J. Korean Math. Soc. **58** (2021), No. 1, pp. 109–122
https://doi.org/10.4134/JKMS.j190851
pISSN: 0304-9914 / eISSN: 2234-3008

SINGULAR MINIMAL TRANSLATION GRAPHS IN EUCLIDEAN SPACES

Muhittin Evren Aydin, Ayla Erdur, and Mahmut Ergüt

Abstract. In this paper, we consider the problem of finding the hypersurface M^n in the Euclidean $(n+1)$-space $\mathbb{R}^{n+1}$ that satisfies an equation of mean curvature type, called singular minimal hypersurface equation. Such an equation physically characterizes the surfaces in the upper halfspace $\mathbb{R}^3_+(\mathbf{u})$ with lowest gravity center, for a fixed unit vector $\mathbf{u} \in \mathbb{R}^3$. We first state that a singular minimal cylinder M^n in $\mathbb{R}^{n+1}$ is either a hyperplane or a α-catenary cylinder. It is also shown that this result remains true when M^n is a translation hypersurface and $\mathbf{u}$ is a horizantal vector. As a further application, we prove that a singular minimal translation graph in $\mathbb{R}^3$ of the form $z = f(x) + g(y + cx)$, $c \in \mathbb{R} - \{0\}$, with respect to a certain horizantal vector $\mathbf{u}$ is either a plane or a α-catenary cylinder.

1. Introduction

Let the pair $(\mathbb{R}^3, g)$ denote the Euclidean 3-space and $\mathbf{u}$ a fixed unit vector in $\mathbb{R}^3$. Given a smooth immersion σ of an oriented surface M^2 into the halfspace

$$\mathbb{R}^3_+(\mathbf{u}) : \left\{ q \in \mathbb{R}^3 : g(q, \mathbf{u}) > 0 \right\}.$$

Let ξ and H be the Gauss map and mean curvature of σ, respectively (see [2]). Then, for some real constant α, the potential α-energy of σ in the direction of $\mathbf{u}$ can be introduced by (see [5], [20–22])

$$E(\sigma) = \int_{M^2} g(q, \mathbf{u})^\alpha \, dM^2, \tag{1}$$

where dM^2 denotes the measure on M^2 with respect to the induced metric tensor from the Euclidean metric g in $\mathbb{R}^3$ and $q = \sigma(p)$, $p \in M^2$.

Denoting $\Sigma : M^2 \times (-\theta, \theta) \to \mathbb{R}^3_+(\mathbf{u})$ a compactly supported variation of σ with variaton vector field ζ, then the first variation of E becomes

$$E'(0) = -\int_{M^2} \left[2Hg(\sigma, \mathbf{u}) - \alpha g(\xi, \mathbf{u}) \right] g(\sigma, \mathbf{u})^{\alpha - 1} g(\xi, \zeta) \, dM^2.$$

Received December 18, 2019; Revised July 9, 2020; Accepted September 21, 2020.
2010 *Mathematics Subject Classification.* Primary 53C42; Secondary 53A10, 53C44.
Key words and phrases. Singular minimal hypersurface, translation hypersurface, cylinder, translation graph, α-catenary.

Taking σ as a critical point of E, it then follows

$$2H = \alpha \frac{g(\xi, \mathbf{u})}{g(\sigma, \mathbf{u})}, \tag{2}$$

called *singular minimal surface equation* with respect to the vector $\mathbf{u}$, see [6]. In the cited paper Dierkes calls a surface M^2 a *singular minimal surface* or α*-minimal surface* with respect to the vector $\mathbf{u}$ if its mean curvature H fulfills Eq. (2).

Eq. (2) is clearly of mean curvature type and an extension of the classic *minimal surface equation* ([18, p. 17]), i.e., the situation $\alpha = 0$. Notice also that Eq. (2) is the Euler equation (see [29, p. 33]) of the variational integral given in Eq. (1).

Let $\gamma = \gamma(s)$ be a curve in $\mathbb{R}^2$ and $\mathbf{u} \in \mathbb{R}^2$ a fixed unit vector. For the curve γ one dimensional case of Eq. (2) writes

$$\kappa(s) = \alpha \frac{g(N(s), \mathbf{u})}{g(\gamma(s), \mathbf{u})}, \tag{3}$$

where κ and N are the *curvature* and unit *principal normal vector field* of γ. Hereinafter the curve γ whose the curvature κ satisfies Eq. (3) is referred to as α*-catenary* [6]. Up to a change of coordinates, we can take $\mathbf{u} = (0,1)$ and γ as the graph of $y = f(s)$. In that case Eq. (3) follows

$$\frac{f''}{1 + f'^2} = \frac{\alpha}{f}. \tag{4}$$

Let the y-axis in $\mathbb{R}^2$ denote the direction of the gravity. What to solve Eq. (4) in the case $\alpha = 1$ is physically to find the curve γ in the upper halfplane $y > 0$ with the lowest gravity center [23]. The solution of Eq. (4) is then the catenary

$$f(s) = \frac{1}{\lambda} \cosh(\lambda s + \mu), \lambda, \mu \in \mathbb{R}, \lambda \neq 0.$$

