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QUASI-CONFORMAL CURVATURE TENSOR ON $N(k)$-QUASI EINSTEIN MANIFOLDS

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Abstract. This paper deals with the study of $N(k)$-quasi Einstein manifolds that satisfies the certain curvature conditions $\mathcal{C}_*\cdot\mathcal{C}_* = 0$, $\mathcal{S}\cdot\mathcal{C}_* = 0$ and $\mathcal{R}\cdot\mathcal{C}_* = f\tilde{Q}(g, \mathcal{C}_*)$, where $\mathcal{C}_*$, $\mathcal{S}$ and $\mathcal{R}$ denotes the quasi-conformal curvature tensor, Ricci tensor and the curvature tensor respectively. Finally, we construct an example of $N(k)$-quasi Einstein manifold.

1. Introduction

An n-dimensional semi-Riemannian or Riemannian manifold (M^n, g) $(n > 2)$, is called an Einstein manifold if its Ricci tensor $\mathcal{S}$ satisfies the criteria

$$\mathcal{S} = \frac{\rho}{n} g,$$

where ρ denotes the scalar curvature of (M^n, g). We can also say an Einstein manifold is a Riemannian or pseudo Riemannian manifold whose Ricci tensor is proportional to the metric. The notion of quasi-Einstein manifold was introduced by M. C. Chaki and R. K. Maity [3]. A non-flat Riemannian manifold (M^n, g) $(n \geq 3)$, is a quasi-Einstein manifold if its Ricci tensor $\mathcal{S}$ satisfies the criteria

$$\mathcal{S}(U, V) = ag(U, V) + b\eta(U)\eta(V) \tag{1}$$

and is not identically zero, where a and b are smooth functions of which $b \neq 0$ and η is a non-zero 1-form such that

$$g(U, \xi) = \eta(U), \qquad g(\xi, \xi) = \eta(\xi) = 1, \tag{2}$$

for all vector field U.
We call η as associated 1-form and ξ as generator of the manifold, which is also an unit vector field. The study of quasi-Einstein manifolds was further continued by Guha [11], De and Ghosh [8], Bejan [2], De and De [6], Debnath and Konar [9] and many others.

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Let $\mathcal{R}$ denotes the Riemannian curvature tensor of a Riemannian manifold M. The k-nullity distribution $N(k)$ [17] of a Riemannian manifold M is defined by

$$N(k): p \longrightarrow N_p(k) = \{W \in T_pM \colon \mathcal{R}(U,V)W = k[g(V,W)U - g(U,W)V]\},$$

where k is a smooth function.
M. M. Tripathi and Jeong Sik-Kim [18] introduced the notion of $N(k)$-quasi Einstein manifolds which is defined as follows: If the generator ξ belongs to the k-nullity distribution $N(k)$, then a quasi-Einstein manifold (M^n, g) is called $N(k)$-quasi Einstein manifold. Here k is not arbitrary.

Lemma 1.1. [15] *In an n-dimensional $N(k)$-quasi Einstein manifold it follows that*

$$k = \frac{a+b}{n-1}. \tag{3}$$

So we note that in an $N(k)$-quasi Einstein manifold [15]

$$\mathcal{R}(U,V)\xi = \frac{a+b}{n-1}[\eta(V)U - \eta(U)V], \tag{4}$$

which is same as

$$\mathcal{R}(\xi,U)V = \frac{a+b}{n-1}[g(U,V)\xi - \eta(V)U]. \tag{5}$$

In [18], Tripathi and Kim proved that an n-dimensional conformally flat quasi-Einstein manifold is an $N\left(\frac{a+b}{n-1}\right)$-quasi Einstein manifold and in particular a 3-dimensional quasi-Einstein manifold is an $N\left(\frac{a+b}{2}\right)$-quasi Einstein manifold. Various geometrical properties of $N(k)$-quasi Einstein manifolds have been discussed by Taleshian and Hosseinzadeh [12, 16], De, De and Gazi [7], Crasmareanu [5], Yildiz, De and Cetinkaya [20], Mallick and De [13] and many others. The above works inspired me to write up a study on this type of manifold.

