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CONTINUOUS DATA ASSIMILATION FOR THE THREE-DIMENSIONAL SIMPLIFIED BARDINA MODEL UTILIZING MEASUREMENTS OF ONLY TWO COMPONENTS OF THE VELOCITY FIELD

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Abstract. We study a continuous data assimilation algorithm for the three-dimensional simplified Bardina model utilizing measurements of only two components of the velocity field. Under suitable conditions on the relaxation (nudging) parameter and the spatial mesh resolution, we obtain an asymptotic in time estimate of the difference between the approximating solution and the unknown reference solution corresponding to the measurements, in an appropriate norm, which shows exponential convergence up to zero.

1. Introduction

Data assimilation is a methodology to estimate weather or ocean variables combining information from observational data with a numerical dynamical (forecast) model. In recent years, data assimilation problems for many important equations in fluid mechanics have been extensively studied by Edriss Titi and his coauthors, see e.g. [2, 4, 12–14, 16, 17, 19, 20, 22, 24]. We also refer the interested reader to [1, 3, 6–8, 20, 23] for some recent results of other authors.

The simplified Bardina model was considered by Layton and Lewandowski [21] being a simpler approximation of the Reynold stress tensor proposed by Bardina et al. [5], which is called the Bardina model. It is noticed that the simplified Bardina model is consistent with other α-models in fluid mechanics, see e.g. [18] and references therein, in the sense that if $\alpha = 0$, we formally recover the classical three-dimensional Navier-Stokes equations. In recent years, there are many results on the existence, convergence and long-time behavior of solutions to the simplified Bardina model, see e.g. [9, 10, 21, 27].

The continuous data assimilation for 3D simplified Bardina model using observations on all three components of the velocity field was studied recently in

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[1]. In this paper, we will investigate a more practical case when the dimension of the observation vector is less than the dimension of the model's state vector. To do this, we will exploit the approach used recently in [13] for 2D Navier-Stokes equations and in [15] for 3D Leray-α model, which is based on ideas in [4] and exploits the divergence free condition for the velocity field. In what follows, we will explain the problem to be investigated.

Suppose that the evolution of u is governed by the three-dimensional simplified Bardina model, subject to periodic boundary conditions on $\Omega = [0, L]^3$:

$$\begin{cases} \dfrac{\partial v}{\partial t} - \nu \Delta v + (u \cdot \nabla) u + \nabla p = f, \\ \nabla \cdot u = \nabla \cdot v = 0, \end{cases} \tag{1}$$

on the interval $[t_0, \infty)$, where the initial data $u(t_0) = u_0$ is unknown. Here $u = u(x, y, z, t)$ represents the velocity of the fluid, called the *filtered velocity*, $v = u - \alpha^2 \Delta u$, and $\alpha > 0$ is a scale parameter with dimension of length. Above, p is a scalar, the pressure, and f is a body force which is assumed, for simplicity, to be time-independent.

Here, the reference solution is given by a solution u of (1) for which the initial data is missing, and in a more practical case than that in [1], the observational data needed to be measured and inserted into the model equation is reduced or subsampled. We required observational measurements of only two components of the 3D velocity vector field.

In this context, we consider the horizontal observational measurements, which are represented by mean of the interpolant operators $I_h(u_1(t))$ and $I_h(u_2(t))$ for $t \in [t_0, T]$, where $I_h(\varphi)$ is an interpolant operator based on the observational measurements of the scalar function φ at a coarse spatial resolution of size h.

We now follow the approach in [13, 15] to introduce the following continuous data assimilation algorithm for finding an approximate solution u^* of the unknown reference solution u. First, we rewrite the simplified Bardina model as follows:

$$\frac{\partial v_1}{\partial t} - \nu \Delta v_1 + u_1 \partial_x u_1 + u_2 \partial_y u_1 + u_3 \partial_z u_1 + \partial_x p = f_1, \tag{2a}$$

$$\frac{\partial v_2}{\partial t} - \nu \Delta v_2 + u_1 \partial_x u_2 + u_2 \partial_y u_2 + u_3 \partial_z u_2 + \partial_y p = f_2, \tag{2b}$$

$$\frac{\partial v_3}{\partial t} - \nu \Delta v_3 + u_1 \partial_x u_3 + u_2 \partial_y u_3 + u_3 \partial_z u_3 + \partial_z p = f_3, \tag{2c}$$

$$\partial_x u_1 + \partial_y u_2 + \partial_z u_3 = \partial_x v_1 + \partial_y v_2 + \partial_z v_3 = 0, \tag{2d}$$

$$v_1 = u_1 - \alpha^2 \Delta u_1, v_2 = u_2 - \alpha^2 \Delta u_2, v_3 = u_3 - \alpha^2 \Delta u_3. \tag{2e}$$

Given an arbitrary initial datum u_0^*, we look for a function u^* satisfying $u^*(t_0) = u_0^*$, the same boundary conditions for u, and the following system:

$$\frac{\partial v_1^*}{\partial t} - \nu \Delta v_1^* + u_1^* \partial_x u_1^* + u_2^* \partial_y u_1^* + u_3^* \partial_z u_1^* + \partial_x p^* \tag{3a}$$

$$= f_1 - \mu \left(I_h(u_1^*) - I_h(u_1) \right),$$

(3b) $$\frac{\partial v_2^*}{\partial t} - \nu \Delta v_2^* + u_1^* \partial_x u_2^* + u_2^* \partial_y u_2^* + u_3^* \partial_z u_2^* + \partial_y p^*$$
$$= f_2 - \mu \left(I_h(u_2^*) - I_h(u_2) \right),$$

(3c) $$\frac{\partial v_3^*}{\partial t} - \nu \Delta v_3^* + u_1^* \partial_x u_3^* + u_2^* \partial_y u_3^* + u_3^* \partial_z u_3^* + \partial_z p^* = f_3,$$

(3d) $$\partial_x u_1^* + \partial_y u_2^* + \partial_z u_3^* = \partial_x v_1^* + \partial_y v_2^* + \partial_z v_3^* = 0,$$

(3e) $$v_1^* = u_1^* - \alpha^2 \Delta u_1^*, v_2^* = u_2^* - \alpha^2 \Delta u_2^*, v_3^* = u_3^* - \alpha^2 \Delta u_3^*,$$

where ν and $f = (f_1, f_2, f_3)$ are the same kinematic viscosity parameter and forcing term from (2), p^* is a modified pressure, and $\mu > 0$ is a relaxation (nudging) parameter. The purpose of μ is to force the coarse spatial scales of u_i^* toward those of the observed data $I_h(u_i)$, $i = 1, 2$. Here, observational data of the horizontal components $I_h(u_1)$ and $I_h(u_2)$ were chosen as an example but any data assimilation algorithm using two out of three components of the velocity field also works. As mentioned in [13,15], one of the advantages of this algorithm is that the initial data u_0^* of the approximate solution can be chosen to be arbitrary.

In this paper, we will consider the following types of interpolant operators. The first interpolant observables given by linear interpolant operators $I_h : H^1(\Omega) \to L^2(\Omega)$, that approximate identity and satisfy the approximation property

(4) $$\|\varphi - I_h(\varphi)\|_{L^2(\Omega)} \le \gamma_0 h \|\varphi\|_{H^1(\Omega)}$$

for every φ in the Sobolev space $H^1(\Omega)$. We also consider a second type of linear interpolant operators $I_h : H^2(\Omega) \to L^2(\Omega)$ that satisfy the approximation property

(5) $$\|\varphi - I_h(\varphi)\|_{L^2(\Omega)} \le \gamma_1 h \|\varphi\|_{H^1(\Omega)} + \gamma_2 h^2 \|\varphi\|_{H^2(\Omega)}$$

for every φ in the Sobolev space $H^2(\Omega)$. We will call the interpolants that satisfy (4) and (5) are of type I and type II, respectively. We present here three examples of such interpolant operators.

The first example is given by the projector onto low Fourier modes. Considering $\phi_k(x) = L^{-3} e^{\frac{2\pi i}{3} k \cdot x}$ with $|k| \le \left\lfloor \frac{1}{2\pi h} \right\rfloor$. Then we define $I_h : H^1(\Omega) \to L^2(\Omega)$ as

$$I_h(\varphi) = P_k \varphi = \sum_{|k| \le \left\lfloor \frac{L}{2\pi h} \right\rfloor} \widehat{\varphi}_k \phi_k(x),$$

where $\varphi(x) = \sum_{k \in \mathbb{Z}^3} \widehat{\varphi}_k \phi_k(x)$. It satisfies inequality (4) (see e.g. [2,15]), i.e., I_h is an interpolant operator of type I.

The second example of interpolant operators of type I is the volume element operator. Dividing $\Omega = [0, L]^3$ into N cubes Ω_k, of same edge, then I_h :

$H^1(\Omega) \to L^2(\Omega)$ given by

$$I_h(\varphi) = \sum_{k=1}^{N} \frac{\chi_{\Omega_k}(x)N}{L^3} \int_{\Omega_k} \varphi(y)dy,$$

with $h = LN^{-1/3}$, i.e., the edge of each Ω_k, and χ_{Ω_k} is the characteristic function of the subdomain Ω_k. One can check that (see e.g. [2, 15]) this interpolant operator satisfies (4).

Finally, we give an example of interpolant operators of type II, which is obtained by observational measurements of velocity on discrete points x_k of cube Ω, that can be divided in N cubes Ω_k, as previous example, with $x_k \in \Omega_k$. Then $I_h : H^2(\Omega) \to L^2(\Omega)$ defined as

$$I_h(\varphi) = \sum_{k=1}^{N} \varphi(x_k)\chi_{\Omega_k}(x)$$

satisfies (5) (see a proof in [2, 4, 15]).

It is noticed that if $I_h : H^2(\Omega) \to L^2(\Omega)$ is an interpolant operator of type I, then it is also an interpolant operator of type II. Although the interpolant operators of type II contain the nodal values, which are very useful in practice because we only need to know measurements of velocity at a discrete set of nodal points in Ω, but it requires that the data need to be smoother (belongs to $H^2(\Omega)$ instead of $H^1(\Omega)$ as in the case of interpolant operators of type I). This leads to the fact that we need to consider the strong solutions for these interpolant operators.

We provide explicit estimates on the spatial resolution h of the observational measurements and the relaxation (nudging) parameter μ, in terms of physical parameters, that are needed in order for the proposed downscaling algorithm to recover the reference resolution. We will show that, under suitable conditions of μ and h, for any initial data u_0^*, the data assimilation equation (3) has a unique solution u^* defined on the whole interval $[t_0, \infty)$, and this approximate solution will converge to the reference solution u of the 3D simplified Bardina model as time goes to ∞. The results obtained in this paper can be seen as an improvement of previous result for the Bardina model in [1] in the sense that we use observations in only any two components and without any measurements on the third component of the velocity field. They are also counterparts of corresponding results for 2D Navier-Stokes equations [13] and the Leray-α model [15].

In what follows, we will describe in more details the procedure of implementing these mathematical results to solve real-world problems, like weather forecasting.

Suppose that $u(t)$ represents a solution of system (3), where the initial data $u(t_0) = u_0$ is missing. The goal of continuous data assimilation is to use low spatial resolution observational measurements, obtained continuously in time, to accurately find the unknown reference solution $u(t)$ from which future predictions can be made.

Step 1. Constructing the data assimilation equation.

Let $I_h(u(t))$ represent an interpolant operator based on the observational measurements of this system at a coarse spatial resolution of size h, for $t \in [0,T]$. The algorithm proposed is to construct an approximate solution $u^*(t)$ that satisfies the data assimilation equation (3).

Here, we only use the observational data of two out of three components of the velocity field, namely $I_h(u_1(t))$ and $I_h(u_2(t))$. The algorithm can be implemented with a variety of finitely many observables: low Fourier modes, nodal values, finite volume averages, or finite elements.

Step 2. Proving the global existence and uniqueness of solutions to the data assimilation equation.

The corresponding solution $u^*(t)$ of the data assimilation equation (3) is called the approximate solution. As mentioned in [13,15], one of the advantages of this algorithm is that the initial data u_0^* of the approximate solution $u^*(t)$ can be chosen to be arbitrary.

Step 3. Proving the convergence of approximate solutions to the reference solution.