Let us now take the surface M^2 as a *generalized cylinder* in $\mathbb{R}^3$ (see [9, p. 439]), i.e., $M^2 = \gamma(s) \times \mathbb{R}\mathbf{w}$, $s \in I \subseteq \mathbb{R}$, where γ is the so-called *base curve*, $\mathbf{w} \in \mathbb{R}^3$ a fixed unit vector. We call it *cylinder*, shortly. Lopez [20] proved that a singular minimal cylinder is an α*-catenary cylinder,* namely a cylinder that takes the base curve as an α-catenary. More generally, in [19, 20], one was proved that a singular minimal translation surface, a graph of the form $z = f(x) + g(y)$, is an α-catenary cylinder, where $\{x, y, z\}$ is the orthogonal coordinate system in $\mathbb{R}^3$.

In this paper, we generalize to higher dimensions the mentioned results in previous paragraph. For this, we concern the following equation

$$nH = \alpha \frac{g(\xi, \mathbf{u})}{g(\sigma, \mathbf{u})}, \ n \geq 2, \tag{5}$$

where $\mathbf{u} \in \mathbb{R}^{n+1}$ is a fixed unit vector, ξ and H the Gauss map and mean curvature of the smooth immersion σ of an oriented hypersurface M^n into the halfspace

$$\mathbb{R}_{+}^{n+1}(\mathbf{u}):\left\{q\in\mathbb{R}^{n+1}:g(q,\mathbf{u})>0\right\}.$$

We call Eq. (5) *singular minimal hypersurface equation* with respect to the vector $\mathbf{u}$.

Let $\{x_1,\ldots,x_{n+1}\}$ denote the orthogonal coordinate system in $\mathbb{R}^{n+1}$. To study Eq. (5) we first take a generalized cylinder M^n in $\mathbb{R}^{n+1}$. By a *generalized cylinder* in $\mathbb{R}^{n+1}$, we mean a hypersurface given in the following form

$$M^n=\{\gamma(s)+t_1\mathbf{w}_1+\cdots+t_{n-1}\mathbf{w}_{n-1}:t_1,\ldots,t_{n-1}\in\mathbb{R},s\in I\subseteq\mathbb{R}\},\tag{6}$$

where $\mathbf{w}_1,\mathbf{w}_2,\ldots,\mathbf{w}_{n-1}$ are some orthonormal vectors in $\mathbb{R}^{n+1}$ and γ (so-called *base curve*) a 2-planar curve lying in $\Gamma=\mathtt{Span}\{\mathbf{w}_1,\ldots,\mathbf{w}_{n-1}\}^{\perp}$. As it was before, we call it *cylinder*, shortly. We prove that besides hyperplanes only singular minimal cylinders in $\mathbb{R}^{n+1}$ with respect to an arbitrary vector are the α-catenary cylinders.

Afterwards we study singular minimal *translation hypersurfaces* in $\mathbb{R}^{n+1}$, the graphs of the form (see [7])

$$x_{n+1}=f_1(x_1)+\cdots+f_n(x_n),$$

where $f_1,\ldots,f_n$ are smooth functions of single variable. We obtain that besides hyperplanes singular minimal translation hypersurfaces in $\mathbb{R}^{n+1}$ with respect to a horizontal vector are α-catenary cylinders. More geometric details on this class of hypersurfaces can be found in [1], [8], [10–12], [14, 15, 24], [25–28].

As a further application, we concern singular minimal translation graphs in $\mathbb{R}^3$ of the form $z=f(x)+g(y+cx)$, $c\in\mathbb{R}-\{0\}$. The study of the surfaces of this kind was initiated by Liu and Yu [17], obtaining explicit equations of minimal ones. These translation graphs, so-called *affine translation surfaces,* belong to the family of surfaces invariant by a group of translations (see Section 5). For some results and progress on those surfaces, see [13, 16], [30–32]. We show that if such a translation graph is singular minimal with respect to the horizontal vector $(1,0,0)$ in $\mathbb{R}^3$, then it is either a plane or a α-catenary cylinder.

2. Preliminaries

This section is devoted to present a short brief for hypersurfaces in $\mathbb{R}^{n+1}$. Further details can be found in [4].