In 1968, Yano and Sawaki [19] defined the quasi-conformal curvature tensor $\mathcal{C}_*$ on a Riemannian manifold (M^n, g) as

$$\begin{aligned} \mathcal{C}_*(U,V)W = {} & a_0\mathcal{R}(U,V)W + a_1[\mathcal{S}(V,W)U \\ & -\mathcal{S}(U,W)V + g(V,W)QU - g(U,W)QV] \\ & - \frac{\rho}{n}\left(\frac{a_0}{n-1} + 2a_1\right)[g(V,W)U - g(U,W)V], \end{aligned} \tag{6}$$

where $\mathcal{S}(U,V) = g(QU,V)$, ρ is the scalar curvature, a_0 and a_1 are arbitrary constants, which are not simultaneously zero. If $a_0 = 1$ and $a_1 = -\frac{1}{n-2}$, then (6) reduces to the conformal curvature tensor. Thus the conformal curvature tensor is a particular case of the tensor $\mathcal{C}_*$. A Riemannian or a semi-Riemannian manifold is called quasi-conformally flat if $\mathcal{C}_* = 0$ for $n > 3$.

The derivation conditions $\mathcal{R}(\xi,U)\cdot\mathcal{R} = 0$ and $\mathcal{R}(\xi,U)\cdot\mathcal{S} = 0$ have been discussed in [18], where $\mathcal{R}$ and $\mathcal{S}$ denotes the curvature tensor and Ricci tensor of the manifold respectively. In 2008, Özgür and Sular [14] studied the derivation conditions $\mathcal{R}(\xi,U)\cdot\mathcal{C} = 0$ and $\mathcal{R}(\xi,U)\cdot\mathcal{C}_* = 0$ on $N(k)$-quasi Einstein manifolds, where $\mathcal{C}$ and $\mathcal{C}_*$ denotes the Weyl conformal and quasi-conformal curvature tensors, respectively.

After studying and analyzing the above papers, we got motivated to work in this area. In the present work we have tried to develop a new concept. This paper is organized as follows: Section 2 is preliminaries that covers various concepts and results of $N(k)$-quasi Einstein manifold and quasi-conformal curvature tensor. Section 3 deals with study of quasi-conformal curvature tensor of an $N(k)$-quasi Einstein manifold. Section 4 is concerned with an $N(k)$-quasi Einstein manifold satisfies $\mathcal{S}(U,\xi)\cdot\mathcal{C}_*=0$. The properties of $\mathcal{C}_*$-pseudosymmetric $N(k)$-quasi Einstein manifolds had been analyzed in section 5. Finally, we give an example of $N(k)$-quasi Einstein manifold.

2. Preliminaries

From (1) and (2) it follows that

$$\rho = an + b \tag{7}$$

and

$$\mathcal{S}(U,\xi) = (a+b)\,\eta(U), \tag{8}$$

where ρ is the scalar curvature and Q is the Ricci operator.

In an n-dimensional $N(k)$-quasi Einstein manifold M, the quasi-conformal curvature tensor $\mathcal{C}_*$ takes the form

$$\begin{aligned}\mathcal{C}_*(U,V)W &= \frac{b}{n}(a_0-2a_1)\left[g(V,W)U - g(U,W)V\right]\\ &\quad + ba_1\left[\eta(V)\eta(W)U - \eta(U)\eta(W)V\right.\\ &\quad \left.+g(V,W)\eta(U)\xi - g(U,W)\eta(V)\xi\right].\end{aligned} \tag{9}$$

Consequently, we have

$$\mathcal{C}_*(\xi,U)V = \frac{b}{n}\left[a_0+(n-2)a_1\right]\left[g(U,V)\xi - \eta(V)U\right], \tag{10}$$

$$\eta(\mathcal{C}_*(U,V)W) = \frac{b}{n}\left[a_0+(n-2)a_1\right]\left[g(V,W)\eta(U) - g(U,W)\eta(V)\right], \tag{11}$$

$$\eta(\mathcal{C}_*(U,V)\xi) = 0 \tag{12}$$

and

$$\eta(\mathcal{C}_*(U,\xi)V) = \frac{b}{n}\left[a_0+(n-2)a_1\right]\left[\eta(V)\eta(U) - g(U,V)\right] = -\eta(\mathcal{C}_*(\xi,U)V), \tag{13}$$

for all vector fields U,V,W on M.