The goal of this step is to find estimates on relaxation parameter $\mu > 0$ and the spatial resolution $h > 0$, in terms of physical parameters of the evolution equation (2), such that the approximate solution $u^*(t)$ approaches the reference solution $u(t)$, with increasing accuracy, as more continuous data in time is supplied. After some large enough time $T > 0$, the solution $u^*(T)$ can then be used as an initial condition in system (2) to make future predictions of the reference solution $u(t)$ for $t > T$, or one can continue with data assimilation equation (3) itself, for as long more measurements are provided.

Since our goal here is to analyze the long-time behavior of solutions, in all the statements below we make the assumption that the reference solution u is a trajectory in the global attractor $\mathcal{A}$ of the 3D simplified Bardina model, which is recalled in Section 2 below. We remark, however, that the same results still hold by assuming that u is a solution of the 3D simplified Bardina model starting at a point $u(t_0) = u_0$ with t_0 large enough so that the uniform bounds (22) and (23) in Section 2 below are also valid for u, up to a multiplicative absolute constant. It is also noticed that all results of the paper are still valid if we assume the external force $f \in L^\infty(t_0,\infty;H)$, where H is the function space defined in Section 2 below.

For external force $f \in H$, we define the Grashof number in three dimensions as follows

$$Gr = \frac{|f|}{\nu^2 \lambda_1^{3/4}}. \tag{6}$$

We are now ready to formulate the main results in this paper.

The two first theorems are about the existence of *weak solutions* to the data assimilation system and its convergence to the reference solution of the simplified Bardina model, with observable data of type I and of type II, respectively.

Theorem 1.1 (Observable data of type I)**.** *Suppose I_h satisfies* (4)*. Let u be a solution in the global attractor of the* 3*D simplified Bardina model* (2) *and choose $\mu > 0$ large enough such that*

$$\mu \geq \frac{c\nu Gr^2}{\lambda_1 \alpha^2}\left(\frac{\lambda_1}{2} + \frac{\nu\lambda_1}{\alpha^2} + \frac{1}{\nu}\right)\exp\left(\frac{54c_5^4\nu Gr^4}{\alpha^4\lambda_1}\right), \tag{7}$$

and $h > 0$ small enough such that $\mu c_0^2 h^2 \leq \nu$.

If $u_0^* \in V$ and $f \in H$, then there exists a unique weak solution u^* of data assimilation equation (3) *on $[t_0, \infty)$ satisfying $u^*(t_0) = u_0^*$ and*

$$u^* \in C([t_0,\infty); V) \cap L^2_{loc}([t_0,\infty); D(A)), \frac{du^*}{dt} \in L^2_{loc}([t_0,\infty); H).$$

Moreover, the solution u^* depends continuously on the initial data u_0^* and it satisfies

$$|u^*(t) - u(t)|^2 + \alpha^2\|u^*(t) - u(t)\|^2 \to 0,$$

at exponential rate, as $t \to \infty$.

Here, the constants c_0 and c_5 appear in (11) *and* (21) *below, and c is a suitable positive constant independent of parameters of the system.*

Theorem 1.2 (Observable data of type II)**.** *Suppose I_h satisfies* (5)*. Let u be a solution in the global attractor of the* 3*D simplified Bardina model* (2) *and choose $\mu > 0$ large enough such that* (7) *holds and $h > 0$ small enough such that $\mu c_0^2 h^2 \leq 2\nu$ and $\mu c_0^4 h^4 \leq 4\nu\alpha^2$.*

If $u_0^* \in V$ and $f \in H$, then there exists a unique weak solution u^* of equation (3) *on $[t_0, \infty)$ satisfying $u^*(t_0) = u_0^*$ and*

$$u^* \in C([t_0,\infty); V) \cap L^2_{loc}([t_0,\infty); D(A)), \frac{du^*}{dt} \in L^2_{loc}([t_0,\infty); H).$$

Moreover, the solution u^* depends continuously on the initial data u_0^* and it satisfies

$$|u^*(t) - u(t)|^2 + \alpha^2\|u^*(t) - u(t)\|^2 \to 0,$$

at exponential rate, as $t \to \infty$.

Here, the constants c_0 and c_5 appear in (11) *and* (21) *below, and c is a suitable positive constant independent of parameters of the system.*

The following theorem shows the existence and uniqueness of *strong solutions* to the data assimilation system (3) with observable data of type II, and its stronger convergence (in $D(A)$) to the reference solution of the simplified Bardina model.

Theorem 1.3. *Suppose I_h satisfies* (5)*. Let u be a solution in the global attractor of the* 3*D simplified Bardina model* (2) *and choose $\mu > 0$ large enough such that*

$$\mu \geq \max\left\{\frac{c\nu Gr^2}{\lambda_1\alpha^2}\left(\frac{\lambda_1}{2}+\frac{\nu\lambda_1}{\alpha^2}+\frac{1}{\nu}\right)\exp\left(\frac{54c_5^4\nu Gr^4}{\alpha^4\lambda_1}\right)+\frac{c\nu Gr^6}{\lambda_1^3\alpha^8}, \nu\lambda_1\right\}, \tag{8}$$

and $h > 0$ small enough such that $\mu c_0^2 h^2 \leq \nu$ and $\mu c_0^4 h^4 \leq 4\nu\alpha^2$.

If $u_0^* \in D(A)$ with

$$|u_0^*|^2 + \alpha^2\|u_0^*\|^2 \leq \frac{2\nu^2 Gr^2}{\lambda_1^{1/2}}, \tag{9}$$

and

$$\|u_0^*\|^2 + \alpha^2|Au_0^*|^2 \leq \frac{2\nu^2 Gr^2}{\lambda_1^{1/2}}\left(\frac{\lambda_1}{2}+\frac{\nu\lambda_1}{\alpha^2}+\frac{1}{\nu}\right)\exp\left(\frac{54c_5^4\nu Gr^4}{\alpha^4\lambda_1}\right), \tag{10}$$

and $f \in H$, then there exists a unique strong solution u^* of data assimilation equation (3) *on $[t_0,\infty)$ satisfying $u^*(t_0) = u_0^*$ and*

$$u^* \in C([t_0,\infty); D(A)) \cap L^2_{loc}([t_0,\infty); D(A^{3/2})), \frac{du^*}{dt} \in L^2_{loc}([t_0,\infty); V),$$

such that

$$\|u^*(t)\|^2 + \alpha^2|Au^*(t)|^2 \leq \frac{22\nu^2 Gr^2}{\lambda_1^{1/2}}\left(\frac{\lambda_1}{2}+\frac{\nu\lambda_1}{\alpha^2}+\frac{1}{\nu}\right)\exp\left(\frac{54c_5^4\nu Gr^4}{\alpha^4\lambda_1}\right) + \frac{384000(16e+2)c_3^4c_4^4}{\nu^4\lambda_1\alpha^6}\left(\frac{2\nu^2Gr^2}{\lambda_1^{1/2}}\right)^3$$

for all $t > t_0$. Moreover, the solution u^* depends continuously on the initial data u_0^* and it satisfies

$$\|u^*(t) - u(t)\|^2 + \alpha^2|Au^*(t) - Au(t)|^2 \to 0,$$

at exponential rate, as $t \to \infty$.

Here, the constants c_0, c_3, c_4 and c_5 appear in (11)*,* (18)*,* (19) *and* (21) *below, and c is the suitable positive constant independent of parameters of the system.*

The rest of the paper is organized as follows. In Section 2, for convenience of the reader, we recall the functional setting and some results on the 3D simplified Bardina model which will be used in the proof of main results. Section 3 is devoted to proving Theorem 1.1, Theorem 1.2 and Theorem 1.3.

2. Preliminaries

Let $\Omega = [0, L]^3$ be a periodic box, for some $L > 0$ fixed. We denote by $\mathcal{V}$ the set of all vector valued trigonometric polynomials defined in Ω, which are divergence-free and have average zero. We denote by $L^2(\Omega)$, $W^{s,p}(\Omega)$, and $H^s(\Omega) \equiv W^{s,2}(\Omega)$ the usual Sobolev spaces in three-dimensions. Denote also by H and V the closures of $\mathcal{V}$ in the $(L^2(\Omega))^3$ and $(H^1(\Omega))^3$, respectively. Then H and V are Hilbert spaces with inner products given by

$$(u,v) = \sum_{i=1}^{3} \int_\Omega u_i v_i dxdydz \text{ and } ((u,v)) = \sum_{i,j=1}^{3} \int_\Omega \partial_j u_i \partial_j v_i dxdydz,$$

respectively, and the associated norms

$$|u| = (u,u)^{1/2} \text{ and } \|u\| = ((u,u))^{1/2}.$$

We denote $\mathcal{P} : (\dot{L}^2(\Omega))^3 \to H$ the Leray projector, where $(\dot{L}^2(\Omega))^3$ is the set of all functions belonging to $(L^2(\Omega))^3$ with zero average and by $A = -\mathcal{P}\Delta$ the Stokes operator, with domain $D(A) = (H^2(\Omega))^3 \cap V$. In the case of periodic boundary conditions, $A = -\Delta|_{D(A)}$. The Stokes operator A is a positive self-adjoint operator with compact inverse. Hence there exists a complete orthonormal set of eigenfunctions $\{w_j\}_{j=1}^\infty \subset H$, such that $Aw_j = \lambda_j w_j$ and

$$0 < \lambda_1 \le \lambda_2 \le \cdots, \quad \lambda_j \to \infty \text{ as } j \to \infty.$$

Furthermore, inequality (4) implies that

$$|u - I_h(u)|^2 \le c_0^2 h^2 \|u\|^2 \tag{11}$$

for every $u \in V$, where $c_0 = \gamma_0$, and inequality (5) implies that

$$|u - I_h(u)|^2 \le \frac{1}{2} c_0^2 h^2 \|u\|^2 + \frac{1}{4} c_0^4 h^4 |Au|^2 \tag{12}$$

for every $u \in D(A)$, for some c_0 that depends only on γ_0, γ_1 and γ_2.

We have the following versions of the Poincaré inequality:

$$\|u\|_{V'}^2 \le \lambda_1^{-1} |u|^2, \ \forall u \in H, \tag{13}$$

$$|u|^2 \le \lambda_1^{-1} \|u\|^2, \ \forall u \in V, \tag{14}$$

$$\|u\|^2 \le \lambda_1^{-1} |Au|^2, \ \forall u \in D(A). \tag{15}$$

In three-dimensions, we also have the Agmon inequality:

$$\|u\|_{L^\infty(\Omega)} \le c_1 \|u\|^{1/2} |Au|^{1/2}, \ \forall u \in D(A), \tag{16}$$

and Ladyzhenskaya inequalities:

$$\|u\|_{L^4(\Omega)} \le c_2 |u|^{1/4} \|u\|^{3/4}, \ \forall u \in V, \tag{17}$$

$$\|u\|_{L^3(\Omega)} \le c_3 |u|^{1/2} \|u\|^{1/2}, \ \forall u \in V, \tag{18}$$

and Sobolev inequality:

$$\|u\|_{L^6(\Omega)} \le c_4 \|u\|, \ \forall u \in D(A). \tag{19}$$

Following the classical notation for the viscous simplified Bardina model, for every $u, v \in \mathcal{V}$, we write $B(u,v) = \mathcal{P}[(u \cdot \nabla)v]$. The bilinear operator B can be extended continuously from $V \times V$ with values in V'.

Let us now recall some algebraic properties of the nonlinear term $B(u,v)$ that play an important role in our analysis. These results may be found in any of the references [11, 25, 26]. For $u, v, w \in V$ we have that

$$\langle B(u,v), w\rangle_{V',V} = -\langle B(u,w), v\rangle_{V',V},$$

and consequently

$$\langle B(u,v), v\rangle_{V',V} = 0. \tag{20}$$

Furthermore,

$$|\langle B(u,v), w\rangle_{V',V}| \leq c_5 \|u\|^{1/2}|Au|^{1/2}|v|\|w\|, \ \forall u \in D(A), v \in H, w \in V. \tag{21}$$

We now prove some asymptotic estimates for solutions to the simplified Bardina model (2).

