective. Let B be a t-invertible t-ideal of R. We may assume that $B \subseteq R$. Then $B = (u_1, \ldots, u_k)_v$ for some $u_i \in R \subseteq D[X]_{N_v}$, and hence there is an $h \in N_v$ such that $hu_i \subseteq D[X]$ for $i = 1, \ldots, k$. Let $A = ((hu_1, \ldots, hu_k)D[X])_t$. Then A is a t-invertible t-ideal of $D[X]$ and $hB = (AR)_t$. Thus, $\varphi(cl(A)) = cl(B)$.

Next, we show that $ker(\varphi) = H$. Note that $H = \{cl(Q_f) \mid f \in U\}$ by Lemma 3.1; hence $H \subseteq ker(\varphi)$ because $(Q_f R)_t = \bigcap_{Q \in \Omega} Q_f D[X]_Q = R$ for all $f \in U$ by (3) and [24, Theorem 3.5]. Conversely, assume that A is a t-invertible t-ideal of $D[X]$ such that $(AR)_t$ is principal. Since D is a PvMD, there are a $u \in K(X)$ and a t-invertible t-ideal I of D such that $A = uID[X]$. Since $I^{-1} = (a, b)_v$ for some $0 \neq a, b \in K$, if we let $h = aX + b$, then $Q_h = hID[X]$, and so $cl(A) = cl(Q_h)$ and $(Q_h R)_t$ is principal. Note that $R_{D \setminus \{0\}} = K[X]_{\langle U \rangle}$; hence $((Q_h R)_t)_{D \setminus \{0\}} = hK[X]_{\langle U \rangle}$, and thus $(Q_h R)_t = \frac{hf}{g} R$ for some $f, g \in \langle U \rangle$. Note also that $\text{Max}(D[X]_{N_v}) = \{PD[X]_{N_v} \mid P \in t\text{-Max}(D)\}$ [24, Proposition 2.1]; so $D[X]_{N_v} = ((Q_h R)_t)_{N_v} = \frac{hf}{g} D[X]_{N_v}$, and thus $c(g)_t = c(hf)_t = (c(h)c(f))_t$. Hence, if we let $f = f_1 \cdots f_n$ and $g = g_1 \cdots g_m$ for $f_i, g_j \in U$, then $(c(g_1) \cdots c(g_m))_t = (c(h)c(f_1) \cdots c(f_n))_t$ or $(c(g_1)^{-1} \cdots c(g_m)^{-1})_t = (c(h)^{-1} c(f_1)^{-1} \cdots c(f_n)^{-1})_t$. Thus,

$$\begin{aligned}
\sum_i cl(Q_{g_i}) &= \sum_i cl(c(g_i)^{-1}[X]) \\
&= cl((c(g_1)^{-1} \cdots c(g_m)^{-1})_t[X]) \\
&= cl((c(h)^{-1} c(f_1)^{-1} \cdots c(f_n)^{-1})_t[X]) \\
&= cl(c(h)^{-1}[X]) + \sum_i cl(c(f_i)^{-1}[D]) \\
&= cl(Q_h) + \sum_j cl(Q_{f_j}).
\end{aligned}$$

Therefore, $cl(A) = cl(Q_h) = \sum_i cl(Q_{g_i}) - \sum_j cl(Q_{f_j}) \in H$. □

We say that a nonzero ideal I of D is *t-locally principal* if ID_P is principal for all $P \in t$-Max(D). It is known that a t-invertible ideal is t-locally principal, and if D is of finite t-character, then a nonzero t-locally principal ideal is t-invertible [9, Corollary 2.2].

Corollary 3.3. *Let the notation be as in Theorem 3.2. Then D is a ring of Krull type* (*resp., an independent ring of Krull type, a generalized Krull domain, a Krull domain, a TV-PvMD*) *if and only if R is.*

Proof. It is known that D is a ring of Krull type (resp., an independent ring of Krull type, a generalized Krull domain, a Krull domain, a TV-PvMD) if and only if $D[X]$ is (Theorem 1.6 and Corollary 1.8).

($\Rightarrow$) Since R is t-linked over $D[X]$ by Theorem 3.2(1), R is a ring of Krull type (resp., an independent ring of Krull type, a generalized Krull domain, a Krull domain) by Lemma 1.9. For the TV-PvMD property, assume that D is a TV-PvMD. Then D, and hence R, is an independent ring of Krull type and $t\text{-Max}(R) = \{PD[X]_{N_v} \cap R \mid P \in t\text{-Max}(D)\} \cup \{fK[X]_{\langle U \rangle} \cap R \mid f \text{ is irreducible in } K[X] \text{ but } f \notin U\}$; $R_{PD[X]_{N_v} \cap R} = D[X]_{PD[X]}$ for all $P \in t$-Max(D); and $R_{fK[X]_{\langle U \rangle} \cap R} = K[X]_{fK[X]}$ for all $f \in D[X]$ that is irreducible in $K[X]$ but $f \notin U$. Hence, each maximal t-ideal Q of R is t-locally principal, and thus Q is t-invertible. Therefore, R is a TV-PvMD.

