

A MODIFIED PREY-PREDATOR MODEL WITH COUPLED RATES OF CHANGE

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ABSTRACT. The prey–predator model is one of the most influential mathematical models in ecology and evolutionary biology. In this study, we considered a modified prey–predator model, which describes the rate of change for each species. The effects of modifications to the classical prey–predator model are investigated here. The conditions required for the existence of the first integral and the stability of the fixed points are studied. In particular, it is shown that the first integral exists only for a subset of the model parameters, and the phase portraits around the fixed points exhibit physically relevant phenomena over a wide range of the parameter space. The results show that adding coupling terms to the classical model widely expands the dynamics with great potential for applicability in real-world phenomena.

1. INTRODUCTION

The prey-predator system was first introduced by Lotka in 1920 as a model to analyze prey-predator interactions in biology [1]; it was later applied by Volterra to treat similar problems in ecology [2]. This model describes a theoretical phenomenon in which the population of one species decreases as the population of the another species increases. References. [3, 4] describe in depth the significance of the classic model.

However, in the natural world, while there exist cases in which populations change at a slowly, there are also cases where a population will suddenly become extinct or increase due to the influence of the environment. For example, additional food for a predator can cause a fear effect of the predator on the prey, and thus reduce reproduction rate of the prey.

Recently, there has been a considerable progress in the study of the prey-predator system with the fear effect [5]. It has been found that the costs of anti-predator behavior induced by the fear effect significantly affects the population of the prey species [6, 7]. The fear effect induces various changes, including habitat changes, hunting, and increased caution, which leads to a

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reduction in reproduction rate of the prey [8]. In this work, we consider the fear effect when the predator is provided with additional food. Additional food for the predator causes an increase in the predator population, and eventually this causes a decrease in the prey population [5, 9]. By modifying the prey–predator model, we consider large and/or rapid changes in species, which reflect the environmental factors in the real world. We achieve this by adding a term encoding information regarding the rate of change of the population to the classical model, and we study the difference between this modified model and the classical model.

The objective of our work is to study the behaviors of the modified prey–predator model and establish the effects of several parameters that play an important role in the model. We also analyze the stability behavior of fixed points. We find that while the original model represents closed curves, the modified model represents open curves or spiral dynamics around the fixed points. It is possible to understand the difference between the classical and modified models through the existence condition of the first integral and the existence of fixed point which were not present before.

The remainder of this paper is organized as follows:

In section 2, the stability of the classical model is discussed. Section 3 introduces the modified model (3.1) and its first integral. The stability behavior of the modified model is analyzed in section 4.

2. THE CLASSICAL PREY-PREDATOR MODEL

The original prey-predator model is described by the following system of differential equations, where x and y are the populations of prey and predator, respectively.

$$M_{old} : \begin{cases} x' = \alpha x - \beta xy, \\ y' = \delta xy - \gamma y \end{cases} \tag{2.1}$$

The parameters $\alpha, \beta, \delta, \gamma$ are positive constants and the primes denote time derivatives. From Eq. (2.1), we can immediately obtain the following:

$$\frac{dy}{dx} = -\frac{y}{x} \frac{\delta x - \gamma}{\beta y - \alpha} \tag{2.2}$$

After modifying Eq. (2.2) to the following variable-separated form, we obtain,

$$\frac{\beta y - \alpha}{y} dy + \frac{\delta x - \gamma}{x} dx = 0,$$

Thus, we can obtain the first integral V of the model M_{old} :

$$V = \delta x - \gamma \ln x + \beta y - \alpha \ln y.$$

We consider the fixed points of the system M_{old} , which can be obtained by solving the following expressions:

$$\begin{cases} x(\alpha - \beta y) = 0 \\ -y(\gamma - \delta x) = 0 \end{cases}$$

By solving the above equations, we can immediately obtain the two fixed points of M_{old} , these are found to be at $(0, 0)$ and $(\frac{\gamma}{\delta}, \frac{\alpha}{\beta})$. Next, we investigate the stability of the fixed points. We note that the Jacobian matrix J of the system M_{old} is given by,

$$J(x, y) = \begin{bmatrix} \alpha - \beta y & -\beta x \\ \delta y & \delta x - \gamma \end{bmatrix}.$$

For the fixed point at $(0, 0)$, the Jacobian matrix is given by:

$$J(0, 0) = \begin{bmatrix} \alpha & 0 \\ 0 & -\gamma \end{bmatrix},$$

which implies that this fixed point is hyperbolic, since one real eigenvalue α is positive and the other one $-\gamma$ is negative, i.e., it is a saddle point.

In the case of the other fixed point at $(\frac{\gamma}{\delta}, \frac{\alpha}{\beta})$, the Jacobian matrix is given by,

$$J\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right) = \begin{bmatrix} 0 & -\frac{\beta\gamma}{\delta} \\ \frac{\alpha\delta}{\beta} & 0 \end{bmatrix},$$

The eigenvalues of this matrix can be obtained as $\pm i\sqrt{\alpha\gamma}$, which indicates that this system behaves periodically, acting as an oscillator.