Theorem 2.1. *Let $f \in H$. If $u_0 \in V$, then the system (2) has a unique weak solution u that satisfies $u(t_0) = u_0$ and*

$$u \in C([t_0,\infty); V) \cap L^2_{loc}([t_0,\infty); D(A)), \frac{du}{dt} \in L^2_{loc}([t_0,\infty); H).$$

Moreover, if $u_0 \in D(A)$, then the system (2) has a unique strong solution u that satisfies $u(t_0) = u_0$ and

$$u \in C([t_0,\infty); D(A)) \cap L^2_{loc}([t_0,\infty); D(A^{3/2})), \frac{du}{dt} \in L^2_{loc}([t_0,\infty); V).$$

Furthermore, the semigroup $S(t) : V \to V$ associated to (2), has a global attractor $\mathcal{A}$ in V. Additionally, for any $u \in \mathcal{A}$, we have

$$|u|^2 + \alpha^2\|u\|^2 \leq \frac{2\nu^2 Gr^2}{\lambda_1^{1/2}}, \tag{22}$$

and

$$\|u\|^2 + \alpha^2|Au|^2 \leq \frac{2\nu^2 Gr^2}{\lambda_1^{1/2}}\left(\frac{\lambda_1}{2} + \frac{\nu\lambda_1}{\alpha^2} + \frac{1}{\nu}\right)\exp\left(\frac{54c_5^4\nu Gr^4}{\alpha^4\lambda_1}\right), \tag{23}$$

where $Gr = \nu^{-2}\lambda_1^{-3/4}|f|$ is the Grashof number.

Proof. The existence and uniqueness of weak/strong solutions to (2) and the existence of the global attractor $\mathcal{A}$ were already known (see, e.g. [1,9,21]). Here we only prove estimates (22) and (23).

Multiplying (2) by u, then integrating over Ω and using (20), we get

$$\frac{1}{2}\frac{d}{dt}(|u|^2 + \alpha^2\|u\|^2) + \nu(\|u\|^2 + \alpha^2|Au|^2) \leq \langle f, u\rangle_{V',V}.$$

By using the Cauchy inequality and the Poincaré inequality (13), we obtain

$$\langle f, u\rangle_{V',V} \leq \|f\|_{V'}\|u\| \leq \frac{|f|^2}{2\nu\lambda_1} + \frac{\nu}{2}\|u\|^2.$$

Thus,

$$\frac{d}{dt}(|u|^2+\alpha^2\|u\|^2)+\nu(\|u\|^2+\alpha^2|Au|^2)\le\frac{|f|^2}{\nu\lambda_1}. \tag{24}$$

Using (14) and (15), we deduce from the above inequality that

$$\frac{d}{dt}(|u|^2+\alpha^2\|u\|^2)+\nu\lambda_1(|u|^2+\alpha^2\|u\|^2)\le\frac{|f|^2}{\nu\lambda_1}. \tag{25}$$

Applying the Gronwall inequality to (25) we conclude that for all $t\ge t_0$:

$$\begin{aligned}|u(t)|^2+\alpha^2\|u(t)\|^2 &\le (|u(t_0)|^2+\alpha^2\|u(t_0)\|^2)e^{-\nu\lambda_1(t-t_0)}\\ &\quad+\frac{|f|^2}{\nu^2\lambda_1^2}\left(1-e^{-\nu\lambda_1(t-t_0)}\right).\end{aligned}$$

Because of this and the definition of Gr, there exists a time $T_1>t_0$, which depends on the norm of u_0, such that for all $t\ge T_1$, we have the estimate (22).

Integrating (24) from t to $t+1$ and using (22) we get for all $t\ge T_1$:

$$\int_t^{t+1}(\|u(s)\|^2+\alpha^2|Au(s)|^2)ds\le\frac{|f|^2}{\nu^2\lambda_1}+\frac{M_0}{\nu}, \tag{26}$$

where $M_0:=\frac{2\nu^2Gr^2}{\lambda_1^{1/2}}$. Multiplying (2) by Au and integrating over Ω, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|u\|^2+\alpha^2|Au|^2)+\nu(|Au|^2+\alpha^2\|Au\|^2)\\ &\le (B(u,u),Au)+\langle f,Au\rangle_{V',V}.\end{aligned} \tag{27}$$

By the Cauchy inequality and the Poincaré inequality (13), we have

$$\langle f,Au\rangle_{V',V}\le\|f\|_{V'}\|Au\|\le\frac{1}{\nu\alpha^2\lambda_1}|f|^2+\frac{\nu\alpha^2}{4}\|Au\|^2. \tag{28}$$

Using (21) and the Young inequality, we arrive at

$$|(B(u,u),Au)|\le c_5\|u\|^{3/2}|Au|^{3/2}\le\frac{27c_5^4}{4\nu^3}\|u\|^6+\frac{\nu}{4}|Au|^2. \tag{29}$$

Substituting (28) and (29) into (27), we deduce that

$$\frac{d}{dt}(\|u\|^2+\alpha^2|Au|^2)+\nu(|Au|^2+\alpha^2\|Au\|^2)\le\frac{27c_5^4}{2\nu^3}\|u\|^6+\frac{2}{\nu\alpha^2\lambda_1}|f|^2.$$

Using (22), we have for all $t\ge T_1$,

$$\frac{d}{dt}(\|u\|^2+\alpha^2|Au|^2)\le\frac{27c_5^4M_0^2}{2\nu^3\alpha^4}(\|u\|^2+\alpha^2|Au|^2)+\frac{2}{\nu\alpha^2\lambda_1}|f|^2. \tag{30}$$

Applying the uniform Gronwall lemma, we obtain from (26) and (30) that

$$\|u(t)\|^2+\alpha^2|Au(t)|^2\le\left(\frac{|f|^2}{\nu^2\lambda_1}+\frac{M_0}{\nu}+\frac{2}{\nu\alpha^2\lambda_1}|f|^2\right)\exp\left(\frac{27c_5^4M_0^2}{2\nu^3\alpha^4}\right)$$

for all $t\ge T_1+1$. Hence, we get (23).

We have proved asymptotic estimates (22) and (23) (i.e., for t large enough) for any solution u of (2). Due to the invariance of the global attractor $\mathcal{A}$, these estimates are valid for all $u \in \mathcal{A}$. □

3. Proof of main results

3.1. Proof of Theorem 1.1

Denote $\tilde{u} = u^* - u$, $\tilde{v} = v^* - v$ with $v^* = u^* + \alpha^2 A u^*, v = u + \alpha^2 A u$, thus $\tilde{v} = \tilde{u} + \alpha^2 A\tilde{u}$. Subtracting (3) from (2) to obtain

$$\begin{aligned}
&\text{(31a)} && \frac{\partial \tilde{v}_1}{\partial t} - \nu\Delta\tilde{v}_1 + (u^*\cdot\nabla)\tilde{u}_1 \\
& && + \tilde{u}_1\partial_x u_1 + \tilde{u}_2\partial_y u_1 + \tilde{u}_3\partial_z u_1 + \partial_x(p^* - p) = -\mu I_h(\tilde{u}_1),\\
&\text{(31b)} && \frac{\partial \tilde{v}_2}{\partial t} - \nu\Delta\tilde{v}_2 + (u^*\cdot\nabla)\tilde{u}_2 \\
& && + \tilde{u}_1\partial_x u_2 + \tilde{u}_2\partial_y u_2 + \tilde{u}_3\partial_z u_2 + \partial_y(p^* - p) = -\mu I_h(\tilde{u}_2),\\
&\text{(31c)} && \frac{\partial \tilde{v}_3}{\partial t} - \nu\Delta\tilde{v}_3 + (u^*\cdot\nabla)\tilde{u}_3 \\
& && + \tilde{u}_1\partial_x u_3 + \tilde{u}_2\partial_y u_3 + \tilde{u}_3\partial_z u_3 + \partial_z(p^* - p) = 0,\\
&\text{(31d)} && \partial_x\tilde{u}_1 + \partial_y\tilde{u}_2 + \partial_z\tilde{u}_3 = \partial_x\tilde{v}_1 + \partial_y\tilde{v}_2 + \partial_z\tilde{v}_3 = 0.
\end{aligned}$$

Here we have used the facts that

$$\begin{aligned}
& u_1^*\partial_x u_1^* + u_2^*\partial_y u_1^* + u_3^*\partial_z u_1^* - u_1\partial_x u_1 - u_2\partial_y u_1 - u_3\partial_z u_1 \\
=\ & (u^*\cdot\nabla)\tilde{u}_1 + \tilde{u}_1\partial_x u_1 + \tilde{u}_2\partial_y u_1 + \tilde{u}_3\partial_z u_1,\\
& u_1^*\partial_x u_2^* + u_2^*\partial_y u_2^* + u_3^*\partial_z u_2^* - u_1\partial_x u_2 - u_2\partial_y u_2 - u_3\partial_z u_2 \\
=\ & (u^*\cdot\nabla)\tilde{u}_2 + \tilde{u}_1\partial_x u_2 + \tilde{u}_2\partial_y u_2 + \tilde{u}_3\partial_z u_2,\\
& u_1^*\partial_x u_3^* + u_2^*\partial_y u_3^* + u_3^*\partial_z u_3^* - u_1\partial_x u_3 - u_2\partial_y u_3 - u_3\partial_z u_3 \\
=\ & (u^*\cdot\nabla)\tilde{u}_3 + \tilde{u}_1\partial_x u_3 + \tilde{u}_2\partial_y u_3 + \tilde{u}_3\partial_z u_3.
\end{aligned}$$

Because of the global existence and uniqueness of the weak solution u of the system (2), the existence and uniqueness of the difference $\tilde{u}$ will imply the global existence and uniqueness of the weak solution u^* of the system (3). In the proof below, we will derive formal *a priori* bounds on $\tilde{u}$ which form the key elements for showing the global existence of the weak solution $\tilde{u}$ of the system (31), under appropriate conditions on μ and h. The convergence of the solution $\tilde{u}$ to 0 will also be established. Uniqueness can then be obtained using similar energy estimates.

Here we will omit the rigorous details and provide only the formal *a priori* estimates. We can justify the estimates by the Galerkin approximation procedure and then passing to the limit while using the relevant compactness theorems (see, e.g., [11, 25, 26]).

Multiplying (31a), (31b) and (31c) by $\tilde{u}_1$, $\tilde{u}_2$ and $\tilde{u}_3$ respectively, then integrating over Ω we obtain

$$\frac{1}{2}\frac{d}{dt}\left(|\tilde{u}_1|^2+\alpha^2\|\tilde{u}_1\|^2\right)+\nu\left(\|\tilde{u}_1\|^2+\alpha^2|A\tilde{u}_1|^2\right)$$
$$\le |J_1| - (\partial_x(p^*-p),\tilde{u}_1) - \mu(I_h(\tilde{u}_1),\tilde{u}_1), \tag{32a}$$
$$\frac{1}{2}\frac{d}{dt}\left(|\tilde{u}_2|^2+\alpha^2\|\tilde{u}_2\|^2\right)+\nu\left(\|\tilde{u}_2\|^2+\alpha^2|A\tilde{u}_2|^2\right)$$
$$\le |J_2| - (\partial_y(p^*-p),\tilde{u}_2) - \mu(I_h(\tilde{u}_2),\tilde{u}_2), \tag{32b}$$
$$\frac{1}{2}\frac{d}{dt}\left(|\tilde{u}_3|^2+\alpha^2\|\tilde{u}_3\|^2\right)+\nu\left(\|\tilde{u}_3\|^2+\alpha^2|A\tilde{u}_3|^2\right)$$
$$\le |J_3| - (\partial_z(p^*-p),\tilde{u}_3), \tag{32c}$$

where

$$\begin{aligned}
J_1 &:= J_{1a}+J_{1b}+J_{1c} := (\tilde{u}_1\partial_x u_1,\tilde{u}_1)+(\tilde{u}_2\partial_y u_1,\tilde{u}_1)+(\tilde{u}_3\partial_z u_1,\tilde{u}_1),\\
J_2 &:= J_{2a}+J_{2b}+J_{2c} := (\tilde{u}_1\partial_x u_2,\tilde{u}_2)+(\tilde{u}_2\partial_y u_2,\tilde{u}_2)+(\tilde{u}_3\partial_z u_2,\tilde{u}_2),\\
J_3 &:= J_{3a}+J_{3b}+J_{3c} := (\tilde{u}_1\partial_x u_3,\tilde{u}_3)+(\tilde{u}_2\partial_y u_3,\tilde{u}_3)+(\tilde{u}_3\partial_z u_3,\tilde{u}_3).
\end{aligned}$$