Let $\sigma:M^n\to\mathbb{R}^{n+1}$ be a smooth immersion of an oriented hypersurface M^n. Then the *Gauss map* $\xi:M^n\longrightarrow\mathbb{S}^n$ maps M^n to the unit hypersphere $\mathbb{S}^n$ of $\mathbb{R}^{n+1}$. The differential $d\xi$ of the Gauss map ξ is called *Weingarten map.* For some vectors $\mathbf{v}$ and $\mathbf{w}$ tangent to M^n at the point $p\in M^n$, the *shape operator* A_p is given by

$$\tilde{g}(A_p(\mathbf{v}),\mathbf{w})=\tilde{g}(d\xi(\mathbf{v}),\mathbf{w}),$$

where $\tilde{g}$ is the induced metric tensor on M^n from the Euclidean metric g on $\mathbb{R}^{n+1}$. The *second fundamental form* h of σ is given in terms of the shape operator A by

$$g(\xi, h(\mathbf{v}, \mathbf{w})) = \tilde{g}(A_p(\mathbf{v}), \mathbf{w}).$$

The *mean curvature* of σ at p is defined by

$$H(p) = \frac{1}{n} \mathtt{tr} A_p,$$

where $\mathtt{tr}$ denotes the trace of A_p. A hypersurface is called *minimal* if H vanishes identically.

The following result is well-known (see [3]).

Proposition 2.1. *For a graph of $\mathbb{R}^{n+1}$ of the form $x_{n+1} = f(x_1, \ldots, x_n)$, we have*

1. *the unit normal vector field is*

$$\xi = \frac{-1}{\phi} (f_{x_1}, \ldots, f_{x_n}, -1),$$

where $\phi = \sqrt{1 + \sum_{j=1}^{n} (f_{x_j})^2}$ and $f_{x_j} = \frac{\partial f}{\partial x_j}$,

2. *the components of the induced metric tensor (or the first fundamental form) are*

$$g_{ij} = \delta_{ij} + f_{x_i} f_{x_j},$$

where δ_{ij} is Kronocker's Delta,

3. *the components of the second fundamental form are*

$$h_{ij} = \frac{f_{x_i x_j}}{\phi},$$

4. *the matrix $[a_{ij}]$ of the shape operator is*

$$a_{ij} = \sum_l h_{il} g^{lj} = \frac{f_{x_i x_j}}{\phi} - \sum_l \frac{f_{x_i x_l} f_{x_l} f_{x_j}}{\phi^3},$$

where $f_{x_i x_j} = \frac{\partial^2 f}{\partial x_i \partial x_j}$ and $[g^{lj}] = [g_{lj}]^{-1}$,

5. *the mean curvature H is*

$$H = \frac{1}{n} \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left(\frac{f_{x_j}}{\phi} \right).$$

3. Generalized cylinders

Let $\mathbf{w}_1, \ldots, \mathbf{w}_{n-1}$ be some fixed orthonormal vectors in $\mathbb{R}^{n+1}$ and $\gamma = \gamma(s)$, $s \in I \subseteq \mathbb{R}$ a unit speed curve lying in the 2-plane $\Gamma = \mathtt{Span}\{\mathbf{w}_1, \ldots, \mathbf{w}_{n-1}\}^{\perp}$. Consider the cylinder in $\mathbb{R}^{n+1}$ given by Eq. (6). Denoting $\times$ the cross product in $\mathbb{R}^{n+1}$ the unit normal vector field ξ of M^n becomes

$$\xi(s) = \mathbf{w}_1 \times \cdots \times \mathbf{w}_{n-1} \times \gamma'(s) = \sum_{i=1}^{n+1} \det(\mathbf{e}_i, \mathbf{w}_1, \ldots, \mathbf{w}_{n-1}, \gamma'(s)) \mathbf{e}_i,$$

where $\{\mathbf{e}_1, \ldots, \mathbf{e}_{n+1}\}$ is the standard basis of $\mathbb{R}^{n+1}$ and $\gamma' = \frac{d\gamma}{ds}$. Then, as the components of the fundamental forms of M^n, we get $g_{ij} = \delta_{ij}$, $1 \leq i, j \leq n$, and

$$h_{ij} = \begin{cases} \kappa & \text{for } i = j = n, \\ 0 & \text{otherwise,} \end{cases}$$

where $\kappa = \kappa(s)$, $s \in I$, denotes the curvature of γ. The mean curvature H of M^n turns to $H(s) = \frac{\kappa(s)}{n}$.