3. FIRST INTEGRAL OF THE MODIFIED MODEL

The previous model has a limitation in that it cannot explain some of the phenomena observed in nature. For this reason, we consider an extension of this model. We aim to consider cases in which the two populations increase and decrease rapidly, as well as the slower variation that are captured in the original model. The following equations represent the modified prey–predator model, which link the rate of changes in the populations of the two species:

$$M_{\text{new}} : \begin{cases} x' = \alpha x - \beta xy + \epsilon xy', \\ y' = \delta xy - \gamma y + \lambda x'y, \end{cases} \quad (3.1)$$

where the parameters α, β, γ and δ are positive constants as before, whereas the new parameters ϵ and λ are arbitrary nonzero ones. We also consider the case in which the coupling effect of the rate of changes in the populations completely replaces that of the coupling terms of the previous model (i.e. $\beta = \delta=0$). Additionally note that the sign of parameter ϵ indicates that increases (decreases) in y' induces decreases (increases) in x' .

Rearranging Eq. (3.1), and assuming $1 - \epsilon\lambda xy \neq 0$, we obtain,

$$M_{\text{new}}^* : \begin{cases} x' = \frac{\alpha x - \beta xy + \epsilon\delta x^2 y - \epsilon\gamma xy}{1 - \epsilon\lambda xy}, \\ y' = \frac{\delta xy - \gamma y + \alpha\lambda xy - \beta\lambda xy^2}{1 - \epsilon\lambda xy}. \end{cases} \quad (3.2)$$

The possible phase portraits for several cases of ϵ and λ are shown in Fig. 1.

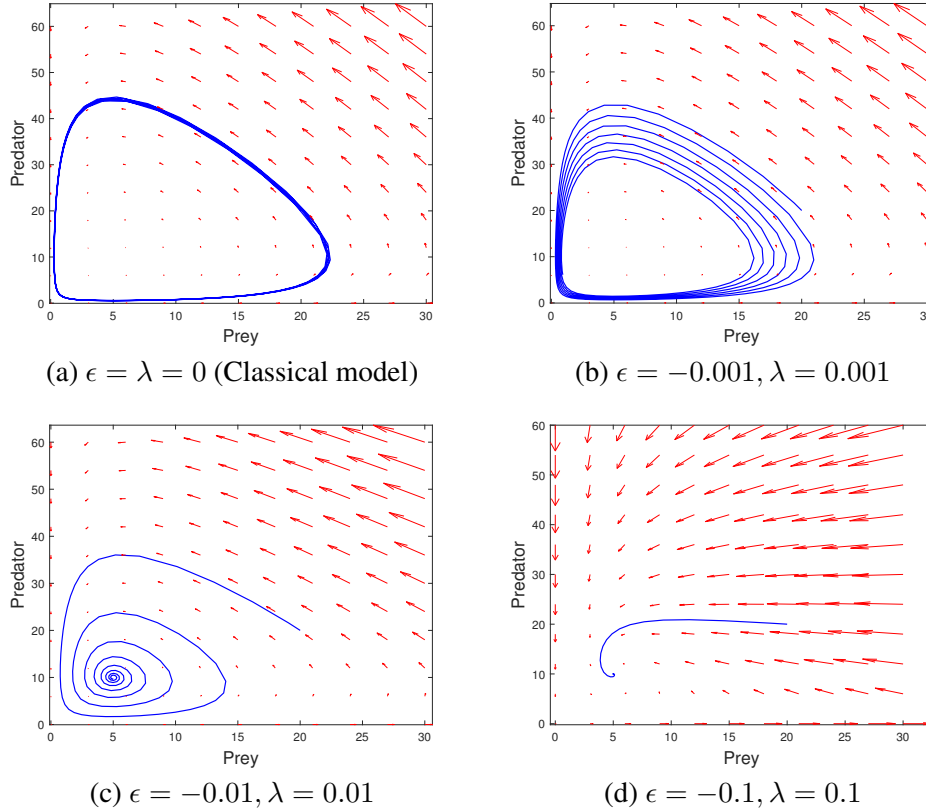


FIGURE 1. Phase plots of the system described by Eq. (3.1) for various values of the new parameters ϵ and λ at $\alpha = 0.1, \beta = 0.01, \delta = 0.02$ and $\gamma = 0.1$.

Here, we consider the first integral of the modified system M_{new}^* . From Eq. (3.2), we obtain

$$\frac{dy}{dx} = \frac{\delta xy - \gamma y + \alpha \lambda xy - \beta \lambda xy^2}{\alpha x - \beta xy + \epsilon \delta x^2 y - \epsilon \gamma xy} = \frac{y(\delta x - \gamma + \alpha \lambda x - \beta \lambda xy)}{x(\alpha - \beta y + \epsilon \delta xy - \epsilon \gamma y)}, \text{ i.e.}$$

$$-\frac{\delta x - \gamma + \alpha \lambda x - \beta \lambda xy}{x} dx + \frac{\alpha - \beta y + \epsilon \delta xy - \epsilon \gamma y}{y} dy = 0. \tag{3.3}$$

Since this equation is not in the variable-separated form, it is not trivial to obtain the first integral of this system. Thus, we consider the conditions in which the first integral exists.