($\Leftarrow$) Note that $D[X]_{N_v} = R_{N_v}$; so $D[X]_{N_v}$ is t-linked over R. Thus, $D[X]_{N_v}$ is a ring of Krull type (resp., an independent ring of Krull type, a generalized Krull domain, a Krull domain), and so is D. Now, assume that R is a TV-PvMD. Then D is an independent ring of Krull type by the previous sentence. Note that $R_{PD[X]_{N_v} \cap R} = D[X]_{PD[X]} = D[X]_{N_v PD[X]_{N_v}}$ for all $P \in t$-Max(D). Hence, if P is a maximal t-ideal of D, then $PD[X]_{N_v}$ is t-locally principal, and since $D[X]_{N_v}$ is of finite t-character, $PD[X]_{N_v}$ is t-invertible. Since each maximal ideal of $D[X]_{N_v}$ is a t-ideal [24, Corollary 2.3], $PD[X]_{N_v}$ is invertible. Thus, P is t-invertible [24, Corollary 2.5]. Therefore, D is a TV-PvMD. □

Corollary 3.4. *Let G an abelian group. Then the following statements hold.*

(1) *There is a ring of Krull type D such that $Cl(D) = G$ but D is not an independent ring of Krull type.*
(2) *There is an independent ring of Krull type D such that $Cl(D) = G$ but D is neither a generalized Krull domain nor a TV-PvMD.*
(3) *There is a generalized Krull domain D such that $Cl(D) = G$ but D is not a Krull domain.*
(4) *There is a TV-PvMD D such that $Cl(D) = G$ but D is not a Krull domain.*

Proof. Since G is an abelian group, there is an index set Λ such that $G = \mathbb{Z}^{(\Lambda)}/H$ for some subgroup H of $\mathbb{Z}^{(\Lambda)}$. Let D be a ring of Krull type (resp., a generalized Krull domain, a TV-PvMD) that is not an independent ring of Krull type (resp., a Krull domain, a Krull domain) and $Cl(D) = \{0\}$ (cf. Example 2.7). Then, by Corollary 2.4, Theorems 2.6 and 3.2, and Corollary 3.3, we can use D to construct a ring of Krull type (resp., a generalized Krull domain, a TV-PvMD) R such that $Cl(R) = G$ but R is not an independent ring of Krull type (resp., a Krull domain, a Krull domain). The same argument also shows that there is an independent ring of Krull type R such that $Cl(R) = G$ but R is neither a generalized Krull domain nor a TV-PvMD. □

Let $\{X_\alpha\}$ be a nonempty set of indeterminates over D and $N_v = \{f \in D[\{X_\alpha\}] \mid c(f)_v = D\}$. It is known that $Cl(D[\{X_\alpha\}]_{N_v}) = \{0\}$, and D is a PvMD if and only if $D[\{X_\alpha\}]_{N_v}$ is a Prüfer domain. We next show that if D is a PvMD, then there is a Prüfer domain R such that $D[\{X_\alpha\}] \subseteq R \subseteq D[\{X_\alpha\}]_{N_v}$ and $Cl(R) = Cl(D)$. For this, let S be a saturated multiplicative set of D, and let $N(S) = \{d \in D \mid (d, s)_v = D$ for all $s \in S\}$. We say that S is *splitting* if each nonzero $d \in D$ can be written as $d = sz$ for some $s \in S$ and $z \in N(S)$. It is known that if S is splitting, then $Cl(D) = Cl(D_S) \oplus Cl(D_{N(S)})$ [1, Corollary 3.8] and AD_S is a t-ideal for all t-ideals A of D [1, Corollary 3.5].

Theorem 3.5. *Let S be the saturated multiplicative set of $D[\{X_\alpha\}]$ generated by all nonconstant prime polynomials, and $R = D[\{X_\alpha\}]_S$.*

(1) *S is a splitting set such that $c(f)_v = D$ for all $f \in S$.*
(2) *$Cl(D) = Cl(R)$ if and only if D is integrally closed.*
(3) *t-Max$(R) = \{PD[\{X_\alpha\}]_S \mid P \in t$-Max$(D)\} \cup \{Q_S \mid Q \in t$-Max$(D[\{X_\alpha\}])$, $Q \cap D = (0)$ and $Q \cap S = \emptyset\}$.*
(4) *If $|\{X_\alpha\}| = \infty$, then t-Max$(R) =$ Max(R).*
(5) *If $|\{X_\alpha\}| = \infty$, then D is a PvMD if and only if R is a Prüfer domain.*
(6) *$R = D[\{X_\alpha\}]_{N_v} \cap K[\{X_\alpha\}]_S$, where $N_v = \{f \in D[\{X_\alpha\}] \mid c(f)_v = D\}$.*