Theorem 3.1. *The only singular minimal cylinders in $\mathbb{R}^{n+1}$ with respect to an arbitrary fixed vector $\mathbf{u}$ are either the hyperplanes parallel to $\mathbf{u}$ or the α-catenary cylinders whose the rulings are orthogonal to $\mathbf{u}$.*

Proof. Eq. (5) leads to

$$\kappa(s) = \alpha \frac{g(\mathbf{w}_1 \times \cdots \times \mathbf{w}_{n-1} \times \gamma'(s), \mathbf{u})}{\sum_{i=1}^{n-1} g(\mathbf{w}_i, \mathbf{u}) t_i + g(\gamma(s), \mathbf{u})}$$

or equivalently

$$\kappa(s) \sum_{i=1}^{n-1} g(\mathbf{w}_i, \mathbf{u}) t_i + \kappa(s) g(\gamma(s), \mathbf{u}) - \alpha g(\mathbf{w}_1 \times \cdots \times \mathbf{w}_{n-1} \times \gamma'(s), \mathbf{u}) = 0. \tag{7}$$

If we take partial derivative of Eq. (7) with respect to t_i, $i = 1, \ldots, n-1$, we find

$$\kappa(s) g(\mathbf{w}_i, \mathbf{u}) = 0 \tag{8}$$

and

$$\kappa(s) g(\gamma(s), \mathbf{u}) - \alpha g(\mathbf{w}_1 \times \cdots \times \mathbf{w}_{n-1} \times \gamma'(s), \mathbf{u}) = 0. \tag{9}$$

We distinguish two cases for Eq. (8).

Case 1. $\kappa(s) = 0$ on I, identically. Then γ is a straight-line and Eq. (9) follows that M^n is a hyperplane parallel to $\mathbf{u}$ given that $\{\mathbf{w}_1, \ldots, \mathbf{w}_{n-1}, \gamma'(s), \mathbf{u}\}$ are linearly dependent at every $s \in I$.

Case 2. $\kappa(s) \neq 0$ for each $s \in I$. Then $g(\mathbf{w}_i, \mathbf{u}) = 0$ and we get that $\mathbf{u}$ is parallel to Γ. Hence Eq. (9) can be rewritten as

$$\kappa(s) = \alpha \frac{g(\mathbf{w}_1 \times \cdots \times \mathbf{w}_{n-1} \times \gamma'(s), \mathbf{u})}{g(\gamma(s), \mathbf{u})} = \alpha \frac{g(N(s), \mathbf{u})}{g(\gamma(s), \mathbf{u})}, \tag{10}$$

where $N(s)$ denotes the principal unit normal vector to γ at $s \in I$. Eq. (10) implies that γ is an α-catenary lying in Γ and M^n a α-catenary cylinder such that the rulings are orthogonal to $\mathbf{u}$ given $g(\mathbf{w}_i, \mathbf{u}) = 0$. $\square$

4. Translation hypersurfaces

A *translation hypersurface* M^n in $\mathbb{R}^{n+1}$ can be described as the sum of n curves $\gamma_1, \ldots, \gamma_n$, so-called *translating curves.* Then M^n locally parameterizes as

$$\sigma\left(x_1, \ldots, x_n\right) = \gamma_1\left(x_1\right) + \cdots + \gamma_n\left(x_n\right).$$

If the translating curves $\gamma_1, \ldots, \gamma_n$ lie in orthogonal 2-planes mutually, then, up to a change of coordinates, M^n becomes the graph of the form

$$x_{n+1} = f_1\left(x_1\right) + \cdots + f_n\left(x_n\right),$$

where $f_1, \ldots, f_n$ are smooth functions of single variable. We mean this graph by a translation hypersurface throughout the section.

By Proposition 2.1 the Gauss map ξ and mean curvature H of M^n are

$$\xi = \frac{\left(-f_1', \ldots, -f_n', 1\right)}{\left[1 + \sum_{i=1}^n \left(f_i'\right)^2\right]^{1/2}} \tag{11}$$

and

$$H = \frac{\sum_{i=1}^n \left(1 + \sum_{j\neq i}^n \left(f_j'\right)^2\right) f_i''}{n\left[1 + \sum_{i=1}^n \left(f_i'\right)^2\right]^{3/2}}, \tag{12}$$

where $f_i' = \frac{df_i}{dx_i}$ and $f_i'' = \frac{d^2 f_i}{dx_i^2}$, $i = 1, \ldots, n$. We have the following result:

Theorem 4.1. *Let M^n be a singular minimal translation hypersurface in $\mathbb{R}^{n+1}$ with respect to a horizantal vector $\mathbf{u}$. Then it is either a hyperplane parallel to $\mathbf{u}$ or a α-catenary cylinder whose the rulings are horizontal straight-lines orthogonal to $\mathbf{u}$.*