3. The quasi-conformal curvature tensor of an $N(k)$-quasi Einstein manifold

In this section we consider an n-dimensional $N(k)$-quasi Einstein manifold M satisfying the condition $(\mathcal{C}_*(\xi,U)\cdot\mathcal{C}_*)(V,W)G=0$. Then we have

$$\begin{aligned}&\mathcal{C}_*(\xi,U)\,\mathcal{C}_*(V,W)G - \mathcal{C}_*(\mathcal{C}_*(\xi,U)V,W)G\\ &-\mathcal{C}_*(V,\mathcal{C}_*(\xi,U)W)G - \mathcal{C}_*(V,W)\,\mathcal{C}_*(\xi,U)G = 0.\end{aligned} \tag{14}$$

Using (10) in (14) we have

$$\frac{b}{n}\left[a_0+(n-2)\,a_1\right]\left[g\left(U,\mathcal{C}_*\left(V,W\right)G\right)\xi-\eta\left(\mathcal{C}_*\left(V,W\right)G\right)U\right.$$
$$-g\left(U,V\right)\mathcal{C}_*\left(\xi,W\right)G+\eta\left(V\right)\mathcal{C}_*\left(U,W\right)G$$
$$-g\left(U,W\right)\mathcal{C}_*\left(V,\xi\right)G+\eta\left(W\right)\mathcal{C}_*\left(V,U\right)G$$
$$\left.-g\left(U,G\right)\mathcal{C}_*\left(V,W\right)\xi+\eta\left(G\right)\mathcal{C}_*\left(V,W\right)U\right]=0.$$

In an $N(k)$-quasi Einstein manifold $b\neq 0$. So we obtain the following:

$$\left[a_0+(n-2)\,a_1\right]\left[g\left(U,\mathcal{C}_*\left(V,W\right)G\right)\xi-\eta\left(\mathcal{C}_*\left(V,W\right)G\right)U\right.$$
$$-g\left(U,V\right)\mathcal{C}_*\left(\xi,W\right)G+\eta\left(V\right)\mathcal{C}_*\left(U,W\right)G$$
$$-g\left(U,W\right)\mathcal{C}_*\left(V,\xi\right)G+\eta\left(W\right)\mathcal{C}_*\left(V,U\right)G$$
$$\left.-g\left(U,G\right)\mathcal{C}_*\left(V,W\right)\xi+\eta\left(G\right)\mathcal{C}_*\left(V,W\right)U\right]=0.$$

Then either $a_0+(n-2)\,a_1=0$ or,

$$g\left(U,\mathcal{C}_*\left(V,W\right)G\right)\xi-\eta\left(\mathcal{C}_*\left(V,W\right)G\right)U$$
$$-\,g\left(U,V\right)\mathcal{C}_*\left(\xi,W\right)G+\eta\left(V\right)\mathcal{C}_*\left(U,W\right)G$$
$$-\,g\left(U,W\right)\mathcal{C}_*\left(V,\xi\right)G+\eta\left(W\right)\mathcal{C}_*\left(V,U\right)G$$
$$-\,g\left(U,G\right)\mathcal{C}_*\left(V,W\right)\xi+\eta\left(G\right)\mathcal{C}_*\left(V,W\right)U=0. \tag{15}$$

Assume that $a_0+(n-2)\,a_1\neq 0$. Taking the inner product on both sides of (15) with ξ we get

$$g\left(U,\mathcal{C}_*\left(V,W\right)G\right)-\eta\left(\mathcal{C}_*\left(V,W\right)G\right)\eta\left(U\right)$$
$$-\,g\left(U,V\right)\eta\left(\mathcal{C}_*\left(\xi,W\right)G\right)+\eta\left(V\right)\eta\left(\mathcal{C}_*\left(U,W\right)G\right)$$
$$-\,g\left(U,W\right)\eta\left(\mathcal{C}_*\left(V,\xi\right)G\right)+\eta\left(W\right)\eta\left(\mathcal{C}_*\left(V,U\right)G\right)$$
$$-\,g\left(U,G\right)\eta\left(\mathcal{C}_*\left(V,W\right)\xi\right)+\eta\left(G\right)\eta\left(\mathcal{C}_*\left(V,W\right)U\right)=0. \tag{16}$$

Now using the equations (11) - (13) in (16) we have

$$g\left(U,\mathcal{C}_*\left(V,W\right)G\right)=\frac{b}{n}\left[a_0+(n-2)\,a_1\right]\left[g\left(U,V\right)g\left(W,G\right)-g\left(U,W\right)g\left(V,G\right)\right].$$