Theorem 3.1. *For the modified prey-predator system of the system M_{new}^* , if $\beta \lambda - \epsilon \delta = 0$, the first integral exists.*

Proof. Let $V(x, y)$ be the first integral of the system M_{new}^* . $\frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy = 0$ must be satisfied along each solution curve that satisfies Eq. (3.2). Comparing this with Eq. (3.3), we assume that there exists a function V satisfying

$$\begin{aligned}\frac{\partial V}{\partial x} &= -\alpha\lambda + \beta\lambda y - \delta + \frac{\gamma}{x} \\ \frac{\partial V}{\partial y} &= \frac{\alpha}{y} - \beta + \epsilon\delta x - \epsilon\gamma\end{aligned}\tag{3.4}$$

We set $V = F + C(x)$, where $C(x)$ is an arbitrary differentiable function and

$$F = \int \frac{\partial V}{\partial y} dy = \int \left(\frac{\alpha}{y} - \beta + \epsilon\delta x - \epsilon\gamma \right) dy = \alpha \ln y - \beta y + \epsilon\delta xy - \epsilon\gamma y,$$

thus, $V = \alpha \ln y - \beta y + \epsilon\delta xy - \epsilon\gamma y + C(x)$. Then, by comparing the partial derivative of V with respect to x and using Eq. (3.4), we obtain

$$\begin{aligned}\frac{\partial V}{\partial x} &= \epsilon\delta y + C'(x) \\ &= -\alpha\lambda + \beta\lambda y - \delta + \frac{\gamma}{x}.\end{aligned}$$

Thus, it must hold that

$$(\beta\lambda - \epsilon\delta)y = 0 \quad \text{and} \quad C'(x) = -\alpha\lambda - \delta + \frac{\gamma}{x}.$$

Therefore, since $\beta\lambda - \epsilon\delta = 0$, we can obtain the first integral V of the system M_{new}^* which is given by,

$$V = \alpha \ln y - \beta y + \epsilon\delta xy - \epsilon\gamma y - (\alpha\lambda + \delta)x + \gamma \ln x.\tag{3.5}$$

□

From Theorem 3.1, we see that, even though we have perturbed the original prey-predator model by coupling the rates of change of the populations, if the ratio of this couplings satisfies $\frac{\beta}{\delta} = \frac{\epsilon}{\lambda}$, the first integral exists in the modified system. (See Fig. 2.) However, in the singular case of $\beta = \delta = 0$, as shown in Fig. 3, the topological properties of the solution curves in the modified system are significantly different from those of the original model, even though the first integral of the modified system exists.

In addition, for a given pair of positive parameters β and δ , considering $\lambda = \frac{\delta}{\beta}\epsilon$, we see that if $\epsilon < 0, \lambda > 0$, the existence of the first integral cannot be guaranteed; this signifies that the sign of the parameters ϵ and λ has an important role in guaranteeing the existence of the first integral. If $\epsilon \cdot \lambda > 0$ (i.e. ϵ and λ have the same sign), we see that the first integral of the modified system exists whenever $\lambda = \frac{\delta}{\beta}\epsilon$. However, we note that, even in these well-behaved cases, in order for the dynamics of the modified system to have physical sense, interpretations of the signs of the new parameters must be obtained. For example, when $\epsilon > 0$ as shown in Fig. 2, we can assume that the rate of change in the population of the prey exhibits an additional reaction (on top of that described by the original model) with the same sign to the change in population of the predator. Thus, when considering the most simple unilateral predator-prey relationship, excluding cooperation, competition, or reversible predator-prey relationship between two groups, it is natural to say that the population change of the predator affects the

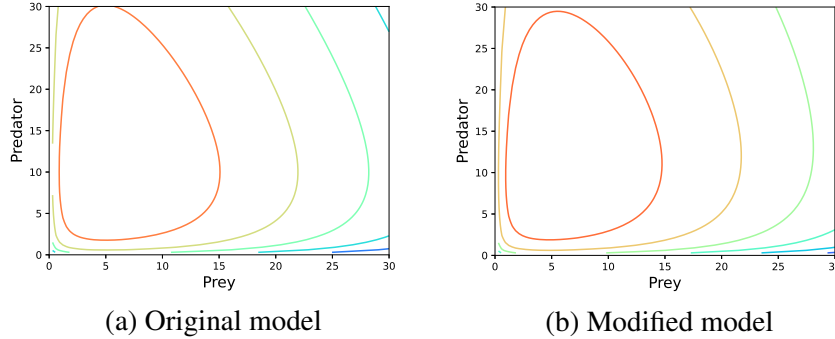


FIGURE 2. Level curves of the first integrals for $\epsilon > 0$. The parameters used in this plot are $\alpha = 0.1, \beta = 0.01, \delta = 0.02, \gamma = 0.1, \epsilon = 0.005$ and $\lambda = 0.01$; these parameters satisfy the condition $\beta\lambda - \epsilon\delta = 0$.

population change of the prey in the opposite direction, (an increase (decrease) in the predator population causes a decrease (increase) in the prey population), and the population change of the prey affects the population change of the predator in the same direction. For this reason, in this paper, we focus predominantly on the case of $\epsilon < 0$ and $\lambda > 0$.