Proof. Without lose of generality we can take $\mathbf{u}$ as $(1, 0, \ldots, 0)$. Eqs. (5), (11) and (12) then follow

$$\sum_{i=1}^n \left(1 + \sum_{i\neq j}^n \left(f_j'\right)^2\right) f_i'' = \frac{-\alpha f_1'}{x_1}\left(1 + \sum_{i=1}^n \left(f_i'\right)^2\right). \tag{13}$$

A hyperplane parallel to $\mathbf{u}$ obviously satisfies Eq. (13). We have to take $\alpha f_1' \neq 0$ in Eq. (13) because M^n is minimal otherwise, which is not our case. Let us assume that $f_1'' = 0$, or equivalently $f_1' = const. \neq 0$, in Eq. (13). Then the partial derivative of Eq. (13) with respect to x_1 leads to $\alpha = 0$. This is the case we already ignore and hereinafter $f_1'' \neq 0$ is assumed. Next taking partial derivative of Eq. (13) with respect to x_k, $k \neq 1$, gives

$$2f_k' f_k'' \sum_{k\neq i}^n f_i'' + \left(1 + \sum_{k\neq i}^n \left(f_i'\right)^2\right) f_k''' = \frac{-2\alpha f_1'}{x_1} f_k' f_k''. \tag{14}$$

It can be seen that $f_k'' = 0$, $k = 2, 3, \ldots, n$, is a solution of Eq. (14). This means that $f_k(x_k) = \lambda_k x_k + \mu_k$, $\lambda_k, \mu_k \in \mathbb{R}$, and M^n is a cylinder that can be written as

$$\sigma(x_1, x_2, \ldots, x_n) = \left(x_1, 0, \ldots, 0, f_1(x_1) + \sum_{i=2}^{n} \mu_k\right) + x_2(0, 1, \ldots, 0, \lambda_2) + \cdots + x_n(0, 0, \ldots, 1, \lambda_n),$$

which refers to the statement in the hypothesis of the theorem due to Theorem 3.1. In order to complete the proof of the theorem, it is needed to show that Eq. (14) has no other solution than $f_k'' = 0$. Assuming now $f_k'' \neq 0$ and dividing Eq. (14) with $2f_k' f_k''$, we have

$$\sum_{k \neq i}^{n} f_i'' + \left(1 + \sum_{k \neq i}^{n} (f_i')^2\right) \frac{f_k'''}{2f_k' f_k''} = \frac{-\alpha f_1'}{x_1}. \tag{15}$$

Taking partial derivative of Eq. (15) with respect to x_k, $k \neq 1$, gives

$$\left(1 + \sum_{k \neq i}^{n} (f_i')^2\right) \left(\frac{f_k'''}{2f_k' f_k''}\right)' = 0,$$

or equivalently,

$$f_k''' = 2\nu_k f_k' f_k'' \tag{16}$$

for some constant ν_k. We distinguish two cases:

- $\nu_k = 0$, i.e., $f_k'' = \lambda_k$, for some nonzero constant λ_k, $k = 2, 3, \ldots, n$. After substituting this into Eq. (13), one can be rewritten as

$$G_1(x_1) + G_2(x_1)(f_2')^2 + G_3(x_1)(f_3')^2 + \cdots + G_n(x_1)(f_n')^2 = 0, \tag{17}$$

  where

$$\begin{aligned} G_1(x_1) &= f_1'' + \tfrac{\alpha f_1'}{x_1} + \tfrac{\alpha (f_1')^3}{x_1} + \left[(f_1')^2 + 1\right] \textstyle\sum_{i=2}^{n} \lambda_i, \\ G_2(x_2) &= f_1'' + \tfrac{\alpha f_1'}{x_1} + \textstyle\sum_{i=3}^{n} \lambda_i, \\ G_3(x_3) &= f_1'' + \tfrac{\alpha f_1'}{x_1} + \textstyle\sum_{3 \neq i=2}^{n} \lambda_i, \\ &\vdots \\ G_n(x_1) &= f_1'' + \tfrac{\alpha f_1'}{x_1} + \textstyle\sum_{i=2}^{n-1} \lambda_i. \end{aligned} \tag{18}$$

  Because $f_k'' \neq 0$, $k = 2, 3, \ldots, n$, taking partial derivative of Eq. (17) with respect to x_k gives that the functions $G_1, \ldots, G_n$ are all zero. If we subtract second equality in Eq. (18) from third one, then we find $\lambda_2 = \lambda_3$. Analogously if we subtract third equality in Eq. (18) from fourth one, then we find $\lambda_3 = \lambda_4$. Hence by maintaining same procedure for other equalities in Eq. (18) we obtain $\lambda_2 = \lambda_3 = \cdots = \lambda_n$.