Then using (6) and (7) we can write

$$a_0\mathcal{R}\left(V,W,G,U\right)+a_1\left[\mathcal{S}\left(W,G\right)g\left(V,U\right)\right.$$
$$\left.-\mathcal{S}\left(V,G\right)g\left(W,U\right)+g\left(W,G\right)\mathcal{S}\left(V,U\right)-g\left(V,G\right)\mathcal{S}\left(W,U\right)\right]$$
$$-\,\frac{an+b}{n}\left(\frac{a_0}{n-1}+2a_1\right)\left[g\left(W,G\right)g\left(V,U\right)-g\left(V,G\right)g\left(W,U\right)\right]$$
$$=\frac{b}{n}\left[a_0+(n-2)\,a_1\right]\left[g\left(U,V\right)g\left(W,G\right)-g\left(U,W\right)g\left(V,G\right)\right]. \tag{17}$$

Contracting (17) over U and V we obtain

$$\mathcal{S}\left(W,G\right)=(a+b)\,g\left(W,G\right).$$

This is a contradiction as M^n is not Einstein. Thus we have $a_0+(n-2)\,a_1=0$. Conversely, if $a_0+(n-2)\,a_1=0$, then in view of (10) the manifold satisfies $\mathcal{C}_*\left(\xi,U\right)\cdot\mathcal{C}_*=0$.

Thus we can state the following theorem:

THEOREM 3.1. *Let M be an n-dimensional $N(k)$-quasi Einstein manifold. Then M satisfies the condition $\mathcal{C}_*(\xi, U) \cdot \mathcal{C}_* = 0$ if and only if $a_0 + (n-2)a_1 = 0$.*

4. $N(k)$-quasi Einstein manifold satisfying $\mathcal{S}(U, \xi) \cdot \mathcal{C}_* = 0$

Let us suppose that an $N(k)$-quasi Einstein manifold (M^n, g) satisfying the condition

$$(\mathcal{S}(U, \xi) \cdot \mathcal{C}_*)(V, W)G = 0. \tag{18}$$

Now, $(\mathcal{S}(U, \xi) \cdot \mathcal{C}_*)(V, W)G = ((U \wedge_{\mathcal{S}} \xi).\mathcal{C}_*)(V, W)G$, where the endomorphism $(U \wedge_{\mathcal{S}} V)W$ is defined by

$$(U \wedge_{\mathcal{S}} V)W = \mathcal{S}(V, W)U - \mathcal{S}(U, W)V. \tag{19}$$

Then (18) takes the form,

$$\begin{aligned}&(U \wedge_{\mathcal{S}} \xi)\mathcal{C}_*(V, W)G - \mathcal{C}_*((U \wedge_{\mathcal{S}} \xi)V, W)G\\ &-\mathcal{C}_*(V, (U \wedge_{\mathcal{S}} \xi)W)G - \mathcal{C}_*(V, W)(U \wedge_{\mathcal{S}} \xi)G = 0.\end{aligned} \tag{20}$$

From (19) and (20), we get

$$\begin{aligned}&\mathcal{S}(\xi, \mathcal{C}_*(V, W)G)U - \mathcal{S}(U, \mathcal{C}_*(V, W)G)\xi\\ &- \mathcal{S}(\xi, V)\mathcal{C}_*(U, W)G + \mathcal{S}(U, V)\mathcal{C}_*(\xi, W)G\\ &- \mathcal{S}(\xi, W)\mathcal{C}_*(V, U)G + \mathcal{S}(U, W)\mathcal{C}_*(V, \xi)G\\ &- \mathcal{S}(\xi, G)\mathcal{C}_*(V, W)U + \mathcal{S}(U, G)\mathcal{C}_*(V, W)\xi = 0.\end{aligned} \tag{21}$$

Using (1) and (8) in (21), we have

$$\begin{aligned}&(a+b)\eta(\mathcal{C}_*(V, W)G)U - ag(U, \mathcal{C}_*(V, W)G)\xi - b\eta(U)\eta(\mathcal{C}_*(V, W)G)\xi\\ &- (a+b)\eta(V)\mathcal{C}_*(U, W)G + [ag(U, V) + b\eta(U)\eta(V)]\mathcal{C}_*(\xi, W)G\\ &- (a+b)\eta(W)\mathcal{C}_*(V, U)G + [ag(U, W) + b\eta(U)\eta(W)]\mathcal{C}_*(V, \xi)G\\ &- (a+b)\eta(G)\mathcal{C}_*(V, W)U + [ag(U, G) + b\eta(U)\eta(G)]\mathcal{C}_*(V, W)\xi = 0.\end{aligned} \tag{22}$$