The following corollary illustrates what happens in the modified system when the original coupling terms are set to zero ($\beta = \delta = 0$) while retaining the new coupling terms of the modified system; understanding this dynamics helps explain the effects of the new coupling terms.

Corollary 3.2. *When $\beta = \delta = 0$, if $\epsilon < 0$ and $\lambda > 0$, there exists no fixed points of the system M_{new}^* , other than the trivial fixed point at $(0, 0)$.*

Proof. Suppose that $\beta = \delta = 0$. Then the system M_{new}^* can be simplified into the following equations:

$$x' = \frac{\alpha x - \epsilon \gamma x y}{1 - \epsilon \lambda x y}, \quad y' = \frac{-\gamma y + \alpha \lambda x y}{1 - \epsilon \lambda x y} \tag{3.6}$$

In order to find the fixed points, we need to find the values of (x, y) that satisfies:

$$x(\alpha - \epsilon \gamma y) = 0 \quad \text{and} \quad y(\alpha \lambda x - \gamma) = 0;$$

these expressions lead to the following 4 cases:

- (i) $x = 0$ and $y = 0$, (ii) $x = 0$ and $\alpha \lambda x - \gamma = 0$,
- (iii) $y = 0$ and $\alpha - \epsilon \gamma y = 0$, (iv) $x = \frac{\gamma}{\alpha \lambda}$ and $y = \frac{\alpha}{\epsilon \gamma}$.

Recall that γ and α are positive in the original predator-prey model, which means that (ii) and (iii) cases cannot be satisfied. Furthermore, since the population of species $x = \frac{\gamma}{\alpha \lambda}$ and

$y = \frac{\alpha}{\epsilon\gamma}$ must be non-negative, case (iv) is not realizable. Thus, only the first case holds, i.e., there exists only the trivial fixed point $(0, 0)$. □

Note that, by Theorem 3.1 and Eq. (3.5), this modified system ($\beta = \delta = 0$) has the first integral V given by $V = \alpha \ln y - \epsilon\gamma y - \alpha\lambda x + \gamma \ln x$. In addition, by Corollary 3.2, this system has no fixed point in the first quadrant of \mathbb{R}^2 (i.e., $x > 0, y > 0$). These two properties imply that this new system has no periodic solutions, that is, all the periodic solutions in the original predator-prey model disappear, and since

$$\frac{dy}{dx} = \frac{\alpha\lambda - \frac{\gamma}{x}}{\frac{\alpha}{y} - \epsilon\gamma},$$

all the solution curves of this new system are oblique curves similar to $y = \frac{1}{x}$ with a slight deformation such that they have one critical point, as shown in Fig. 3.

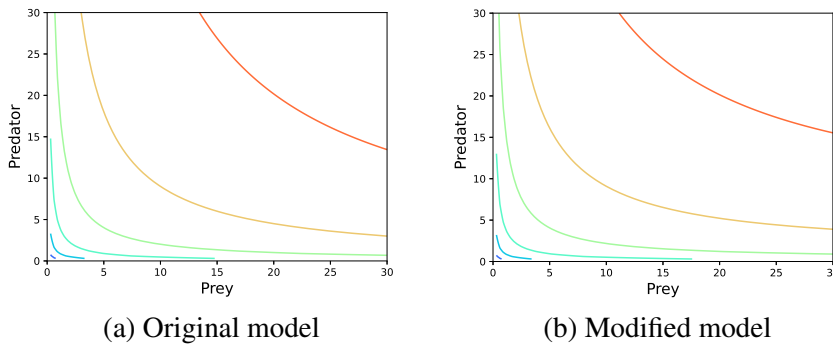


FIGURE 3. Level curves of the first integrals for $\beta = \delta = 0, \epsilon < 0$ and $\lambda > 0$. The parameters used in this plot are $\alpha = \gamma = 0.1, \epsilon = -0.01$, and $\lambda = 0.01$;

Next, we consider the following equations which are equivalent to the system M_{new} :

$$\begin{cases} x' = \alpha x - x(\beta y - \epsilon y'), \\ y' = (\delta x + \lambda x')y - \gamma y \end{cases} \tag{3.7}$$

Comparing these equations with Eq. (2.1), we see that the modified system M_{new} is a model obtained from the original predator-prey model, M_{old} , by adding additional coupling terms of the population change rates to those of the original model. Thus, the systems given by Eq. (2.1) and Eq. (3.6) are the two extremes of the modified model considered here. The question thus arises, how different are the dynamics of the new model given by Eq. (3.7) from those of the original predator-prey model? This is the subject of the next section.

4. STABILITY OF THE MODIFIED PREY-PREDATOR MODEL

In the previous section, we added additional coupling terms describing a dependence on the population change rates to the original predator-prey model, and we observed what changes occur in the dynamics of the modified model in some extreme cases. Here, we focus on the changes in the fixed points and their properties, and we consider what happens in the modified model over a wider range of model parameters.