Put $\tilde{\lambda} = \lambda_k$, $k = 2, 3, \ldots, n$. The following can be obtained by some equality in Eq. (18) (except the first one)

$$f_1'' + \frac{\alpha f_1'}{x_1} = \tilde{\lambda}(2-n). \tag{19}$$

Substituting Eq. (19) into the first equality in Eq. (18) leads to

$$\tilde{\lambda}(n-1)\left(f_1'\right)^2 + \frac{\alpha\left(f_1'\right)^3}{x_1} + \tilde{\lambda} = 0. \tag{20}$$

By taking derivative of Eq. (20) with respect to x_1 and then dividing x_1 we derive

$$f_1''\left[2\tilde{\lambda}(n-1)\frac{f_1'}{x_1} + 3\alpha\left(\frac{f_1'}{x_1}\right)^2\right] - \left(\frac{f_1'}{x_1}\right)^3 = 0. \tag{21}$$

From Eq. (19) we get $f_1'' = \tilde{\lambda}(2-n) - \frac{\alpha f_1'}{x_1}$ and considering this into Eq. (21) yields a polynomial equation of $\left(\frac{f_1'}{x_1}\right)$ in which the leading coefficient is $-3\alpha^2 - 1$. This leads to a contradiction.

- $\nu_k \neq 0$. Hence Eq. (15) reduces to

$$\sum_{k\neq i}^{n} f_i'' + \nu_k\left(1 + \sum_{k\neq i}^{n}\left(f_i'\right)^2\right) = \frac{-\alpha f_1'}{x_1}. \tag{22}$$

Taking partial derivative of Eq. (22) with respect to x_l, $1 \neq l \neq k$, yields

$$f_l''' + 2\nu_k f_l' f_l'' = 0. \tag{23}$$

Because Eq. (16) hold for $k = 2, 3, \ldots, n$, we have $f_l''' = 2\nu_l f_l' f_l''$. By substituting this into Eq. (23), we obtain

$$\nu_k + \nu_l = 0. \tag{24}$$

On the other hand, integrating Eq. (16) gives

$$f_k'' = \nu_k\left(f_k'\right)^2 + \mu_k. \tag{25}$$

Substituting Eqs. (24) and (25) into Eq. (22) leads to

$$f_1'' = -\left[\nu_k\left(f_1'\right)^2 + \frac{\alpha f_1'}{x_1} + \varepsilon\right], \tag{26}$$

where $\varepsilon = \nu_k + \sum_{k\neq i=2}^{n} \mu_i$. After substituting Eqs. (25) and (26) into Eq. (13) we can rearrange it as

$$\left(\sum_{i=2}^{n}\left[\left(\nu_i - \nu_k\right)\left(f_i'\right)^2 + \mu_i\right] - \nu_k\right)\left(f_1'\right)^2 \tag{27}$$

$$+\sum_{i=2}^{n}\left(1+\sum_{j\neq i=2}^{n}\left(f_j'\right)^2\right)\left(\nu_i\left(f_i'\right)^2+\mu_i\right)-\varepsilon\left(1+\sum_{i=2}^{n}\left(f_i'\right)^2\right)$$

$$=\frac{-\alpha\left(f_1'\right)^3}{x_1}.$$

The partial derivatives of Eq. (27) with respect to x_1 and x_l, $1\neq l\neq k$ lead to

$$\nu_k-\nu_l=0. \tag{28}$$

Comparing Eqs. (24) and (28) contradicts with $\nu_k\neq 0$. □

5. An application in 3-dimensional case

Let M^2 be a translation graph in $\mathbb{R}^3$ of the form

$$z=f\left(x\right)+g\left(y+cx\right),\ c\in\mathbb{R}-\{0\}, \tag{29}$$

for some smooth functions f and g. By the change of parameters $\tilde{x}=x$ and $\tilde{y}=y+cx$, we can choose a local parameterization on M^2 as

$$\sigma: I\times J\subset\mathbb{R}^2\to\mathbb{R}^3$$

and

$$\sigma\left(\tilde{x},\tilde{y}\right)=\left(\tilde{x},\tilde{y}-c\tilde{x},f\left(\tilde{x}\right)+g\left(\tilde{y}\right)\right). \tag{30}$$

Eq. (30) follows that one can be written as a sum of two planar curves, i.e.

$$M^2=\gamma\left(\tilde{x}\right)+\eta\left(\tilde{y}\right),$$

where

$$\gamma: I\subset\mathbb{R}\to\mathbb{R}^3,\ \gamma\left(\tilde{x}\right)=\left(\tilde{x},-c\tilde{x},f\left(\tilde{x}\right)\right)$$

and

$$\eta: J\subset\mathbb{R}\to\mathbb{R}^3,\ \eta\left(\tilde{y}\right)=\left(0,\tilde{y},g\left(\tilde{y}\right)\right).$$

Notice that γ and η lie in the non-orthogonal planes to each other. Hence M^2 turns to an extension of classical translation surface.