Taking the inner product on both sides of (22) with ξ, we obtain

$$\begin{aligned}&a\eta(\mathcal{C}_*(V, W)G)\eta(U) - ag(U, \mathcal{C}_*(V, W)G) - (a+b)\eta(V)\eta(\mathcal{C}_*(U, W)G)\\ &+ [ag(U, V) + b\eta(U)\eta(V)]\eta(\mathcal{C}_*(\xi, W)G) - (a+b)\eta(W)\eta(\mathcal{C}_*(V, U)G)\\ &+ [ag(U, W) + b\eta(U)\eta(W)]\eta(\mathcal{C}_*(V, \xi)G) - (a+b)\eta(G)\eta(\mathcal{C}_*(V, W)U)\\ &+ [ag(U, G) + b\eta(U)\eta(G)]\eta(\mathcal{C}_*(V, W)\xi) = 0.\end{aligned} \tag{23}$$

Using (9) and (11) - (13) in (23) we get

$$\begin{aligned}&aba_1[g(U, V)g(W, G) - g(U, W)g(V, G) - g(U, V)\eta(W)\eta(G)\\ &+g(W, U)\eta(V)\eta(G) - g(W, G)\eta(V)\eta(U) + g(V, G)\eta(W)\eta(U)]\\ &- \frac{b^2}{n}[a_0 + (n-2)a_1][g(W, U)\eta(V)\eta(G) - g(V, U)\eta(W)\eta(G)] = 0.\end{aligned} \tag{24}$$

Putting $W = \xi$ in (24), we obtain

$$\frac{b^2}{n}[a_0 + (n-2)a_1]\eta(G)[\eta(U)\eta(V) - g(U, V)] = 0. \tag{25}$$

Since in an $N(k)$-quasi Einstein manifold $b \neq 0$, the 1-form η is non-zero and $g(U,V) \neq \eta(U)\eta(V)$, from equation (25) it follows that $a_0 + (n-2)a_1 = 0$. Again, if we take $a_0 + (n-2)a_1 = 0$, then the converse is trivial.
This leads to the following theorem:

THEOREM 4.1. *An n-dimensional $N(k)$-quasi Einstein manifold M satisfies $\mathcal{S}(U,\xi)\cdot \mathcal{C}_* = 0$ if and only if $a_0 + (n-2)a_1 = 0$.*

Therefore, by Theorem 3.1. and 4.1. we can state the following corollary:

COROLLARY 4.2. *Let (M^n, g) be an n-dimensional $N(k)$-quasi Einstein manifold. Then the following statements are equivalent:*

(i) $\mathcal{C}_*(\xi, U)\cdot \mathcal{C}_* = 0$,
(ii) $\mathcal{S}(U,\xi)\cdot \mathcal{C}_* = 0$,
(iii) $a_0 + (n-2)a_1 = 0$,

for every vector field U on (M^n, g).

5. $\mathcal{C}_*$-pseudosymmetric $N(k)$-quasi Einstein manifolds

In [14], Özgür and Sular studied the condition $\mathcal{R}(\xi,U)\cdot \mathcal{C}_* = 0$ for an $N(k)$-quasi Einstein manifolds, where $\mathcal{C}_*$ is the quasi-conformal curvature tensor and $\mathcal{R}$ is the curvature tensor of the manifold. In this section we generalize this condition.

An n-dimensional Riemannian or a semi-Riemannian manifold (M^n, g) is said to be $\mathcal{C}_*$-pseudosymmetric [10] if and only if the tensors $\mathcal{R}\cdot\mathcal{C}_*$ and $\tilde{Q}(g,\mathcal{C}_*)$ defined by

$$\begin{aligned}(\mathcal{R}(U,V)\cdot\mathcal{C}_*)(W,G)H &= \mathcal{R}(U,V)\mathcal{C}_*(W,G)H - \mathcal{C}_*(\mathcal{R}(U,V)W,G)H\\ &\quad - \mathcal{C}_*(W,\mathcal{R}(U,V)G)H - \mathcal{C}_*(W,G)\mathcal{R}(U,V)H\end{aligned} \tag{26}$$