Using the Hölder inequality, the Agmon inequality (16) and the Poincaré inequality (15), we have

$$\begin{aligned}
|J_{1a}| &= |(\tilde{u}_1\partial_x u_1,\tilde{u}_1)|\\
&\le \|\tilde{u}_1\|_{L^\infty(\Omega)}|\partial_x u_1||\tilde{u}_1|\\
&\le c_1\|\tilde{u}_1\|^{1/2}|A\tilde{u}_1|^{1/2}|\partial_x u_1||\tilde{u}_1|\\
&\le c_1\lambda_1^{-1/4}|A\tilde{u}_1||\partial_x u_1||\tilde{u}_1|\\
&\le \frac{\nu}{8}\alpha^2|A\tilde{u}_1|^2+\frac{c}{\nu\lambda_1^{1/2}\alpha^2}|\partial_x u_1|^2|\tilde{u}_1|^2\\
&\le \frac{\nu}{8}\left(\|\tilde{u}_1\|^2+\alpha^2|A\tilde{u}_1|^2\right)+\frac{c}{\nu\lambda_1^{1/2}\alpha^2}|\partial_x u_1|^2|\tilde{u}_1|^2.
\end{aligned}\tag{33}$$

Using similar analysis as above, we obtain the following estimates

$$\begin{aligned}
|J_{1b}| &= |(\tilde{u}_2\partial_y u_1,\tilde{u}_1)|\\
&\le \frac{\nu}{8}\left(\|\tilde{u}_2\|^2+\alpha^2|A\tilde{u}_2|^2\right)+\frac{c}{\nu\lambda_1^{1/2}\alpha^2}|\partial_y u_1|^2|\tilde{u}_1|^2,
\end{aligned}\tag{34}$$

$$\begin{aligned}
|J_{1c}| &= |(\tilde{u}_3\partial_z u_1,\tilde{u}_1)|\\
&\le \frac{\nu}{20}\left(\|\tilde{u}_3\|^2+\alpha^2|A\tilde{u}_3|^2\right)+\frac{c}{\nu\lambda_1^{1/2}\alpha^2}|\partial_z u_1|^2|\tilde{u}_1|^2,
\end{aligned}\tag{35}$$

$$\begin{aligned}
|J_{2a}| &= |(\tilde{u}_1\partial_x u_2,\tilde{u}_2)|\\
&\le \frac{\nu}{8}\left(\|\tilde{u}_1\|^2+\alpha^2|A\tilde{u}_1|^2\right)+\frac{c}{\nu\lambda_1^{1/2}\alpha^2}|\partial_x u_2|^2|\tilde{u}_2|^2,
\end{aligned}\tag{36}$$

$$
\begin{aligned}
|J_{2b}| &= |(\tilde{u}_2 \partial_y u_2, \tilde{u}_2)| \\
&\leq \frac{\nu}{8}\left(\|\tilde{u}_2\|^2 + \alpha^2 |A\tilde{u}_2|^2\right) + \frac{c}{\nu \lambda_1^{1/2} \alpha^2} |\partial_y u_2|^2 |\tilde{u}_2|^2,
\end{aligned} \tag{37}
$$

$$
\begin{aligned}
|J_{2c}| &= |(\tilde{u}_3 \partial_z u_2, \tilde{u}_2)| \\
&\leq \frac{\nu}{20}\left(\|\tilde{u}_3\|^2 + \alpha^2 |A\tilde{u}_3|^2\right) + \frac{c}{\nu \lambda_1^{1/2} \alpha^2} |\partial_z u_2|^2 |\tilde{u}_2|^2,
\end{aligned} \tag{38}
$$

$$
\begin{aligned}
|J_{3a}| &= |(\tilde{u}_1 \partial_x u_3, \tilde{u}_3)| \\
&\leq \frac{\nu}{20}\left(\|\tilde{u}_3\|^2 + \alpha^2 |A\tilde{u}_3|^2\right) + \frac{c}{\nu \lambda_1^{1/2} \alpha^2} |\partial_x u_3|^2 |\tilde{u}_1|^2,
\end{aligned} \tag{39}
$$

$$
\begin{aligned}
|J_{3b}| &= |(\tilde{u}_2 \partial_y u_3, \tilde{u}_3)| \\
&\leq \frac{\nu}{20}\left(\|\tilde{u}_3\|^2 + \alpha^2 |A\tilde{u}_3|^2\right) + \frac{c}{\nu \lambda_1^{1/2} \alpha^2} |\partial_y u_3|^2 |\tilde{u}_2|^2.
\end{aligned} \tag{40}
$$

Next, using integration by parts and the divergence free condition (31d) we obtain

$$
\begin{aligned}
J_{3c} = (\tilde{u}_3 \partial_z u_3, \tilde{u}_3) &= -(u_3, \partial_z(\tilde{u}_3^2)) \\
&= -2(u_3, \tilde{u}_3 \partial_z \tilde{u}_3) \\
&= 2(u_3, \tilde{u}_3(\partial_x \tilde{u}_1 + \partial_y \tilde{u}_2)) \\
&=: 2J_{3d}.
\end{aligned}
$$

Integrating by parts once again implies that

$$
\begin{aligned}
J_{3d} &= (u_3, \tilde{u}_3(\partial_x \tilde{u}_1 + \partial_y \tilde{u}_2)) \\
&= -(u_3, \tilde{u}_1 \partial_x \tilde{u}_3) - (u_3, \tilde{u}_2 \partial_y \tilde{u}_3) - (\partial_x u_3, \tilde{u}_1 \tilde{u}_3) - (\partial_y u_3, \tilde{u}_2 \tilde{u}_3) \\
&=: J_{3d1} + J_{3d2} + J_{3d3} + J_{3d4}.
\end{aligned}
$$

Using Hölder inequality, Agmon inequality (16) and the Poincaré inequality (15), we have

$$
\begin{aligned}
|J_{3d1}| &= |(u_3, \tilde{u}_1 \partial_x \tilde{u}_3)| \\
&\leq \|u_3\|_{L^\infty(\Omega)} |\tilde{u}_1| |\partial_x \tilde{u}_3| \\
&\leq c_1 \|u_3\|^{1/2} |Au_3|^{1/2} |\tilde{u}_1| |\partial_x \tilde{u}_3| \\
&\leq c_1 \lambda_1^{-1/4} |Au_3| |\tilde{u}_1| |\partial_x \tilde{u}_3| \\
&\leq \frac{\nu}{20} |\partial_x \tilde{u}_3|^2 + \frac{c}{\nu \lambda_1^{1/2}} |Au_3|^2 |\tilde{u}_1|^2 \\
&\leq \frac{\nu}{20}\left(\|\tilde{u}_3\|^2 + \alpha^2 |A\tilde{u}_3|^2\right) + \frac{c}{\nu \lambda_1^{1/2}} |Au_3|^2 |\tilde{u}_1|^2,
\end{aligned}
$$

and similarly,

$$
|J_{3d2}| = |(u_3, \tilde{u}_2 \partial_y \tilde{u}_3)|
$$

$$\le \frac{\nu}{20}|\partial_y \tilde{u}_3|^2 + \frac{c}{\nu\lambda_1^{1/2}}|Au_3|^2|\tilde{u}_2|^2$$
$$\le \frac{\nu}{20}\left(\|\tilde{u}_3\|^2 + \alpha^2|A\tilde{u}_3|^2\right) + \frac{c}{\nu\lambda_1^{1/2}}|Au_3|^2|\tilde{u}_2|^2.$$

By a similar argument as in (33), we can show that

$$\begin{aligned}
|J_{3d3}| &= |(\partial_x u_3, \tilde{u}_1\tilde{u}_3)| \\
&\le \|\tilde{u}_3\|_{L^\infty(\Omega)}|\partial_x u_3||\tilde{u}_1| \\
&\le c_1\|\tilde{u}_3\|^{1/2}|A\tilde{u}_3|^{1/2}|\partial_x u_3||\tilde{u}_1| \\
&\le c_1\lambda_1^{-1/4}|A\tilde{u}_3||\partial_x u_3||\tilde{u}_1| \\
&\le \frac{\nu}{20}\alpha^2|A\tilde{u}_3|^2 + \frac{c}{\nu\lambda_1^{1/2}\alpha^2}|\partial_x u_3|^2|\tilde{u}_1|^2 \\
&\le \frac{\nu}{20}\left(\|\tilde{u}_3\|^2 + \alpha^2|A\tilde{u}_3|^2\right) + \frac{c}{\nu\lambda_1^{1/2}\alpha^2}|\partial_x u_3|^2|\tilde{u}_1|^2,
\end{aligned}$$

and

$$\begin{aligned}
|J_{3d4}| &= |(\partial_y u_3, \tilde{u}_2\tilde{u}_3)| \\
&\le \frac{\nu}{20}\left(\|\tilde{u}_3\|^2 + \alpha^2|A\tilde{u}_3|^2\right) + \frac{c}{\nu\lambda_1^{1/2}\alpha^2}|\partial_y u_3|^2|\tilde{u}_2|^2.
\end{aligned}$$

Thus,

$$|J_{3d}| \le \frac{\nu}{5}\left(\|\tilde{u}_3\|^2 + \alpha^2|A\tilde{u}_3|^2\right) + \frac{c}{\nu\lambda_1^{1/2}\alpha^2}(\|u_3\|^2 + \alpha^2|Au_3|^2)(|\tilde{u}_1|^2 + |\tilde{u}_2|^2).$$

This yield

$$\begin{aligned}
&|J_{3c}| = |(\tilde{u}_3\partial_z u_3, \tilde{u}_3)| \\
&\le \frac{2\nu}{5}\left(\|\tilde{u}_3\|^2 + \alpha^2|A\tilde{u}_3|^2\right) + \frac{c}{\nu\lambda_1^{1/2}\alpha^2}(\|u_3\|^2 + \alpha^2|Au_3|^2)(|\tilde{u}_1|^2 + |\tilde{u}_2|^2).
\end{aligned} \tag{41}$$

Using Young's inequality, (11) and the assumption $\mu c_0^2 h^2 \le \nu$, we have with $i = 1, 2$ that

$$\begin{aligned}
-\mu(I_h(\tilde{u}_i), \tilde{u}_i) =&\ -\mu(I_h(\tilde{u}_i) - \tilde{u}_i, \tilde{u}_i) - \mu|\tilde{u}_i|^2 \\
\le&\ \mu c_0 h\|\tilde{u}_i\||\tilde{u}_i| - \mu|\tilde{u}_i|^2 \\
\le&\ \frac{\mu c_0^2 h^2}{2}\|\tilde{u}_i\|^2 + \frac{\mu}{2}|\tilde{u}_i|^2 - \mu|\tilde{u}_i|^2 \\
\le&\ \frac{\nu}{2}\|\tilde{u}_i\|^2 - \frac{\mu}{2}|\tilde{u}_i|^2.
\end{aligned} \tag{42}$$

Also we note that

$$(\partial_x(p^* - p), \tilde{u}_1) + (\partial_y(p^* - p), \tilde{u}_2) + (\partial_z(p^* - p), \tilde{u}_3) = 0, \tag{43}$$

due to integration by parts, the boundary conditions, and the divergence free condition (31d). Combining all the bounds (33)-(43) and denoting $|\tilde{u}_H|^2 = |\tilde{u}_1|^2 + |\tilde{u}_2|^2$, we obtain

$$\frac{d}{dt}\left(|\tilde{u}|^2 + \alpha^2\|\tilde{u}\|^2\right) + \frac{\nu}{2}\left(\|\tilde{u}\|^2 + \alpha^2|A\tilde{u}|^2\right) \leq \left(\frac{c}{\nu\lambda_1^{1/2}\alpha^2}(\|u\|^2 + \alpha^2|Au|^2) - \mu\right)|\tilde{u}_H|^2,$$

or, using Poincaré inequalities (14) and (15), we have

$$\frac{d}{dt}\left(|\tilde{u}|^2 + \alpha^2\|\tilde{u}\|^2\right) + \frac{\nu\lambda_1}{2}\left(|\tilde{u}|^2 + \alpha^2\|\tilde{u}\|^2\right) + \beta(t)|\tilde{u}_H|^2 \leq 0,$$

where

$$\beta(t) := \mu - \frac{c}{\nu\lambda_1^{1/2}\alpha^2}(\|u\|^2 + \alpha^2|Au|^2).$$

Since u is a solution in the global attractor $\mathcal{A}$, we can use the bound from (23). Using the assumption (7), we have

$$\frac{d}{dt}\left(|\tilde{u}|^2 + \alpha^2\|\tilde{u}\|^2\right) + \min\left\{\frac{\nu\lambda_1}{2}, \frac{\mu}{2}\right\}\left(|\tilde{u}|^2 + \alpha^2\|\tilde{u}\|^2\right) \leq 0$$

for $t > t_0$. By the Gronwall inequality, we obtain

$$|\tilde{u}(t)|^2 + \alpha^2\|\tilde{u}(t)\|^2 \to 0,$$

at an exponential rate, as $t \to \infty$.