Let us put $f'=\frac{df}{d\tilde{x}}$ and $g'=\frac{dg}{d\tilde{y}}$, etc. Thereby we have the following result:

Theorem 5.1. *If a translation graph in $\mathbb{R}^3$ of the form* (29) *is a singular minimal surface with respect to the horizantal vector $\mathbf{u}=(1,0,0)$ then it is either a plane parallel to $\mathbf{u}$ or a α-catenary cylinder whose the rulings are horizontal straight-lines orthogonal to $\mathbf{u}$.*

Proof. If M^2 is a singular minimal surface, then Eq. (2) writes

$$\frac{\left[1+\left(g'\right)^2\right]f''+\left[1+c^2+\left(f'\right)^2\right]g''}{1+\left(f'+cg'\right)^2+\left(g'\right)^2}=-\alpha\frac{f'+cg'}{\tilde{x}}. \tag{31}$$

We distinguish several cases:

- $f''=0$. Hence Eq. (31) reduces to

$$\frac{\left[1+c^2+f_0^2\right]g''}{1+\left(f_0+cg'\right)^2+\left(g'\right)^2}=-\alpha\frac{f_0+cg'}{\tilde{x}}, \tag{32}$$

where $f'=f_0$ is some constant. Because $\alpha\neq 0$, the partial derivative of Eq. (32) with respect to $\tilde{x}$ gives

$$f_0+cg'=0. \tag{33}$$

Eq. (33) implies $g''=0$. Putting $g'=g_0$ for some nonzero constant g_0, we conclude

$$f_0+cg_0=0,$$

which yields that

$$z\left(x,y\right)=f_0x+g_0\left(y+cx\right)+\mu=g_0y+\mu$$

for some constant μ. This leads M^2 to a plane parallel to $\mathbf{u}$.

- $f''\neq 0$ and $g''=0$. Then we have $g\left(y+cx\right)=\lambda\left(cx+y\right)+\mu$ for some constants λ,μ. Therefore M^2 turns to a cylinder of the form

$$\begin{aligned}\sigma\left(x,y\right)&=\left(x,y,f\left(x\right)+\lambda\left(cx+y\right)+\mu\right)\\&=\left(x,0,f\left(x\right)+\lambda cx+\mu\right)+y\left(0,1,\lambda\right).\end{aligned}$$

Due to [20, Theorem 1], we obtain that M^2 is an α-catenary cylinder whose the rulings are parallel to the vector $(0,1,\lambda)$.

- $f''g''\neq 0$. By taking partial derivative of Eq. (31) with respect to $\tilde{y}$, we have

$$\begin{aligned}2g'g''f''+\left[1+c^2+\left(f'\right)^2\right]g'''=&\frac{-\alpha c}{\tilde{x}}\left[1+\left(f'+cg'\right)^2+\left(g'\right)^2\right]g''\\&-2\alpha\frac{f'+cg'}{\tilde{x}}\left[cf'+\left(c^2+1\right)g'\right]g''.\end{aligned} \tag{34}$$

Dividing Eq. (34) with g'' leads to

$$\begin{aligned}2g'f''+\left[1+c^2+\left(f'\right)^2\right]\frac{g'''}{g''}=&\frac{-\alpha c}{\tilde{x}}\left[1+\left(f'+cg'\right)^2+\left(g'\right)^2\right]\\&-2\alpha\frac{f'+cg'}{\tilde{x}}\left[cf'+\left(c^2+1\right)g'\right].\end{aligned} \tag{35}$$

Assume now that $\frac{g'''}{g''}=\lambda$ for some constant λ. Then Eq. (35) turns to

$$\begin{aligned}2\tilde{x}f''g'+\left[1+c^2+\left(f'\right)^2\right]\lambda\tilde{x}=&-\alpha c\left[1+\left(f'+cg'\right)^2+\left(g'\right)^2\right]\\&-2\alpha\left(f'+cg'\right)\left[cf'+\left(c^2+1\right)g'\right].\end{aligned} \tag{36}$$