We first examine the fixed points of the modified model. From Eq. (3.7), we see that, in order to obtain the fixed points, the following equations must be satisfied:

$$\begin{cases} x(\alpha - \beta y + \epsilon \delta x y - \epsilon \gamma y) = 0 \\ y(\delta x - \gamma + \alpha \lambda x - \beta \lambda x y) = 0 \end{cases} \tag{4.1}$$

From Eq. (4.1), we immediately see that (0, 0) is also the trivial fixed point of the modified model. Next, rearranging Eq. (4.1) to

$$\begin{cases} x((\alpha - \beta y) + \epsilon y(\delta x - \gamma)) = 0 \\ y((\delta x - \gamma) + \lambda x(\alpha - \beta y)) = 0, \end{cases}$$

we find an additional fixed point at $(\frac{\gamma}{\delta}, \frac{\alpha}{\beta})$, this signifies that the non-trivial fixed point of the original model is retained in the modified model. That is, the additional coupling of the population change rates does not remove the non-trivial fixed point of the original model.

To obtain other possible fixed points, we consider the following equations:

$$\alpha - \beta y + \epsilon \delta x y - \epsilon \gamma y = 0 \tag{4.2}$$

$$\delta x - \gamma + \alpha \lambda x - \beta \lambda x y = 0. \tag{4.3}$$

By multiplying Eqs. (4.2) and (4.3) by $\beta \lambda$ and $\epsilon \delta$ respectively, we obtain

$$\alpha \beta \lambda - \beta^2 \lambda y + \epsilon \delta \beta \lambda x y - \beta \epsilon \gamma \lambda y = 0 \tag{4.4}$$

$$\epsilon \delta^2 x - \gamma \epsilon \delta + \alpha \epsilon \delta \lambda x - \epsilon \delta \beta \lambda x y = 0. \tag{4.5}$$

Here, adding Eqs. (4.4) and (4.5), we obtain

$$(\alpha \beta \lambda - \gamma \epsilon \delta) + \epsilon \delta (\delta + \alpha \lambda) x - \beta \lambda (\beta + \epsilon \gamma) y = 0,$$

i.e.,

$$y = \frac{(\alpha \beta \lambda - \gamma \epsilon \delta) + \epsilon \delta (\delta + \alpha \lambda) x}{\beta \lambda (\beta + \epsilon \gamma)}. \tag{4.6}$$

Next, by substituting Eq. (4.6) into Eq. (4.3), we obtain an expression that must be satisfied by any remaining fixed points:

$$\delta x - \gamma + \alpha \lambda x - \frac{(\alpha \beta \lambda - \gamma \epsilon \delta) x + \epsilon \delta (\delta + \alpha \lambda) x^2}{\beta + \epsilon \gamma} = 0,$$

i.e.,

$$\delta (\beta + \epsilon \gamma) x - \gamma (\beta + \epsilon \gamma) + \alpha \lambda (\beta + \epsilon \gamma) x - (\alpha \beta \lambda - \gamma \epsilon \delta) x - \epsilon \delta (\delta + \alpha \lambda) x^2 = 0. \tag{4.7}$$

From Eq. (4.7), we obtain,

$$(\epsilon(\delta + \alpha\lambda)x - \beta - \gamma\epsilon)(\delta x - \gamma) = 0.$$

Consequently, we obtain two fixed points; $(\frac{\beta+\gamma\epsilon}{\epsilon(\delta+\alpha\lambda)}, \frac{\alpha\lambda+\delta}{\lambda(\beta+\epsilon\gamma)})$ and $(\frac{\gamma}{\delta}, \frac{\alpha}{\beta})$; one is the nontrivial fixed point of the original model and the other is a new fixed point. Then here, note that it is required that another constraint $(\beta + \epsilon\gamma) < 0$, for the x -coordinate value of the new fixed point to be positive, to make sense, if $\epsilon < 0$ and $\lambda > 0$. However, in this case, the y -coordinate value of the new fixed point is negative, which means that the first fixed point $(\frac{\beta+\gamma\epsilon}{\epsilon(\delta+\alpha\lambda)}, \frac{\alpha\lambda+\delta}{\lambda(\beta+\epsilon\gamma)})$ can not be involved in the domain of nontrivial fixed points. Therefore, when $\epsilon < 0$ and $\lambda > 0$, our modified model has only one nontrivial fixed point $(\frac{\gamma}{\delta}, \frac{\alpha}{\beta})$ that is the same one as in the original model:

Theorem 4.1. *The modified prey-predator model M_{new}^* formally has three fixed points. If $\epsilon < 0$ and $\lambda > 0$, there exists only one non-trivial fixed point, which is the same one as in the original prey-predator model M_{old} .*

Remark 4.2. On the other hand, if we can find some specific phenomena with $\epsilon > 0$ to which our modified model can be applied, we can obtain one more fixed point $\mathbf{f}_s = (\frac{\beta+\gamma\epsilon}{\epsilon(\delta+\alpha\lambda)}, \frac{\alpha\lambda+\delta}{\lambda(\beta+\epsilon\gamma)})$. Of course, we can consider the other cases: $\epsilon > 0$ and $\lambda < 0$; $\epsilon > 0$ and $\lambda > 0$; $\epsilon < 0$ and $\lambda < 0$. However, even though these all are expected to show interesting dynamics, the most important key is to discover real objects and their interactions that meet those conditions. We focus on the closest extended model ($\epsilon < 0$ and $\lambda > 0$) to the original one, in the sense that the predation-prey relationship in coupling term is maintained in the same way as before. Moreover, the new fixed point $\mathbf{f}_s = (\frac{\beta+\gamma\epsilon}{\epsilon(\delta+\alpha\lambda)}, \frac{\alpha\lambda+\delta}{\lambda(\beta+\epsilon\gamma)})$ has one more serious problem: Even though \mathbf{f}_s is included in the first quadrant of the xy -plane, \mathbf{f}_s is a singular point to make the system defined by Eq. (3.2) singular, since $(1 - \epsilon\lambda xy)|_{\mathbf{f}_s} = 0$, which means that we can not handle the dynamics near \mathbf{f}_s directly by Eq. (3.2) in that case.