3.2. Proof of Theorem 1.2

Using Young's inequality, (12) and the assumptions $\mu c_0^2 h^2 \leq 2\nu$ and $\mu c_0^4 h^4 \leq 4\nu\alpha^2$, we have with $i = 1, 2$ that

$$\begin{aligned} -\mu(I_h(\tilde{u}_i), \tilde{u}_i) &= -\mu(I_h(\tilde{u}_i) - \tilde{u}_i, \tilde{u}_i) - \mu|\tilde{u}_i|^2 \\ &\leq \mu|I_h(\tilde{u}_i) - \tilde{u}_i||\tilde{u}_i| - \mu|\tilde{u}_i|^2 \\ &\leq \frac{\mu}{2}|I_h(\tilde{u}_i) - \tilde{u}_i|^2 + \frac{\mu}{2}|\tilde{u}_i|^2 - \mu|\tilde{u}_i|^2 \\ &\leq \frac{\mu c_0^2 h^2}{4}\|\tilde{u}_i\|^2 + \frac{\mu c_0^4 h^4}{8}|A\tilde{u}_i|^2 + \frac{\mu}{2}|\tilde{u}_i|^2 - \mu|\tilde{u}_i|^2 \\ &\leq \frac{\nu}{2}\|\tilde{u}_i\|^2 + \frac{\nu\alpha^2}{2}|A\tilde{u}_i|^2 - \frac{\mu}{2}|\tilde{u}_i|^2 \\ &= \frac{\nu}{2}(\|\tilde{u}_i\|^2 + \alpha^2|A\tilde{u}_i|^2) - \frac{\mu}{2}|\tilde{u}_i|^2. \end{aligned} \tag{44}$$

Now, the remainder of the proof is similar to that of Theorem 1.1, where (42) is replaced with (44). Therefore, we omit the details here.

3.3. Proof of Theorem 1.3

For brevity, as in the proof of Theorem 1.1, we will omit the rigorous details and provide only the formal *a priori* estimates.

Since

$$\|u_0^*\|^2 + \alpha^2|Au_0^*|^2 \leq \frac{2\nu^2 Gr^2}{\lambda_1^{1/2}}\left(\frac{\lambda_1}{2} + \frac{\nu\lambda_1}{\alpha^2} + \frac{1}{\nu}\right)\exp\left(\frac{54c_5^4\nu Gr^4}{\alpha^4\lambda_1}\right) := M_1,$$

by the continuity of the map $t \mapsto \|u^*(t)\|^2 + \alpha^2|Au^*(t)|^2$, there exists a short time interval $[t_0, \tilde{T})$ such that

$$\|u^*(t)\|^2 + \alpha^2|Au^*(t)|^2 \leq 11M_1 + \frac{384000(16e+2)c_3^4c_4^4}{\nu^4\lambda_1\alpha^6}\left(\frac{2\nu^2Gr^2}{\lambda_1^{1/2}}\right)^3 \tag{45}$$

for all $t \in [t_0, \tilde{T})$. Assume $[t_0, \tilde{T})$ is the maximal finite time interval such that (45) holds. We will show, by contradiction, that $\tilde{T} = \infty$. Assume that $\tilde{T} < \infty$, then it is clear that

$$\limsup_{t\to\tilde{T}^-}(\|u^*(t)\|^2 + \alpha^2|Au^*(t)|^2) = 11M_1 + \frac{384000(16e+2)c_3^4c_4^4}{\nu^4\lambda_1\alpha^6}\left(\frac{2\nu^2Gr^2}{\lambda_1^{1/2}}\right)^3.$$

Multiplying (31a), (31b) and (31c) by $A\tilde{u}_1$, $A\tilde{u}_2$ and $A\tilde{u}_3$ respectively, then integrating over Ω we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\tilde{u}_1\|^2 + \alpha^2|A\tilde{u}_1|^2\right) + \nu\left(|A\tilde{u}_1|^2 + \alpha^2\|A\tilde{u}_1\|^2\right)\\
&\leq |K_1| - (\partial_x(p^*-p), A\tilde{u}_1) - \mu(I_h(\tilde{u}_1), A\tilde{u}_1),
\end{aligned}\tag{46a}$$

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\tilde{u}_2\|^2 + \alpha^2|A\tilde{u}_2|^2\right) + \nu\left(|A\tilde{u}_2|^2 + \alpha^2\|A\tilde{u}_2\|^2\right)\\
&\leq |K_2| - (\partial_y(p^*-p), A\tilde{u}_2) - \mu(I_h(\tilde{u}_2), A\tilde{u}_2),
\end{aligned}\tag{46b}$$

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\tilde{u}_3\|^2 + \alpha^2|A\tilde{u}_3|^2\right) + \nu\left(|A\tilde{u}_3|^2 + \alpha^2\|A\tilde{u}_3\|^2\right)\\
&\leq |K_3| - (\partial_z(p^*-p), A\tilde{u}_3),
\end{aligned}\tag{46c}$$

where

$$\begin{aligned}
K_1 &:= K_{1a} + K_{1b} + K_{1c} + K_{1d} + K_{1e} + K_{1f}\\
&:= (u_1^*\partial_x\tilde{u}_1, A\tilde{u}_1) + (u_2^*\partial_y\tilde{u}_1, A\tilde{u}_1) + (u_3^*\partial_z\tilde{u}_1, A\tilde{u}_1)\\
&\quad + (\tilde{u}_1\partial_x u_1, A\tilde{u}_1) + (\tilde{u}_2\partial_y u_1, A\tilde{u}_1) + (\tilde{u}_3\partial_z u_1, A\tilde{u}_1),\\
K_2 &:= K_{2a} + K_{2b} + K_{2c} + K_{2d} + K_{2e} + K_{2f}\\
&:= (u_1^*\partial_x\tilde{u}_2, A\tilde{u}_2) + (u_2^*\partial_y\tilde{u}_2, A\tilde{u}_2) + (u_3^*\partial_z\tilde{u}_2, A\tilde{u}_2)\\
&\quad + (\tilde{u}_1\partial_x u_2, A\tilde{u}_2) + (\tilde{u}_2\partial_y u_2, A\tilde{u}_2) + (\tilde{u}_3\partial_z u_2, A\tilde{u}_2),\\
K_3 &:= K_{3a} + K_{3b} + K_{3c} + K_{3d} + K_{3e} + K_{3f}\\
&:= (u_1^*\partial_x\tilde{u}_3, A\tilde{u}_3) + (u_2^*\partial_y\tilde{u}_3, A\tilde{u}_3) + (u_3^*\partial_z\tilde{u}_3, A\tilde{u}_3)
\end{aligned}$$

$$+ (\tilde{u}_1 \partial_x u_3, A\tilde{u}_3) + (\tilde{u}_2 \partial_y u_3, A\tilde{u}_3) + (\tilde{u}_3 \partial_z u_3, A\tilde{u}_3).$$

Using the Hölder inequality, the Agmon inequality (16) and the Poincaré inequality (15), we have

$$\begin{aligned}
|K_{1a}| &= |(u_1^* \partial_x \tilde{u}_1, A\tilde{u}_1)| \\
&\le \|u_1^*\|_{L^\infty(\Omega)} |\partial_x \tilde{u}_1| |A\tilde{u}_1| \\
&\le c_1 \|u_1^*\|^{1/2} |Au_1^*|^{1/2} |\partial_x \tilde{u}_1| |A\tilde{u}_1| \\
&\le c_1 \lambda_1^{-1/4} |Au_1^*| |\partial_x \tilde{u}_1| |A\tilde{u}_1| \\
&\le \frac{\nu}{20} |A\tilde{u}_1|^2 + \frac{c}{\nu \lambda_1^{1/2}} |Au_1^*|^2 |\partial_x \tilde{u}_1|^2 \\
&\le \frac{\nu}{20} \left(|A\tilde{u}_1|^2 + \alpha^2 \|A\tilde{u}_1\|^2 \right) + \frac{c}{\nu \lambda_1^{1/2}} |Au_1^*|^2 \|\tilde{u}_1\|^2. \qquad (47)
\end{aligned}$$

Using similar analysis as above, we obtain the following estimates

$$\begin{aligned}
|K_{1b}| &= |(u_2^* \partial_y \tilde{u}_1, A\tilde{u}_1)| \\
&\le \frac{\nu}{20} |A\tilde{u}_1|^2 + \frac{c}{\nu \lambda_1^{1/2}} |Au_2^*|^2 |\partial_y \tilde{u}_1|^2 \\
&\le \frac{\nu}{20} \left(|A\tilde{u}_1|^2 + \alpha^2 \|A\tilde{u}_1\|^2 \right) + \frac{c}{\nu \lambda_1^{1/2}} |Au_2^*|^2 \|\tilde{u}_1\|^2, \qquad (48)
\end{aligned}$$

$$\begin{aligned}
|K_{1c}| &= |(u_3^* \partial_z \tilde{u}_1, A\tilde{u}_1)| \\
&\le \frac{\nu}{20} |A\tilde{u}_1|^2 + \frac{c}{\nu \lambda_1^{1/2}} |Au_3^*|^2 |\partial_z \tilde{u}_1|^2 \\
&\le \frac{\nu}{20} \left(|A\tilde{u}_1|^2 + \alpha^2 \|A\tilde{u}_1\|^2 \right) + \frac{c}{\nu \lambda_1^{1/2}} |Au_3^*|^2 \|\tilde{u}_1\|^2. \qquad (49)
\end{aligned}$$

Using the Hölder inequality, the Ladyzhenskaya inequality (17) and the Poincaré inequality (14), we have

$$\begin{aligned}
|K_{1d}| &= |(\tilde{u}_1 \partial_x u_1, A\tilde{u}_1)| \\
&\le \|\tilde{u}_1\|_{L^4(\Omega)} \|\partial_x u_1\|_{L^4(\Omega)} |A\tilde{u}_1| \\
&\le c_2^2 |\tilde{u}_1|^{1/4} \|\tilde{u}_1\|^{3/4} |\partial_x u_1|^{1/4} \|\partial_x u_1\|^{3/4} |A\tilde{u}_1| \\
&\le c_2^2 \lambda_1^{-1/4} \|\tilde{u}_1\| \|\partial_x u_1\| |A\tilde{u}_1| \\
&\le \frac{\nu}{20} |A\tilde{u}_1|^2 + \frac{c}{\nu \lambda_1^{1/2}} \|\partial_x u_1\|^2 \|\tilde{u}_1\|^2 \\
&\le \frac{\nu}{20} \left(|A\tilde{u}_1|^2 + \alpha^2 \|A\tilde{u}_1\|^2 \right) + \frac{c}{\nu \lambda_1^{1/2}} |Au_1|^2 \|\tilde{u}_1\|^2. \qquad (50)
\end{aligned}$$

Using similar analysis as above, we obtain

$$|K_{1e}| = |(\tilde{u}_2 \partial_y u_1, A\tilde{u}_1)|$$

$$
\begin{aligned}
&\le \frac{\nu}{20}|A\tilde{u}_1|^2 + \frac{c}{\nu\lambda_1^{1/2}}\|\partial_y u_1\|^2\|\tilde{u}_2\|^2 \\
&\le \frac{\nu}{20}\left(|A\tilde{u}_1|^2 + \alpha^2\|A\tilde{u}_1\|^2\right) + \frac{c}{\nu\lambda_1^{1/2}}|Au_1|^2\|\tilde{u}_2\|^2. \qquad (51)
\end{aligned}
$$