If we take partial derivative of Eq. (36) with respect to $\tilde{y}$ and then divide it with $2g''$, we find

$$\tilde{x}f''+\alpha\left(3c^2+1\right)f'=-3\alpha c\left(c^2+1\right)g'. \tag{37}$$

The partial derivative of Eq. (37) with respect to $\tilde{y}$ yields a contradiction. Hence we conclude $\left(\frac{g'''}{g''}\right)' \neq 0$. Next taking partial derivative of Eq. (35) with respect to $\tilde{y}$ and dividing it with $2g''$

$$f'' + \frac{1}{2}\left[1 + c^2 + (f')^2\right]\left[\left(\frac{g'''}{g''}\right)' / g''\right] = \frac{-\alpha c}{\tilde{x}}\left[cf' + \left(c^2+1\right)g'\right] - \frac{\alpha\left(c^2+1\right)}{\tilde{x}}\left[f' + cg'\right]. \tag{38}$$

The partial derivative of Eq. (38) with respect to $\tilde{y}$ yields

$$\left[1 + c^2 + (f')^2\right]\left[\left(\frac{g'''}{g''}\right)' / g''\right]' = \frac{-4\alpha c\left(c^2+1\right)}{\tilde{x}} g'', \tag{39}$$

which implies that both hand-sides cannot vanish. Hence Eq. (39) leads to

$$1 + c^2 + (f')^2 = \frac{-4\alpha c\left(c^2+1\right)}{\lambda_1 \tilde{x}} \tag{40}$$

and

$$\left[\left(\frac{g'''}{g''}\right)' / g''\right]' = \lambda_1 g'' \tag{41}$$

for nonzero constant λ_1. Integrating of Eq. (41) gives

$$g'' = \frac{\lambda_1}{6}(g')^3 + \frac{\lambda_2}{2}(g')^2 + \lambda_3 g' + \lambda_4 \tag{42}$$

for some constants $\lambda_2, \lambda_3, \lambda_4$. Substituting Eq. (42) into Eq. (31) gives

$$\left[1 + (g')^2\right] f'' + \left[1 + c^2 + (f')^2\right]\left[\frac{\lambda_1}{6}(g')^3 + \frac{\lambda_2}{2}(g')^2 + \lambda_3 g' + \lambda_4\right] = -\alpha \frac{f' + cg'}{\tilde{x}}\left[1 + (f' + cg')^2 + (g')^2\right]. \tag{43}$$

Eq. (43) is a polynomial equation of g' in which the coefficient of the term of degree 1 satisfies

$$\lambda_3\left[1 + c^2 + (f')^2\right] + \frac{\alpha c}{\tilde{x}} + \frac{3\alpha c}{\tilde{x}}(f')^2 = 0. \tag{44}$$

Putting Eq. (40) into Eq. (44) yields

$$3(f')^2 - \frac{4\lambda_3\left(c^2+1\right)}{\lambda_1} + 1 = 0,$$

which implies $f'' = 0$. This is not our case. □

6. Conclusions

In Sections 4 and 5 the results on singular minimal translation (hyper) surfaces were obtained by taking $\mathbf{u}$ as a horizontal vector. Point out that the vector $\mathbf{u}$ is parallel to the hyperplane $x_{n+1} = 0$ where the hypersurface is a graph. The observation of these graphs is still open when $\mathbf{u}$ is a vertical vector, that is, $\mathbf{u}$ is normal to the hyperplane $x_{n+1} = 0$.

Moreover, the translation graph considering in Section 5 can be directly extended to higher dimensions as

(45)
$$x_{n+1}(x_1, \ldots, x_n) = f_1(x_1) + \cdots + f_{n-1}(x_{n-1}) + g\left(x_n + \sum_{i=1}^{n-1} c_i x_i\right), \ n \geq 2,$$

for smooth functions $f_1, \ldots, f_{n-1}, g$. This is explicitly a generalization in arbitrary dimensions of a classical translation hypersurface. Taking the vector $\mathbf{u}$ as a horizontal or a vertical vector, gives another interesting problem to find a singular minimal translation graph in $\mathbb{R}^{n+1}$ of the form Eq. (45).

As a final remark these problems also could be considered by taking the vector $\mathbf{u}$ as arbitrary.

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Muhittin Evren Aydin
Department of Mathematics
Firat University
Elazig 23200, Turkey
Email address: meaydin@firat.edu.tr

Ayla Erdur
Department of Mathematics
Tekirdag Namik Kemal University
Tekirdag 59100, Turkey
Email address: aerdur@nku.edu.tr

Mahmut Ergüt
Department of Mathematics
Tekirdag Namik Kemal University
Tekirdag 59100, Turkey
Email address: mergut@nku.edu.tr