Next, we analyze the stability of the fixed points in the modified model M_{new}^* . First, we note that, from Eq. (3.2), the Jacobian matrix J of our modified model is given by,

$$J(x, y) = \begin{bmatrix} \frac{\alpha - \beta y - \epsilon \gamma y + 2\epsilon \delta x y - \epsilon^2 \delta \lambda x^2 y^2}{(1 - \epsilon \lambda x y)^2} & \frac{\alpha \epsilon \lambda x^2 + \epsilon \delta x^2 - \beta x - \epsilon \gamma x}{(1 - \epsilon \lambda x y)^2} \\ \frac{\delta y + \alpha \lambda y - \beta \lambda y^2 - \epsilon \gamma \lambda y^2}{(1 - \epsilon \lambda x y)^2} & \frac{\delta x + \alpha \lambda x - \gamma - 2\beta \lambda x y + \beta \epsilon \lambda^2 x^2 y^2}{(1 - \epsilon \lambda x y)^2} \end{bmatrix}.$$

So, in the case of the trivial fixed point at $(0, 0)$, we have

$$J(0, 0) = \begin{bmatrix} \alpha & 0 \\ 0 & -\gamma \end{bmatrix},$$

which indicates that the fixed point at $(0, 0)$ is hyperbolic, since J has two real eigenvalues: one is positive ($\alpha > 0$) and the other is negative ($\eta_2 = -\gamma < 0$), which are exactly the same as in the original model.

In the case of the other fixed point at $(\frac{\gamma}{\delta}, \frac{\alpha}{\beta})$, the Jacobian matrix of the modified model at this point is given by:

$$\begin{aligned}
 J\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right) &= \frac{\beta^2 \delta^2}{(\beta\delta - \alpha\epsilon\gamma\lambda)^2} \begin{bmatrix} \frac{\alpha\epsilon\gamma(\beta\delta - \alpha\epsilon\gamma\lambda)}{\beta^2 \delta} & \frac{\gamma(\alpha\epsilon\gamma\lambda - \beta\delta)}{\delta^2} \\ \frac{\alpha(\beta\delta - \alpha\epsilon\gamma\lambda)}{\beta^2} & \frac{\alpha\gamma\lambda(\alpha\epsilon\gamma\lambda - \beta\delta)}{\beta\delta^2} \end{bmatrix} \\
 &= \frac{1}{\beta\delta - \alpha\epsilon\gamma\lambda} \begin{bmatrix} \alpha\epsilon\gamma\delta & -\beta^2\gamma \\ \alpha\delta^2 & -\alpha\beta\gamma\lambda \end{bmatrix}.
 \end{aligned} \tag{4.8}$$

From Eq. (4.8), we obtain the characteristic equation of J as follows:

$$\begin{aligned}
 p(\eta) &= \left(\frac{\alpha\epsilon\gamma\delta}{\beta\delta - \alpha\epsilon\gamma\lambda} - \eta\right) \left(\frac{-\alpha\beta\gamma\lambda}{\beta\delta - \alpha\epsilon\gamma\lambda} - \eta\right) + \frac{\alpha\gamma\beta^2\delta^2}{(\beta\delta - \alpha\epsilon\gamma\lambda)^2} = 0, \\
 \text{i.e. } &(\beta\delta - \alpha\epsilon\gamma\lambda)\eta^2 - \alpha\gamma(\epsilon\delta - \beta\lambda)\eta + \alpha\gamma\beta\delta = 0.
 \end{aligned}$$

Solving the above equation, we obtain the two eigenvalues of $J\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)$ as,

$$\eta = \frac{\alpha\gamma(\epsilon\delta - \beta\lambda) \pm \sqrt{\alpha^2\gamma^2(\epsilon\delta - \beta\lambda)^2 - 4\alpha\gamma\beta\delta(\beta\delta - \alpha\epsilon\gamma\lambda)}}{2(\beta\delta - \alpha\epsilon\gamma\lambda)}. \tag{4.9}$$

Here, we introduce three new parameters as follows for the convenience of further discussion:

$$A = \alpha\gamma(\epsilon\delta - \beta\lambda), \quad B = \alpha^2\gamma^2(\epsilon\delta - \beta\lambda)^2 - 4\alpha\gamma\beta\delta(\beta\delta - \alpha\epsilon\gamma\lambda), \quad \text{and } C = \beta\delta - \alpha\epsilon\gamma\lambda.$$

The eigenvalues in Eq. (4.9) can then be expressed in a simplified form:

$$\eta_1 = \frac{A - \sqrt{B}}{2C}, \quad \eta_2 = \frac{A + \sqrt{B}}{2C}.$$

Here, we investigate the stability of the non-trivial fixed point at $(\frac{\gamma}{\delta}, \frac{\alpha}{\beta})$. Thus, in summary, we can obtain the following theorem, which indicates how the stability of this non-trivial fixed point is determined by selecting the parameters, ϵ and λ , in the $\epsilon\lambda$ -parameter space.