Next, integrating by parts we obtain

$$
\begin{aligned}
K_{1f} &= (\tilde{u}_3\partial_z u_1, A\tilde{u}_1) \\
&= -(\partial_x\tilde{u}_3\partial_z u_1, \partial_x\tilde{u}_1) - (\tilde{u}_3\partial_x\partial_z u_1, \partial_x\tilde{u}_1) \\
&\quad - (\partial_y\tilde{u}_3\partial_z u_1, \partial_y\tilde{u}_1) - (\tilde{u}_3\partial_y\partial_z u_1, \partial_y\tilde{u}_1) \\
&\quad - (\partial_z\tilde{u}_3\partial_z u_1, \partial_z\tilde{u}_1) - (\tilde{u}_3\partial_z\partial_z u_1, \partial_z\tilde{u}_1) \\
&:= K_{1f1} + K_{1f2} + K_{1f3} + K_{1f4} + K_{1f5} + K_{1f6}.
\end{aligned}
$$

Using the Hölder inequality, the Ladyzhenskaya inequality (17), the Agmon inequality (16) and the Poincaré inequalities (14) and (15), we can show that

$$
\begin{aligned}
|K_{1f1}| &= |(\partial_x\tilde{u}_3\partial_z u_1, \partial_x\tilde{u}_1)| \\
&\le \|\partial_x\tilde{u}_3\|_{L^4(\Omega)}\|\partial_z u_1\|_{L^4(\Omega)}\|\partial_x\tilde{u}_1\|_{L^2(\Omega)} \\
&\le c_2^2|\partial_x\tilde{u}_3|^{1/4}\|\partial_x\tilde{u}_3\|^{3/4}|\partial_z u_1|^{1/4}\|\partial_z u_1\|^{3/4}|\partial_x\tilde{u}_1| \\
&\le c_2^2\lambda_1^{-1/4}\|\partial_x\tilde{u}_3\|\|\partial_z u_1\||\partial_x\tilde{u}_1| \\
&\le c_2^2\lambda_1^{-1/4}|A\tilde{u}_3||Au_1|\|\tilde{u}_1\| \\
&\le \frac{\nu}{120}\left(|A\tilde{u}_3|^2 + \alpha^2\|A\tilde{u}_3\|^2\right) + \frac{c}{\nu\lambda_1^{1/2}}|Au_1|^2\|\tilde{u}_1\|^2,
\end{aligned}
$$

and

$$
\begin{aligned}
|K_{1f2}| &= |(\tilde{u}_3\partial_x\partial_z u_1, \partial_x\tilde{u}_1)| \\
&\le \|\tilde{u}_3\|_{L^\infty(\Omega)}\|\partial_x\partial_z u_1\|_{L^2(\Omega)}\|\partial_x\tilde{u}_1\|_{L^2(\Omega)} \\
&\le c_1\|\tilde{u}_3\|^{1/2}|A\tilde{u}_3|^{1/2}|\partial_x\partial_z u_1||\partial_x\tilde{u}_1| \\
&\le c_1\|\tilde{u}_3\|^{1/2}|A\tilde{u}_3|^{1/2}|Au_1|\|\tilde{u}_1\| \\
&\le c_1\lambda_1^{-1/4}|A\tilde{u}_3||Au_1|\|\tilde{u}_1\| \\
&\le \frac{\nu}{120}\left(|A\tilde{u}_3|^2 + \alpha^2\|A\tilde{u}_3\|^2\right) + \frac{c}{\nu\lambda_1^{1/2}}|Au_1|^2\|\tilde{u}_1\|^2.
\end{aligned}
$$

Using similar analysis as above, we obtain

$$
\begin{aligned}
|K_{1f3}| &= |(\partial_y\tilde{u}_3\partial_z u_1, \partial_y\tilde{u}_1)| \\
&\le \frac{\nu}{120}\left(|A\tilde{u}_3|^2 + \alpha^2\|A\tilde{u}_3\|^2\right) + \frac{c}{\nu\lambda_1^{1/2}}|Au_1|^2\|\tilde{u}_1\|^2, \\
|K_{1f4}| &= |(\tilde{u}_3\partial_y\partial_z u_1, \partial_y\tilde{u}_1)| \\
&\le \frac{\nu}{120}\left(|A\tilde{u}_3|^2 + \alpha^2\|A\tilde{u}_3\|^2\right) + \frac{c}{\nu\lambda_1^{1/2}}|Au_1|^2\|\tilde{u}_1\|^2,
\end{aligned}
$$

$$
\begin{aligned}
|K_{1f5}| &= |(\partial_z\tilde{u}_3\partial_z u_1, \partial_z\tilde{u}_1)| \\
&\le \frac{\nu}{120}\left(|A\tilde{u}_3|^2 + \alpha^2\|A\tilde{u}_3\|^2\right) + \frac{c}{\nu\lambda_1^{1/2}}|Au_1|^2\|\tilde{u}_1\|^2, \\
|K_{1f6}| &= |(\tilde{u}_3\partial_z\partial_z u_1, \partial_z\tilde{u}_1)| \\
&\le \frac{\nu}{120}\left(|A\tilde{u}_3|^2 + \alpha^2\|A\tilde{u}_3\|^2\right) + \frac{c}{\nu\lambda_1^{1/2}}|Au_1|^2\|\tilde{u}_1\|^2.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
|K_{1f}| &= |(\tilde{u}_3\partial_z u_1, A\tilde{u}_1)| \\
&\le \frac{\nu}{20}\left(|A\tilde{u}_3|^2 + \alpha^2\|A\tilde{u}_3\|^2\right) + \frac{c}{\nu\lambda_1^{1/2}}|Au_1|^2\|\tilde{u}_1\|^2.
\end{aligned} \tag{52}
$$

Using similar analysis as above, we obtain the following estimates:

$$
\begin{aligned}
|K_{2a}| &= |(u_1^*\partial_x\tilde{u}_2, A\tilde{u}_2)| \\
&\le \frac{\nu}{20}\left(|A\tilde{u}_2|^2 + \alpha^2\|A\tilde{u}_2\|^2\right) + \frac{c}{\nu\lambda_1^{1/2}}|Au_1^*|^2\|\tilde{u}_2\|^2,
\end{aligned} \tag{53}
$$

$$
\begin{aligned}
|K_{2b}| &= |(u_2^*\partial_y\tilde{u}_2, A\tilde{u}_2)| \\
&\le \frac{\nu}{20}\left(|A\tilde{u}_2|^2 + \alpha^2\|A\tilde{u}_2\|^2\right) + \frac{c}{\nu\lambda_1^{1/2}}|Au_2^*|^2\|\tilde{u}_2\|^2,
\end{aligned} \tag{54}
$$

$$
\begin{aligned}
|K_{2c}| &= |(u_3^*\partial_z\tilde{u}_2, A\tilde{u}_2)| \\
&\le \frac{\nu}{20}\left(|A\tilde{u}_2|^2 + \alpha^2\|A\tilde{u}_2\|^2\right) + \frac{c}{\nu\lambda_1^{1/2}}|Au_3^*|^2\|\tilde{u}_2\|^2,
\end{aligned} \tag{55}
$$

$$
\begin{aligned}
|K_{2d}| &= |(\tilde{u}_1\partial_x u_2, A\tilde{u}_2)| \\
&\le \frac{\nu}{20}\left(|A\tilde{u}_2|^2 + \alpha^2\|A\tilde{u}_2\|^2\right) + \frac{c}{\nu\lambda_1^{1/2}}|Au_2|^2\|\tilde{u}_1\|^2,
\end{aligned} \tag{56}
$$

$$
\begin{aligned}
|K_{2e}| &= |(\tilde{u}_2\partial_y u_2, A\tilde{u}_2)| \\
&\le \frac{\nu}{20}\left(|A\tilde{u}_2|^2 + \alpha^2\|A\tilde{u}_2\|^2\right) + \frac{c}{\nu\lambda_1^{1/2}}|Au_2|^2\|\tilde{u}_2\|^2,
\end{aligned} \tag{57}
$$

$$
\begin{aligned}
|K_{2f}| &= |(\tilde{u}_3\partial_z u_2, A\tilde{u}_2)| \\
&\le \frac{\nu}{20}\left(|A\tilde{u}_3|^2 + \alpha^2\|A\tilde{u}_3\|^2\right) + \frac{c}{\nu\lambda_1^{1/2}}|Au_2|^2\|\tilde{u}_2\|^2,
\end{aligned} \tag{58}
$$

$$
\begin{aligned}
|K_{3d}| &= |(\tilde{u}_1\partial_x u_3, A\tilde{u}_3)| \\
&\le \frac{\nu}{20}\left(|A\tilde{u}_3|^2 + \alpha^2\|A\tilde{u}_3\|^2\right) + \frac{c}{\nu\lambda_1^{1/2}}|Au_3|^2\|\tilde{u}_1\|^2,
\end{aligned} \tag{59}
$$

$$
|K_{3e}| = |(\tilde{u}_2\partial_y u_3, A\tilde{u}_3)| \tag{60}
$$

$$\leq \frac{\nu}{20}\left(|A\tilde{u}_3|^2+\alpha^2\|A\tilde{u}_3\|^2\right)+\frac{c}{\nu\lambda_1^{1/2}}|Au_3|^2\|\tilde{u}_2\|^2.$$

Using Hölder inequality, Sobolev inequality (19) and Ladyzhenskaya inequality (18), we have

$$\begin{aligned}
|K_{3a}| &= |(u_1^*\partial_x\tilde{u}_3, A\tilde{u}_3)| \\
&\leq |(\tilde{u}_1\partial_x\tilde{u}_3, A\tilde{u}_3)| + |(u_1\partial_x\tilde{u}_3, A\tilde{u}_3)| \\
&\leq \|\tilde{u}_1\|_{L^6(\Omega)}\|\partial_x\tilde{u}_3\|_{L^3(\Omega)}\|A\tilde{u}_3\|_{L^2(\Omega)} + \|u_1\|_{L^6(\Omega)}\|\partial_x\tilde{u}_3\|_{L^3(\Omega)}\|A\tilde{u}_3\|_{L^2(\Omega)} \\
&\leq c_3c_4\|\tilde{u}_1\||\partial_x\tilde{u}_3|^{1/2}\|\partial_x\tilde{u}_3\|^{1/2}|A\tilde{u}_3| + c_3c_4\|u_1\||\partial_x\tilde{u}_3|^{1/2}\|\partial_x\tilde{u}_3\|^{1/2}|A\tilde{u}_3| \\
&\leq \frac{\nu}{80}|A\tilde{u}_3|^2 + \frac{20c_3^2c_4^2}{\nu}\|\tilde{u}_1\|^2|\partial_x\tilde{u}_3|\|\partial_x\tilde{u}_3\| \\
&\quad + \frac{\nu}{80}|A\tilde{u}_3|^2 + \frac{20c_3^2c_4^2}{\nu}\|u_1\|^2|\partial_x\tilde{u}_3|\|\partial_x\tilde{u}_3\| \\
&\leq \frac{\nu}{40}|A\tilde{u}_3|^2 + \frac{\nu}{80}\|\partial_x\tilde{u}_3\|^2 + \frac{8000c_3^4c_4^4}{\nu^3}\|\tilde{u}_1\|^4|\partial_x\tilde{u}_3|^2 \\
&\quad + \frac{\nu}{80}\|\partial_x\tilde{u}_3\|^2 + \frac{8000c_3^4c_4^4}{\nu^3}\|u_1\|^4|\partial_x\tilde{u}_3|^2 \\
&\leq \frac{\nu}{20}|A\tilde{u}_3|^2 + \frac{8000c_3^4c_4^4}{\nu^3}\|\tilde{u}_1\|^4\|\tilde{u}_3\|^2 + \frac{8000c_3^4c_4^4}{\nu^3}\|u_1\|^4\|\tilde{u}_3\|^2 \\
&\leq \frac{\nu}{20}\left(|A\tilde{u}_3|^2+\alpha^2\|A\tilde{u}_3\|^2\right) + \frac{8000c_3^4c_4^4}{\nu^3}\|\tilde{u}\|^6 + \frac{8000c_3^4c_4^4}{\nu^3}\|u\|^4\|\tilde{u}\|^2. \qquad (61)
\end{aligned}$$