and

$$\begin{aligned}\tilde{Q}(g,\mathcal{C}_*)(W,G,H;U,V) &= ((U\wedge V)\cdot\mathcal{C}_*)(W,G)H\\ &= (U\wedge V)\mathcal{C}_*(W,G)H - \mathcal{C}_*((U\wedge V)W,G)H\\ &\quad - \mathcal{C}_*(W,(U\wedge V)G)H - \mathcal{C}_*(W,G)(U\wedge V)H\end{aligned} \tag{27}$$

are linearly dependent, i.e.,

$$(\mathcal{R}(U,V)\cdot\mathcal{C}_*)(W,G)H = f\tilde{Q}(g,\mathcal{C}_*)(W,G,H;U,V), \tag{28}$$

for arbitrary vector fields U,V,W,G,H on M^n and the endomorphism $(U\wedge V)$ is defined by

$$(U\wedge V)W = g(V,W)U - g(U,W)V \tag{29}$$

and f is a smooth function on $\Omega_{\mathcal{C}_*} = \{x\in M^n : \mathcal{C}_* \neq 0 \text{ at } x\}$.
If $f = 0$, then the manifold (M^n, g) reduces to a quasi-conformally semisymmetric manifold (i.e. $\mathcal{R}\cdot\mathcal{C}_* = 0$).

From (26), (27) and (28) we have

$$\begin{aligned}&\mathcal{R}(U,V)\,\mathcal{C}_*(W,G)\,H-\mathcal{C}_*(\mathcal{R}(U,V)\,W,G)\,H\\&-\mathcal{C}_*(W,\mathcal{R}(U,V)\,G)\,H-\mathcal{C}_*(W,G)\,\mathcal{R}(U,V)\,H\\&=f\,[(U\wedge V)\,\mathcal{C}_*(W,G)\,H-\mathcal{C}_*((U\wedge V)\,W,G)\,H\\&-\mathcal{C}_*(W,(U\wedge V)\,G)\,H-\mathcal{C}_*(W,G)\,(U\wedge V)\,H]\,.\end{aligned}\tag{30}$$

Putting $U=\xi$ in (30) and then using (2), (5) and (29), we obtain that

$$\begin{aligned}(k-f)\,[&g(V,\mathcal{C}_*(W,G)\,H)\,\xi-\eta(\mathcal{C}_*(W,G)\,H)\,V\\&-g(V,W)\,\mathcal{C}_*(\xi,G)\,H+\eta(W)\,\mathcal{C}_*(V,G)\,H\\&-g(V,G)\,\mathcal{C}_*(W,\xi)\,H+\eta(G)\,\mathcal{C}_*(W,V)\,H\\&-g(V,H)\,\mathcal{C}_*(W,G)\,\xi+\eta(H)\,\mathcal{C}_*(W,G)\,V]=0,\end{aligned}$$

which implies either $f=k$ or

$$\begin{aligned}&g(V,\mathcal{C}_*(W,G)\,H)\,\xi-\eta(\mathcal{C}_*(W,G)\,H)\,V\\&-g(V,W)\,\mathcal{C}_*(\xi,G)\,H+\eta(W)\,\mathcal{C}_*(V,G)\,H\\&-g(V,G)\,\mathcal{C}_*(W,\xi)\,H+\eta(G)\,\mathcal{C}_*(W,V)\,H\\&-g(V,H)\,\mathcal{C}_*(W,G)\,\xi+\eta(H)\,\mathcal{C}_*(W,G)\,V=0.\end{aligned}\tag{31}$$

Taking the inner product on both sides of (31) with ξ, we get

$$\begin{aligned}&g(V,\mathcal{C}_*(W,G)\,H)-\eta(\mathcal{C}_*(W,G)\,H)\,\eta(V)\\&-g(V,W)\,\eta(\mathcal{C}_*(\xi,G)\,H)+\eta(W)\,\eta(\mathcal{C}_*(V,G)\,H)\\&-g(V,G)\,\eta(\mathcal{C}_*(W,\xi)\,H)+\eta(G)\,\eta(\mathcal{C}_*(W,V)\,H)\\&-g(V,H)\,\eta(\mathcal{C}_*(W,G)\,\xi)+\eta(H)\,\eta(\mathcal{C}_*(W,G)\,V)=0.\end{aligned}\tag{32}$$