Proof. (1) If g is a nonconstant prime polynomial of $D[\{X_\alpha\}]$, then $gD[\{X_\alpha\}]$ is a maximal t-ideal. Hence, $g \notin PD[\{X_\alpha\}]$ for all $P \in t$-Max(D), and thus $c(g)_v = D$. Thus, if $f \in S$, then f is a finite product of such prime polynomials, and thus $c(f)_v = D$. Next, note that $K[\{X_\alpha\}]$ is a UFD; so

$$\bigcap_{n=1}^{\infty} (f_1 \cdots f_n) D[\{X_\alpha\}] \subseteq \bigcap_{n=1}^{\infty} (f_1 \cdots f_n) K[\{X_\alpha\}] = (0)$$

for distinct prime elements $\{f_1, \ldots, f_n, \ldots\} \subseteq S$. Also, $\bigcap_{n=1}^{\infty} f^n D[\{X_\alpha\}] = (0)$ for $f \in D[\{X_\alpha\}] \setminus D$. Thus, S is a splitting set of $D[\{X_\alpha\}]$ [1, Proposition 2.6].

(2) Let $N(S) = \{h \in D[\{X_\alpha\}] \mid (h, f)_v = D[\{X_\alpha\}]$ for all $f \in S\}$. Then $D \setminus \{0\} \subseteq N(S)$ because $c(f)_v = D$ for all $f \in S$ by (1). Hence, $D[\{X_\alpha\}]_{N(S)} = K[\{X_\alpha\}]_{N(S)}$, and so $D[\{X_\alpha\}]_{N(S)}$ is a UFD. Hence,

$$Cl(D[\{X_\alpha\}]) = Cl(D[\{X_\alpha\}]_S) \oplus Cl(D[\{X_\alpha\}]_{N(S)}) = Cl(R).$$

Thus, $Cl(D) = Cl(R)$ if and only if $Cl(D) = Cl(D[\{X_\alpha\}])$, if and only if D is integrally closed [14, Corollary 2.13].

(3) Recall that t-Max$(D[\{X_\alpha\}]) = \{PD[\{X_\alpha\}] \mid P \in t\text{-Max}(D)\} \cup \{Q \in t$-Max$(D[\{X_\alpha\}]) \mid Q \cap D = (0)\}$. Also, since S is a splitting set, $(AR)_t = A_tR$ for all nonzero ideals A of $D[\{X_\alpha\}]$ [1, Corollary 3.5]. Thus, the result follows.

(4) It suffices to show that if Q is a nonzero prime ideal of $D[\{X_\alpha\}]$ such that $Q_t = D[\{X_\alpha\}]$, then $Q \cap S \neq \emptyset$. Note that $c(Q)_t = D$, and hence there is an $f \in Q$ such that $c(f)_v = D$.

Case 1. $Q \cap D \neq (0)$. Choose $0 \neq a \in Q \cap D$. Since $|\{X_\alpha\}| = \infty$, there is an $X \in \{X_\alpha\}$ such that X does not appear in f. Clearly, $(a, f)_v = D[\{X_\alpha\}]$, and so if we let $g = aX + f$, then $g \in Q$ and g is a prime element of $D[\{X_\alpha\}]$.

Case 2. $Q \cap D = (0)$. Then $Q_{D\setminus\{0\}}$ is a prime ideal of $K[\{X_\alpha\}]$ and $\text{ht}(Q_{D\setminus\{0\}}) \geq 2$. Note that $K[\{X_\alpha\}]$ is a UFD. Hence, there is an $0 \neq h \in Q$ such that $hK[\{X_\alpha\}]$ is a prime ideal and $f \notin hK[\{X_\alpha\}]$. Clearly, $(f, h)_v = D[\{X_\alpha\}]$. Choose $X \in \{X_\alpha\}$ such that X does not appear in both f and h, and let $g = hX + f$. Then $g \in Q \cap S$.

(5) ($\Rightarrow$) Let M be a maximal ideal of R. Then $M \cap D[\{X_\alpha\}]$ is a maximal t-ideal of $D[\{X_\alpha\}]$ by (4) above, and hence $D[\{X_\alpha\}]_{M\cap D[\{X_\alpha\}]}$ is a valuation domain by Theorem 1.4. Note that $D[\{X_\alpha\}]_{M\cap D[\{X_\alpha\}]} \subseteq R_M$; so R_M is a valuation domain. Thus, R is a Prüfer domain. ($\Leftarrow$) By (1), $S \subseteq N_v$, and hence $R \subseteq D[\{X_\alpha\}]_{N_v}$. Since R is a Prüfer domain, $D[\{X_\alpha\}]_{N_v}$ is a Prüfer domain [17, Theorem 26.1]. Thus, D is a PvMD by Theorem 1.4.