Theorem 4.3. *The fixed point at $(\frac{\gamma}{\delta}, \frac{\alpha}{\beta})$ is stable, if $\epsilon < 0$ and $\lambda > 0$.*

Proof. In this section, we previously established the permissible ranges of the parameters ϵ and λ as $\epsilon < 0$ and $\lambda > 0$. Thus, it is clear that $C = \beta\delta - \alpha\epsilon\gamma\lambda > 0$ and $A = \alpha\gamma(\epsilon\delta - \beta\lambda) < 0$. Then we consider the case of $B > 0$,

$$\begin{aligned}
 A^2 - B &= (\alpha\gamma(\epsilon\delta - \beta\lambda))^2 - \alpha^2\gamma^2(\epsilon\delta - \beta\lambda)^2 + 4\alpha\gamma\beta\delta(\beta\delta - \alpha\epsilon\gamma\lambda) \\
 &= 4\alpha\gamma\beta\delta(\beta\delta - \alpha\gamma\epsilon\lambda) > 0 \quad \text{for } \epsilon < 0 \quad \text{and } \lambda > 0.
 \end{aligned}$$

Also, in the case of $B < 0$, since the imaginary number \sqrt{B} contributes only to the rotation of the solution trajectories, the stability of the eigenvalue depends entirely on the sign of A . Since $A < 0$, we see that the fixed point is also stable. \square

Figure 4 clearly illustrates the possible dynamics discussed in Theorem 4.3. From Fig. 4, we see that, although the non-trivial fixed point of the original prey-predator model is maintained, its character is significantly altered by the newly added parameters. As stated in Theorem 4.3, by selecting various values for ϵ and λ , the characteristics of the fixed point that was the center of the original model can change; in this modified model one can be observed for sinks, sources, or centers: For example, Fig. 4 (a) shows a case in which the fixed point becomes unstable due to the addition of the extra coupling terms. In the case of Fig. 4 (b), in contrast with the case shown in Fig. 4 (a), the fixed point becomes unstable as a result of the additional coupling terms.

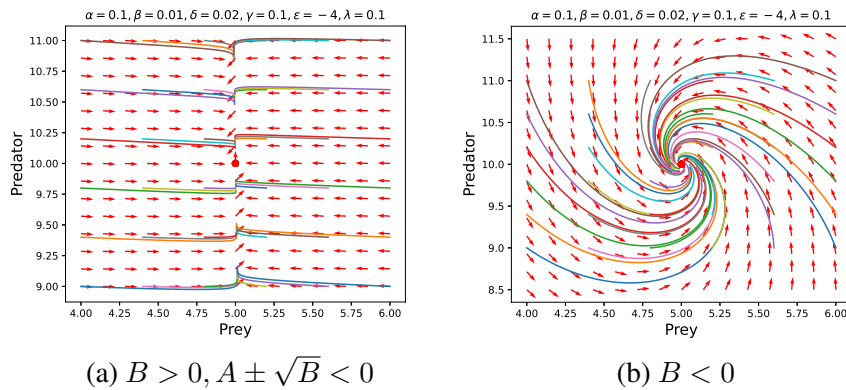


FIGURE 4. Possible phase portraits in the case of $\epsilon < 0$ and $\lambda > 0$ near the fixed point $(\frac{\gamma}{\delta}, \frac{\alpha}{\beta})$: (a) non-spiral sink when $B > 0$ and (b) spiral sink when $B < 0$. For more details, see Fig. 5 (a)

Next, by applying Theorem 4.3, we analyze the dynamics of the modified model, according to the parameter selection on the $\epsilon\lambda$ -plane for a fixed set of parameters α, β, δ and γ . First, we obtain Fig. 5 (a) which shows the partitioned regions representing the possible cases for the stability of the eigenvalues of the Jacobian matrix at the non-trivial fixed point. The partitioned regions are first determined, according to the criteria of $A = 0$ and $B = 0$ (i.e. $\epsilon\delta + \beta\lambda = \pm 2\frac{\beta\delta}{\sqrt{\alpha\gamma}}$) given as follows:

$$\begin{aligned}
 P_1 &= \left\{ (\epsilon, \lambda) \mid A < 0 \text{ and } B > 0 \text{ with } \epsilon\delta + \beta\lambda < -2\frac{\beta\delta}{\sqrt{\alpha\gamma}} \right\}, \\
 P_2 &= \left\{ (\epsilon, \lambda) \mid A < 0 \text{ and } B < 0 \left(-2\frac{\beta\delta}{\sqrt{\alpha\gamma}} < \epsilon\delta + \beta\lambda < 2\frac{\beta\delta}{\sqrt{\alpha\gamma}} \right) \right\}, \\
 P_3 &= \left\{ (\epsilon, \lambda) \mid A < 0 \text{ and } B > 0 \text{ with } \epsilon\delta + \beta\lambda > 2\frac{\beta\delta}{\sqrt{\alpha\gamma}} \right\}.
 \end{aligned}$$