By virtue of (11) - (13) we obtain from (32) that

$$g(V,\mathcal{C}_*(W,G)\,H)=\frac{b}{n}\,[a_0+(n-2)\,a_1]\,[g(V,W)\,g(G,H)-g(V,G)\,g(W,H)]\,.\tag{33}$$

Using (6) and (7), (33) can be written as

$$\begin{aligned}&a_0\mathcal{R}(W,G,H,V)+a_1\,[\mathcal{S}(G,H)\,g(W,V)\\&-\mathcal{S}(W,H)\,g(G,V)+g(G,H)\,\mathcal{S}(W,V)-g(W,H)\,\mathcal{S}(G,V)]\\&-\frac{an+b}{n}\left(\frac{a_0}{n-1}+2a_1\right)[g(G,H)\,g(W,V)-g(W,H)\,g(G,V)]\\&=\frac{b}{n}\,[a_0+(n-2)\,a_1]\,[g(V,W)\,g(G,H)-g(V,G)\,g(W,H)]\,.\end{aligned}\tag{34}$$

Putting $V=W=e_i$ in (34), where $\{e_i\}$, $i=1,2,\ldots,n$ be an orthonormal basis of the tangent space at any point of the manifold (M^n,g) and taking summation over i, $1\le i\le n$, we have

$$[a_0+(n-2)\,a_1]\,[\mathcal{S}(G,H)-(a+b)\,g(G,H)]=0.$$

Since M^n is an $N(k)$-quasi Einstein manifold, $\mathcal{S}(G,H)\neq(a+b)\,g(G,H)$. So we obtain

$$a_0+(n-2)\,a_1=0.$$

Therefore, from (11)

$$\eta\left(\mathcal{C}_*\left(U,V\right)W\right)=0. \tag{35}$$

Using (35) in (32) yields

$$g\left(V,\mathcal{C}_*\left(W,G\right)H\right)=0.$$

This implies that the manifold is quasi-conformally flat. But, in this case $\mathcal{C}_* \neq 0$. Hence $f=k$, i.e., $f=\dfrac{a+b}{n-1}$.

Thus we conclude the following theorem:

THEOREM 5.1. *In a $\mathcal{C}_*$-pseudosymmetric $N\left(k\right)$-quasi Einstein manifold $f=\dfrac{a+b}{n-1}$.*

We know that [1] a quasi-conformally flat manifold is either conformally flat or Einstein.

In [14], authors proved the following corollary:

COROLLARY 5.2. *An $N\left(k\right)$-quasi Einstein manifold is quasi-conformally semisymmetric if and only if either $a+b=0$ or the manifold is conformally flat with $a_0=\left(2-n\right)a_1$.*

Now if we take $f\neq k$, then in view of (1) and (34) we have

$$\begin{aligned}\mathcal{R}\left(W,G,H,V\right)&=\lambda\left[g\left(G,H\right)g\left(W,V\right)-g\left(W,H\right)g\left(G,V\right)\right]\\&\quad+\mu\left[g\left(W,V\right)\eta\left(G\right)\eta\left(H\right)-g\left(G,V\right)\eta\left(W\right)\eta\left(H\right)\right.\\&\quad\left.+g\left(G,H\right)\eta\left(W\right)\eta\left(V\right)-g\left(W,H\right)\eta\left(G\right)\eta\left(V\right)\right],\end{aligned} \tag{36}$$

where $\lambda=\left(k+\dfrac{ba_1}{a_0}\right)$ and $\mu=-\dfrac{ba_1}{a_0}$.

A Riemannian or semi-Riemannian manifold is said to be a manifold of quasi-constant curvature [4] if the curvature tensor $\mathcal{R}$ of type $(0,4)$ satisfies the following condition

$$\begin{aligned}\mathcal{R}\left(U,V,W,G\right)&=p\left[g\left(V,W\right)g\left(U,G\right)-g\left(U,W\right)g\left(V,G\right)\right]\\&\quad+q\left[g\left(U,G\right)\eta\left(V\right)\eta\left(W\right)-g\left(U,W\right)\eta\left(V\right)\eta\left(G\right)\right.\\&\quad\left.+g\left(V,W\right)\eta\left(U\right)\eta\left(G\right)-g\left(V,G\right)\eta\left(U\right)\eta\left(W\right)\right],\end{aligned} \tag{37}$$

where p,q are scalar functions of which $q\neq 0$ and η is a non-zero 1-form defined by

$$g\left(U,\xi\right)=\eta\left(U\right),$$

for all U and ξ being a unit vector field.