(6) Let Ω be the set of all maximal t-ideals Q of $D[\{X_\alpha\}]$ such that $Q \cap D = (0)$ and $Q \cap S = \emptyset$. Then

$$D[\{X_\alpha\}]_Q = (D[\{X_\alpha\}]_S)_{QD[\{X_\alpha\}]_S} = (K[\{X_\alpha\}]_S)_{QK[\{X_\alpha\}]_S}$$

for all $Q \in \Omega$, and thus

$$\begin{aligned} R &= \Big(\bigcap_{P\in t\text{-Max}(D)} D[\{X_\alpha\}]_{PD[\{X_\alpha\}]}\Big) \cap \Big(\bigcap_{Q\in\Omega} D[\{X_\alpha\}]_Q\Big) \\ &= D[\{X_\alpha\}]_{N_v} \cap K[\{X_\alpha\}]_S \end{aligned}$$

(cf. [24, Proposition 2.1] for the last equality). □

An integral domain D is called a *divisorial domain* if every nonzero ideal of D is a v-ideal. Since an invertible ideal is a t-invertible t-ideal, a Prüfer domain that is a TV-PvMD is a divisorial domain. In [21], Heinzer showed that (i) if D is a divisorial domain, then D is an h-local domain and (ii) if D is integrally closed, then D is a divisorial domain if and only if D is an h-local Prüfer domain whose nonzero maximal ideals are invertible. It is clear that a Dedekind domain is an integrally closed divisorial domain.

Corollary 3.6. *Let the notation be as in Theorem 3.5, and assume $|\{X_\alpha\}| = \infty$. Then the following statements hold.*

(1) *D is a ring of Krull type if and only if R is a Prüfer domain of finite character.*
(2) *D is an independent ring of Krull type if and only if R is an h-local Prüfer domain.*

(3) *D is a generalized Krull domain if and only if R is a generalized Krull domain of (Krull) dimension one.*
(4) *D is a Krull domain if and only if R is a Dedekind domain.*
(5) *D is a TV-PvMD if and only if R is an integrally closed divisorial domain.*
(6) *D is a UFD if and only if R is a principal ideal domain.*

Proof. Let Q be a maximal t-ideal of $D[\{X_\alpha\}]$ such that $Q \cap D = (0)$. Then $\text{ht}Q = 1$ and Q is t-invertible by Lemma 1.3, and hence $D[\{X_\alpha\}]_Q$ is a rank-one DVR. Note that $K[\{X_\alpha\}]$ is a UFD; so each nonzero nonunit of $D[\{X_\alpha\}]$ is contained in only finitely many such maximal t-ideals. Also, note that $D[\{X_\alpha\}]_{N_v}$ is a Prüfer domain, $\text{Max}(D[\{X_\alpha\}]_{N_v}) = \{PD[\{X_\alpha\}]_{N_v} \mid P \in t\text{-Max}(D)\}$, and each prime ideal of $D[\{X_\alpha\}]_{N_v}$ is extended from D [24, Proposition 2.1, Theorems 3.1 and 3.7]. Thus, the result follows directly from Theorem 3.5 (cf. the proof of Corollary 3.3). □

Corollary 3.7. *Let G an abelian group. Then the following statements hold.*

(1) *There is a Prüfer domain of finite character D such that $Cl(D) = G$ but D is not an h-local Prüfer domain.*
(2) *There is an h-local Prüfer domain D such that $Cl(D) = G$ but D is neither a generalized Krull domain nor a divisorial domain.*
(3) *There is a generalized Krull domain of dimension one D such that $Cl(D) = G$ but D is not a Dedekind domain.*
(4) *There is a Prüfer domain D in which each nonzero ideal is a v-ideal (i.e., an integrally closed divisorial domain) such that $Cl(D) = G$ but D is not a Dedekind domain.*

Proof. This follows directly from Theorem 3.5, Corollaries 3.4 and 3.6. □

Let $\{X_\alpha\}$ be an infinite set of indeterminates over a PvMD D, S be the multiplicative set of $D[\{X_\alpha\}]$ generated by all nonconstant prime polynomials, and $R = D[\{X_\alpha\}]_S$. In Theorem 3.5, we show that R is a PvMD with $Cl(R) = Cl(D)$ by using the fact that $\{X_\alpha\}$ is infinite. Hence, we have the following question.

Question 3.8. *Let D be a PvMD, X be an indeterminate over D, and $D[X]$ be the polynomial ring over D. Is there a multiplicative set T of $D[X]$ such that $D[X]_T$ is a Prüfer domain with $Cl(D[X]_T) = Cl(D)$?*

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