Using similar analysis as above, we obtain

$$\begin{aligned}
|K_{3b}| &= |(u_2^*\partial_y\tilde{u}_3, A\tilde{u}_3)| \\
&\leq \frac{\nu}{20}\left(|A\tilde{u}_3|^2+\alpha^2\|A\tilde{u}_3\|^2\right) + \frac{8000c_3^4c_4^4}{\nu^3}\|\tilde{u}\|^6 + \frac{8000c_3^4c_4^4}{\nu^3}\|u\|^4\|\tilde{u}\|^2, \qquad (62)
\end{aligned}$$

$$\begin{aligned}
|K_{3c}| &= |(u_3^*\partial_z\tilde{u}_3, A\tilde{u}_3)| \\
&\leq \frac{\nu}{20}\left(|A\tilde{u}_3|^2+\alpha^2\|A\tilde{u}_3\|^2\right) + \frac{8000c_3^4c_4^4}{\nu^3}\|\tilde{u}\|^6 + \frac{8000c_3^4c_4^4}{\nu^3}\|u\|^4\|\tilde{u}\|^2. \qquad (63)
\end{aligned}$$

Next, using integration by parts and the divergence free condition (31d) we obtain

$$\begin{aligned}
K_{3f} = (\tilde{u}_3\partial_z u_3, A\tilde{u}_3) &= -(u_3, \partial_z(\tilde{u}_3A\tilde{u}_3)) \\
&= -(u_3, \tilde{u}_3\partial_z A\tilde{u}_3) - (u_3, A\tilde{u}_3\partial_z\tilde{u}_3) \\
&= -(u_3, \tilde{u}_3A\partial_z\tilde{u}_3) - (u_3, A\tilde{u}_3\partial_z\tilde{u}_3) \\
&= (u_3, \tilde{u}_3A(\partial_x\tilde{u}_1+\partial_y\tilde{u}_2)) + (u_3, A\tilde{u}_3(\partial_x\tilde{u}_1+\partial_y\tilde{u}_2)) \\
&=: K_{3g} + K_{3h}.
\end{aligned}$$

Integrating by parts implies that

$$K_{3g} = (u_3, \tilde{u}_3A(\partial_x\tilde{u}_1+\partial_y\tilde{u}_2))$$

$$
\begin{aligned}
&= (u_3, \tilde{u}_3 \partial_x A\tilde{u}_1) + (u_3, \tilde{u}_3 \partial_y A\tilde{u}_2) \\
&= -(u_3, A\tilde{u}_1 \partial_x \tilde{u}_3) - (u_3, A\tilde{u}_2 \partial_y \tilde{u}_3) - (\partial_x u_3, A\tilde{u}_1 \tilde{u}_3) - (\partial_y u_3, A\tilde{u}_2 \tilde{u}_3) \\
&=: K_{3g1} + K_{3g2} + K_{3g3} + K_{3g4}.
\end{aligned}
$$

Using integration by parts we obtain

$$
\begin{aligned}
K_{3g1} &= (u_3, A\tilde{u}_1 \partial_x \tilde{u}_3) \\
&= (u_3, \partial_x\partial_x \tilde{u}_1 \partial_x \tilde{u}_3) + (u_3, \partial_y\partial_y \tilde{u}_1 \partial_x \tilde{u}_3) + (u_3, \partial_z\partial_z \tilde{u}_1 \partial_x \tilde{u}_3) \\
&= -(\partial_x u_3 \partial_x \tilde{u}_3, \partial_x \tilde{u}_1) - (u_3 \partial_x\partial_x \tilde{u}_3, \partial_x \tilde{u}_1) \\
&\quad - (\partial_y u_3 \partial_x \tilde{u}_3, \partial_y \tilde{u}_1) - (u_3 \partial_y\partial_x \tilde{u}_3, \partial_y \tilde{u}_1) \\
&\quad - (\partial_z u_3 \partial_x \tilde{u}_3, \partial_z \tilde{u}_1) - (u_3 \partial_z\partial_x \tilde{u}_3, \partial_z \tilde{u}_1) \\
&=: K_{3g1-1} + K_{3g1-2} + K_{3g1-3} + K_{3g1-4} + K_{3g1-5} + K_{3g1-6}.
\end{aligned}
$$

Using Hölder inequality, Ladyzhenskaya inequality (17), the Agmon inequality (16) and the Poincaré inequalities (14) and (15), we have

$$
\begin{aligned}
|K_{3g1-1}| &= |(\partial_x u_3 \partial_x \tilde{u}_3, \partial_x \tilde{u}_1)| \\
&\le \|\partial_x u_3\|_{L^4(\Omega)} \|\partial_x \tilde{u}_3\|_{L^4(\Omega)} \|\partial_x \tilde{u}_1\|_{L^2(\Omega)} \\
&\le c_2^2 |\partial_x u_3|^{1/4} \|\partial_x u_3\|^{3/4} |\partial_x \tilde{u}_3|^{1/4} \|\partial_x \tilde{u}_3\|^{3/4} |\partial_x \tilde{u}_1| \\
&\le c_2^2 \lambda_1^{-1/4} \|\partial_x u_3\| \|\partial_x \tilde{u}_3\| |\partial_x \tilde{u}_1| \\
&\le c_2^2 \lambda_1^{-1/4} |Au_3| |A\tilde{u}_3| \|\tilde{u}_1\| \\
&\le \frac{\nu}{480} \left(|A\tilde{u}_3|^2 + \alpha^2 \|A\tilde{u}_3\|^2 \right) + \frac{c}{\nu \lambda_1^{1/2}} |Au_3|^2 \|\tilde{u}_1\|^2,
\end{aligned}
$$

and

$$
\begin{aligned}
|K_{3g1-2}| &= |(u_3 \partial_x\partial_x \tilde{u}_3, \partial_x \tilde{u}_1)| \\
&\le \|u_3\|_{L^\infty(\Omega)} \|\partial_x\partial_x \tilde{u}_3\|_{L^2(\Omega)} \|\partial_x \tilde{u}_1\|_{L^2(\Omega)} \\
&\le c_1 \|u_3\|^{1/2} |Au_3|^{1/2} |\partial_x\partial_x \tilde{u}_3| |\partial_x \tilde{u}_1| \\
&\le c_1 \|u_3\|^{1/2} |Au_3|^{1/2} |A\tilde{u}_3| \|\tilde{u}_1\| \\
&\le c_1 \lambda_1^{-1/4} |Au_3| |A\tilde{u}_3| \|\tilde{u}_1\| \\
&\le \frac{\nu}{480} \left(|A\tilde{u}_3|^2 + \alpha^2 \|A\tilde{u}_3\|^2 \right) + \frac{c}{\nu \lambda_1^{1/2}} |Au_3|^2 \|\tilde{u}_1\|^2.
\end{aligned}
$$

Using similar analysis as above, we obtain

$$
\begin{aligned}
|K_{3g1-3}| &= |(\partial_y u_3 \partial_x \tilde{u}_3, \partial_y \tilde{u}_1)| \\
&\le \frac{\nu}{480} \left(|A\tilde{u}_3|^2 + \alpha^2 \|A\tilde{u}_3\|^2 \right) + \frac{c}{\nu \lambda_1^{1/2}} |Au_3|^2 \|\tilde{u}_1\|^2,
\end{aligned}
$$

$$
|K_{3g1-4}| = |(u_3 \partial_y\partial_x \tilde{u}_3, \partial_y \tilde{u}_1)|
$$

$$\leq \frac{\nu}{480}\left(|A\tilde{u}_3|^2+\alpha^2\|A\tilde{u}_3\|^2\right)+\frac{c}{\nu\lambda_1^{1/2}}|Au_3|^2\|\tilde{u}_1\|^2,$$

$$\begin{aligned}|K_{3g1-5}| &= |(\partial_z u_3\partial_x\tilde{u}_3,\partial_z\tilde{u}_1)|\\ &\leq \frac{\nu}{480}\left(|A\tilde{u}_3|^2+\alpha^2\|A\tilde{u}_3\|^2\right)+\frac{c}{\nu\lambda_1^{1/2}}|Au_3|^2\|\tilde{u}_1\|^2,\end{aligned}$$

$$\begin{aligned}|K_{3g1-6}| &= |(u_3\partial_z\partial_x\tilde{u}_3,\partial_z\tilde{u}_1)|\\ &\leq \frac{\nu}{480}\left(|A\tilde{u}_3|^2+\alpha^2\|A\tilde{u}_3\|^2\right)+\frac{c}{\nu\lambda_1^{1/2}}|Au_3|^2\|\tilde{u}_1\|^2.\end{aligned}$$

Thus,

$$\begin{aligned}|K_{3g1}| &= |(u_3,A\tilde{u}_1\partial_x\tilde{u}_3)|\\ &\leq \frac{\nu}{80}\left(|A\tilde{u}_3|^2+\alpha^2\|A\tilde{u}_3\|^2\right)+\frac{c}{\nu\lambda_1^{1/2}}|Au_3|^2\|\tilde{u}_1\|^2,\end{aligned}$$

and similarly,

$$\begin{aligned}|K_{3g2}| &= |(u_3,A\tilde{u}_2\partial_y\tilde{u}_3)|\\ &\leq \frac{\nu}{80}\left(|A\tilde{u}_3|^2+\alpha^2\|A\tilde{u}_3\|^2\right)+\frac{c}{\nu\lambda_1^{1/2}}|Au_3|^2\|\tilde{u}_2\|^2.\end{aligned}$$

By a similar argument, we can show that

$$\begin{aligned}|K_{3g3}| &= |(\partial_x u_3,A\tilde{u}_1\tilde{u}_3)|\\ &\leq \frac{\nu}{80}\left(|A\tilde{u}_3|^2+\alpha^2\|A\tilde{u}_3\|^2\right)+\frac{c}{\nu\lambda_1^{1/2}}|Au_3|^2\|\tilde{u}_1\|^2,\end{aligned}$$

and

$$\begin{aligned}|K_{3g4}| &= |(\partial_y u_3,A\tilde{u}_2\tilde{u}_3)|\\ &\leq \frac{\nu}{80}\left(|A\tilde{u}_3|^2+\alpha^2\|A\tilde{u}_3\|^2\right)+\frac{c}{\nu\lambda_1^{1/2}}|Au_3|^2\|\tilde{u}_2\|^2.\end{aligned}$$

Thus,

$$\begin{aligned}|K_{3g}| \leq{}& \frac{\nu}{20}\left(|A\tilde{u}_3|^2+\alpha^2\|A\tilde{u}_3\|^2\right)\\ &+\frac{c}{\nu\lambda_1^{1/2}\alpha^2}(\|u_3\|^2+\alpha^2|Au_3|^2)(\|\tilde{u}_1\|^2+\|\tilde{u}_2\|^2).\end{aligned}$$

Now, we have

$$\begin{aligned}K_{3h} &= (u_3,A\tilde{u}_3(\partial_x\tilde{u}_1+\partial_y\tilde{u}_2))\\ &= (u_3,A\tilde{u}_3\partial_x\tilde{u}_1)+(u_3,A\tilde{u}_3\partial_y\tilde{u}_2)\\ &=: K_{3h1}+K_{3h2}.\end{aligned}$$

Using the Hölder inequality, the Agmon inequality (16) and the Poincaré inequality (15), we obtain

$$|K_{3h1}| = |(u_3,A\tilde{u}_3\partial_x\tilde{u}_1)|$$

$$
\begin{aligned}
&\leq \|u_3\|_{L^\infty(\Omega)}\|A\tilde{u}_3\|_{L^2(\Omega)}\|\partial_x\tilde{u}_1\|_{L^2(\Omega)} \\
&\leq c_1\|u_3\|^{1/2}|Au_3|^{1/2}|A\tilde{u}_3||\partial_x\tilde{u}_1| \\
&\leq c_1\lambda_1^{-1/4}|Au_3||A\tilde{u}_3|\|\tilde{u}_1\| \\
&\leq \frac{\nu}{10}\left(|A\tilde{u}_3|^2+\alpha^2\|A\tilde{u}_3\|^2\right)+\frac{c}{\nu\lambda_1^{1/2}}|Au_3|^2\|\tilde{u}_1\|^2,
\end{aligned}
$$

and similarly,

$$
\begin{aligned}
|K_{3h2}| &= |(u_3, A\tilde{u}_3\partial_y\tilde{u}_2)| \\
&\leq \frac{\nu}{10}\left(|A\tilde{u}_3|^2+\alpha^2\|A\tilde{u}_3\|^2\right)+\frac{c}{\nu\lambda_1^{1/2}}|Au_3|^2\|\tilde{u}_2\|^2.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
|K_{3h}| &= (u_3, A\tilde{u}_3(\partial_x\tilde{u}_1+\partial_y\tilde{u}_2)) \\
&\leq \frac{\nu}{5}\left(|A\tilde{u}_3|^2+\alpha^2\|A\tilde{u}_3\|^2\right) \\
&\quad + \frac{c}{\nu\lambda_1^{1/2}\alpha^2}(\|u_3\|^2+\alpha^2|Au_3|^2)(\|\tilde{u}_1\|^2+\|\tilde{u}_2\|^2).
\end{aligned}
$$