From (36) and (37), we can state the following theorem:

THEOREM 5.3. *An n-dimensional $\mathcal{C}_*$-pseudosymmetric $N\left(k\right)$-quasi Einstein manifold $\left(M^n,g\right)$, $(n>2)$ with $f\neq k$ is a manifold of quasi-constant curvature.*

6. Example of $N\left(k\right)$-quasi Einstein manifolds

Let $(x^1,x^2,\ldots,x^n)\in\mathbb{R}^n$, where $\mathbb{R}^n$ is an n-dimensional real number space. We consider a Riemannian metric g on $\mathbb{R}^4=\left(x^1,x^2,x^3,x^4\right)$, by

$$ds^2=g_{ij}dx^idx^j=\left(dx^1\right)^2+\left(x^1\right)^2\left(dx^2\right)^2+\left(x^2\right)^2\left(dx^3\right)^2+\left(dx^4\right)^2, \tag{38}$$

where $i, j = 1, 2, 3, 4$. Using (38), we see the non-vanishing components of Riemannian metric are

$$g_{11} = 1, \ \ g_{22} = \left(x^1\right)^2, \ \ g_{33} = \left(x^2\right)^2, \ \ g_{44} = 1 \tag{39}$$

and its associated components are

$$g^{11} = 1, \ \ g^{22} = \frac{1}{\left(x^1\right)^2}, \ \ g^{33} = \frac{1}{\left(x^2\right)^2}, \ \ g^{44} = 1. \tag{40}$$

Using (39) and (40), we can calculate that the non-vanishing components of Christoffel symbols, curvature tensor and Ricci tensor are given by

$$\Gamma^1_{22} = -x^1, \ \ \Gamma^2_{33} = -\frac{x^2}{\left(x^1\right)^2}, \ \ \Gamma^2_{12} = \frac{1}{x^1}, \ \ \Gamma^3_{23} = \frac{1}{x^2}, \ \ R_{1332} = -\frac{x^2}{x^1}, \ \ S_{12} = -\frac{1}{x^1 x^2}$$

and the other components are obtained by the symmetric properties. It can be easily shown that the scalar curvature r of the resulting manifold $(\mathbb{R}^4, g)$ is zero. We shall now show that this $(\mathbb{R}^4, g)$ is an $N(k)$-quasi Einstein manifold.
Let us consider the associated scalars as follows:

$$a = \frac{1}{x^1\left(x^2\right)^2}, \qquad b = -\frac{2}{\left(x^1\right)^2 x^2}. \tag{41}$$

We choose the 1-form as follows:

$$\eta_i(x) = \begin{cases} \dfrac{1}{\sqrt{2}}, & \text{when } i = 1 \\ \dfrac{x^1}{\sqrt{2}}, & \text{when } i = 2 \\ 0, & \text{otherwise} \end{cases} \tag{42}$$

at any point $x \in \mathbb{R}^4$. Now the equation (1) reduces to the equation

$$S_{12} = a g_{12} + b\eta_1\eta_2, \tag{43}$$

since, for the other cases (1) holds trivially.
From the equations (41), (42) and (43) we get

$$\begin{aligned} \text{Right hand side of (43)} &= a g_{12} + b\eta_1\eta_2 \\ &= \frac{1}{x^1\left(x^2\right)^2}\cdot 0 + \left(-\frac{2}{\left(x^1\right)^2 x^2}\right)\cdot\left(\frac{1}{\sqrt{2}}\right)\cdot\left(\frac{x^1}{\sqrt{2}}\right) \\ &= -\frac{1}{x^1 x^2} = S_{12}. \end{aligned}$$

By Lemma 1.1., here we see that $k = \dfrac{x^1 - 2x^2}{3\left(x^1\right)^2\left(x^2\right)^2}$.
We shall now show that the 1-form η_i are unit.
Here,

$$g^{ij}\eta_i\eta_j = 1.$$

So, $(\mathbb{R}^4, g)$ is an $N\left(\dfrac{x^1 - 2x^2}{3\left(x^1\right)^2\left(x^2\right)^2}\right)$-quasi Einstein manifold.

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