This yields

$$
\begin{aligned}
|K_{3f}| &= |(\tilde{u}_3\partial_z u_3, A\tilde{u}_3)| \\
&\leq \frac{\nu}{4}\left(|A\tilde{u}_3|^2+\alpha^2\|A\tilde{u}_3\|^2\right) \\
&\quad + \frac{c}{\nu\lambda_1^{1/2}\alpha^2}(\|u_3\|^2+\alpha^2|Au_3|^2)(\|\tilde{u}_1\|^2+\|\tilde{u}_2\|^2). \qquad (64)
\end{aligned}
$$

Using the Young inequality, (12) and the assumption $\mu c_0^2 h^2 \leq \nu$, we have with $i = 1, 2$ that

$$
\begin{aligned}
-\mu(I_h(\tilde{u}_i), A\tilde{u}_i) =&\ -\mu(I_h(\tilde{u}_i)-\tilde{u}_i, A\tilde{u}_i)-\mu\|\tilde{u}_i\|^2 \\
\leq&\ \mu|I_h(\tilde{u}_i)-\tilde{u}_i||A\tilde{u}_i|-\mu\|\tilde{u}_i\|^2 \\
\leq&\ \frac{\mu^2}{\nu}|I_h(\tilde{u}_i)-\tilde{u}_i|^2+\frac{\nu}{4}|A\tilde{u}_i|^2-\mu\|\tilde{u}_i\|^2 \\
\leq&\ \frac{\mu^2c_0^2h^2}{2\nu}\|\tilde{u}_i\|^2+\frac{\mu^2c_0^4h^4}{4\nu}|A\tilde{u}_i|^2+\frac{\nu}{4}|A\tilde{u}_i|^2-\mu\|\tilde{u}_i\|^2 \\
\leq&\ \frac{\mu}{2}\|\tilde{u}_i\|^2+\frac{\nu}{4}|A\tilde{u}_i|^2+\frac{\nu}{4}|A\tilde{u}_i|^2-\mu\|\tilde{u}_i\|^2 \\
\leq&\ \frac{\nu}{2}|A\tilde{u}_i|^2-\frac{\mu}{2}\|\tilde{u}_i\|^2 \\
\leq&\ \frac{\nu}{2}(|A\tilde{u}_i|^2+\alpha^2\|A\tilde{u}_i\|^2)-\frac{\mu}{2}\|\tilde{u}_i\|^2. \qquad (65)
\end{aligned}
$$

Also we note that

$$
(\partial_x(p^*-p), A\tilde{u}_1)+(\partial_y(p^*-p), A\tilde{u}_2)+(\partial_z(p^*-p), A\tilde{u}_3)=0, \qquad (66)
$$

due to integration by parts, the boundary conditions, and the divergence free condition (31d). Combining all the bounds (47)-(66) and denoting $\|\tilde{u}_H\|^2 = \|\tilde{u}_1\|^2 + \|\tilde{u}_2\|^2$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\|\tilde{u}\|^2 + \alpha^2|A\tilde{u}|^2\right) + \frac{\nu}{2}\left(|A\tilde{u}|^2 + \alpha^2\|A\tilde{u}\|^2\right) \\
&\leq \left(\frac{c}{\nu\lambda_1^{1/2}\alpha^2}(\|u\|^2 + \alpha^2|Au|^2 + \|u^*\|^2 + \alpha^2|Au^*|^2) - \mu\right)\|\tilde{u}_H\|^2 \\
&\quad + \frac{48000c_3^4c_4^4}{\nu^3}\|\tilde{u}\|^6 + \frac{48000c_3^4c_4^4}{\nu^3}\|u\|^4\|\tilde{u}\|^2
\end{aligned}$$

for all $t \in [t_0, \tilde{T})$. Hence using Poincaré inequalities (14) and (15), we have

$$\frac{d}{dt}\left(\|\tilde{u}\|^2 + \alpha^2|A\tilde{u}|^2\right) + \frac{\nu\lambda_1}{2}\left(\|\tilde{u}\|^2 + \alpha^2|A\tilde{u}|^2\right) + \beta(t)\|\tilde{u}_H\|^2 \leq F(t)$$

for all $t \in [t_0, \tilde{T})$, where

$$\beta(t) := \mu - \frac{c}{\nu\lambda_1^{1/2}\alpha^2}(\|u\|^2 + \alpha^2|Au|^2 + \|u^*\|^2 + \alpha^2|Au^*|^2),$$

and

$$F(t) := \frac{48000c_3^4c_4^4}{\nu^3}\|\tilde{u}\|^6 + \frac{48000c_3^4c_4^4}{\nu^3}\|u\|^4\|\tilde{u}\|^2.$$

Since u is a solution in the global attractor $\mathcal{A}$, we can use the bound (23). Using (45) and the assumption (8), we have

$$\frac{d}{dt}\left(\|\tilde{u}\|^2 + \alpha^2|A\tilde{u}|^2\right) + \min\left\{\frac{\nu\lambda_1}{2}, \frac{\mu}{2}\right\}\left(\|\tilde{u}\|^2 + \alpha^2|A\tilde{u}|^2\right) \leq F(t)$$

for all $t \in [t_0, \tilde{T})$. Because the assumptions of Theorem 1.2 are satisfied if the assumptions of Theorem 1.3 are satisfied, we can use Theorem 1.2. So we have for all $t \in [t_0, \tilde{T})$:

$$|\tilde{u}(t)|^2 + \alpha^2\|\tilde{u}(t)\|^2 \leq (|\tilde{u}(t_0)|^2 + \alpha^2\|\tilde{u}(t_0)\|^2)e^{-\min\{\nu\lambda_1/2,\mu/2\}(t-t_0)}.$$

By (9) and (22), we have for all $t \in [t_0, \tilde{T})$:

$$\begin{aligned}
&\|\tilde{u}(t)\|^2 \\
&\leq \alpha^{-2}(|\tilde{u}(t_0)|^2 + \alpha^2\|\tilde{u}(t_0)\|^2)e^{-\min\{\nu\lambda_1/2,\mu/2\}(t-t_0)} \\
&\leq \alpha^{-2}(|u^*(t_0) - u(t_0)|^2 + \alpha^2\|u^*(t_0) - u(t_0)\|^2)e^{-\min\{\nu\lambda_1/2,\mu/2\}(t-t_0)} \\
&\leq 2\alpha^{-2}(|u^*(t_0)|^2 + \alpha^2\|u^*(t_0)\|^2 + |u(t_0)|^2 + \alpha^2\|u(t_0)\|^2)e^{-\min\{\nu\lambda_1/2,\mu/2\}(t-t_0)} \\
&\leq \frac{4M_0}{\alpha^2}e^{-\min\{\nu\lambda_1/2,\mu/2\}(t-t_0)},
\end{aligned}$$

where

$$M_0 := \frac{2\nu^2 Gr^2}{\lambda_1^{1/2}}.$$

Thus,

$$F(t) \le C_1 e^{-3\min\{\nu\lambda_1/2,\mu/2\}(t-t_0)} + C_2 e^{-\min\{\nu\lambda_1/2,\mu/2\}(t-t_0)}$$

for all $t \in [t_0, \tilde{T})$, where $C_1 := \frac{3072000 c_3^4 c_4^4 M_0^3}{\nu^3 \alpha^6}$ and $C_2 := \frac{192000 c_3^4 c_4^4 M_0^3}{\nu^3 \alpha^6}$.

Now, using the Gronwall inequality, we obtain for all $t \in [t_0, \tilde{T})$:

$$\begin{aligned}
&\|\tilde{u}(t)\|^2 + \alpha^2 |A\tilde{u}(t)|^2 \\
&\le (\|\tilde{u}(t_0)\|^2 + \alpha^2 |A\tilde{u}(t_0)|^2) e^{-\min\{\nu\lambda_1/2,\mu/2\}(t-t_0)} \\
&\quad + \int_{t_0}^{t} e^{-\min\{\nu\lambda_1/2,\mu/2\}(t-t_0) + \min\{\nu\lambda_1/2,\mu/2\}(s-t_0)} \times \\
&\quad \left(C_1 e^{-3\min\{\nu\lambda_1/2,\mu/2\}(s-t_0)} + C_2 e^{-\min\{\nu\lambda_1/2,\mu/2\}(s-t_0)} \right) ds \\
&= (\|\tilde{u}(t_0)\|^2 + \alpha^2 |A\tilde{u}(t_0)|^2) e^{-\min\{\nu\lambda_1/2,\mu/2\}(t-t_0)} \\
&\quad + \tfrac{C_1}{2\min\{\nu\lambda_1/2,\mu/2\}} e^{-\min\{\nu\lambda_1/2,\mu/2\}(t-t_0)} \left(1 - e^{-2\min\{\nu\lambda_1/2,\mu/2\}(t-t_0)} \right) \\
&\quad + C_2 e^{-\min\{\nu\lambda_1/2,\mu/2\}(t-t_0)} (t-t_0).
\end{aligned} \tag{67}$$

Thus,

$$\begin{aligned}
&\|\tilde{u}(t)\|^2 + \alpha^2 |A\tilde{u}(t)|^2 \\
&\le \|\tilde{u}(t_0)\|^2 + \alpha^2 |A\tilde{u}(t_0)|^2 + \frac{C_1}{2\min\{\nu\lambda_1/2, \mu/2\}} + \frac{C_2}{e\min\{\nu\lambda_1/2, \mu/2\}}
\end{aligned}$$

for all $t \in [t_0, \tilde{T})$. Since

$\|\tilde{u}(t_0)\|^2 + \alpha^2 |A\tilde{u}(t_0)|^2 \le 2(\|u(t_0)\|^2 + \alpha^2 |Au(t_0)|^2) + 2(\|u^*(t_0)\|^2 + \alpha^2 |Au^*(t_0)|^2)$,

then by (10) and (23), we have

$$\|\tilde{u}(t)\|^2 + \alpha^2 |A\tilde{u}(t)|^2 \le 4M_1 + \frac{eC_1 + 2C_2}{2\min\{\nu\lambda_1/2, \mu/2\}}$$

for all $t \in [t_0, \tilde{T})$. This implies that

$$\begin{aligned}
\|u^*(t)\|^2 + \alpha^2 |Au^*(t)|^2 &= \|\tilde{u}(t) + u(t)\|^2 + \alpha^2 |A\tilde{u}(t) + Au(t)|^2 \\
&\le 2(\|\tilde{u}(t)\|^2 + \alpha^2 |A\tilde{u}(t)|^2) + 2(\|u(t)\|^2 + \alpha^2 |Au(t)|^2) \\
&\le 8M_1 + \frac{eC_1 + 2C_2}{\min\{\nu\lambda_1/2, \mu/2\}} + 2M_1 \\
&= 10M_1 + \frac{2eC_1 + 4C_2}{\nu\lambda_1}
\end{aligned}$$

for all $t \in [t_0, \tilde{T})$, if we choose $\mu \ge \nu\lambda_1$. This in turn will yield a contradiction since

$$11M_1 + \frac{2eC_1 + 4C_2}{\nu\lambda_1} = 11M_1 + \frac{384000(16e+2)c_3^4 c_4^4}{\nu^4 \lambda_1 \alpha^6} \left(\frac{2\nu^2 Gr^2}{\lambda_1^{1/2}} \right)^3$$

$$= \limsup_{t\to\tilde{T}^-}(\|u^*(t)\|^2 + \alpha^2|Au^*(t)|^2)$$
$$\le 10M_1 + \frac{2eC_1 + 4C_2}{\nu\lambda_1}.$$

This proves that $\tilde{T} = \infty$. Thus, by (67), with $\tilde{T} = \infty$, we obtain

$$\|\tilde{u}(t)\|^2 + \alpha^2|A\tilde{u}(t)|^2 \to 0,$$

at an exponential rate, as $t \to \infty$. The proof is complete.

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