Second, according to the additional criterion of $A^2 - B = 0$, we complete the fully partitioned regions as shown in Fig. 5 (b). Here recall that $A^2 - B = 4\alpha\gamma\beta\delta(\beta\delta - \alpha\gamma\epsilon\lambda)$. The additionally partitioned regions with hyperbolic fixed points cannot appear in the second

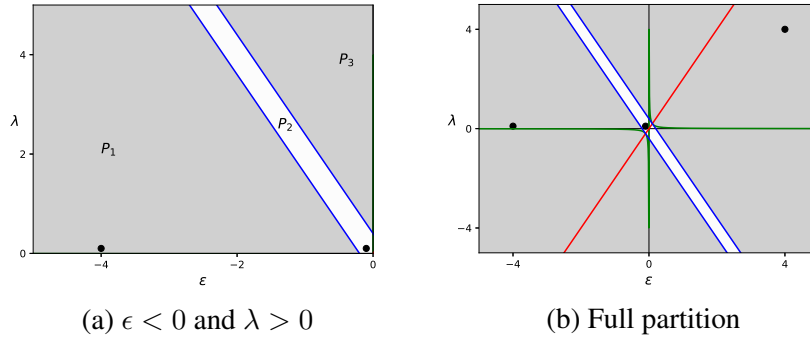


FIGURE 5. Parameter space partitioned according to the dynamics for values of ϵ and λ : (a) Phase space partition when $\epsilon < 0$ and $\lambda > 0$ (the 2 black dots denote the 2 cases shown in Fig. 4), (b) Full phase space partition, for $\alpha = 0.1, \beta = 0.01, \delta = 0.02$ and $\gamma = 0.1$ where the red, green and blue line represents $A = 0, A^2 - B = 0$ and $B = 0$, respectively.

quadrant of the $\epsilon\lambda$ -plane, since $A^2 > B$ and $A < 0$ (i.e., all the signs of the eigenvalues are negative), as shown in Fig. 5 (a). Consequently, in the case of $\epsilon < 0$ and $\lambda > 0$, there exist only three possible regions, which show two distinct dynamics: One is the dynamics characterized by non-spiral sink behavior and the other is that of a spiral sink behavior, as shown in Fig. 4.

On the other hand, if the signs of ϵ and λ change, we have to consider the criterion curves satisfying the conditions, $\beta\delta - \alpha\gamma\epsilon\lambda = 0$ (i.e., $A^2 - B = 0$) and $\epsilon\delta - \beta\lambda = 0$ (i.e. $A = 0$). Since they may appear in the first and third quadrant in the $\epsilon\lambda$ -plane. (For more details, see Fig. 5 (b); these conditions are met by the green and red curves, respectively, in the figure.) Finally, we consider the behavior that occurs for other signs of ϵ and λ . We have previously shown that, if ϵ and λ have different signs, there exists one more non-trivial but singular fixed point \mathbf{f}_s . To consider the dynamics near this non-trivial fixed point in such a model, we need to establish the stability of \mathbf{f}_s . As stated in the previous section, \mathbf{f}_s is a singular point of the system described in Eq. (3.2), thus we cannot obtain the Jacobian matrix J at \mathbf{f}_s ; this strongly implies that the dynamics near \mathbf{f}_s are singular. We also note that the characteristics of the existing fixed point can change: For example, when we select the parameters $\epsilon = 4$ and $\lambda = 4$ in the first quadrant as shown in Fig. 5 (b), the fixed point changes from being stable to being a hyperbolic fixed point, as shown in Fig. 6 (b); as the parameters move from the second quadrant to the partitioned region, including (4, 4) in the first quadrant, $A > 0, B > 0$, and $A^2 < B$ are all satisfied. These observations suggest that, when the signs of ϵ and λ vary, the dynamics might change in a non-trivial manner. To confirm this, we observe numerically what happens near the point \mathbf{f}_s . Figure 6 demonstrates this non-trivial behavior: While the fixed point at $(0, 0)$ is still hyperbolic, the fixed point at $(\frac{\gamma}{\beta}, \frac{\alpha}{\beta})$, which was also present in the original model M_{old} , suddenly becomes hyperbolic and, in addition, another singular fixed point at \mathbf{f}_s appears. Together, these features are likely to create very complex dynamics. We also note that the

singular fixed point \mathbf{f}_s is situated on the singular curve

$$y = \left(\frac{1}{\epsilon\lambda} \right) \frac{1}{x},$$

on which the system given by Eq. (3.2) becomes singular. In Fig. 6 (a), the region of the green curve above \mathbf{f}_s appears to be an unstable manifold, while the lower region appears to be a stable manifold, and the point \mathbf{f}_s looks like the center in the original prey-predator model. However, from Fig. 7, we can see that, in fact, on this curve, intense repelling-and-attracting behavior is taking place near the curve, which strongly supports the hypothesis that there might be highly non-trivial dynamics in this region. Therefore, based on these observations, it is suggested that

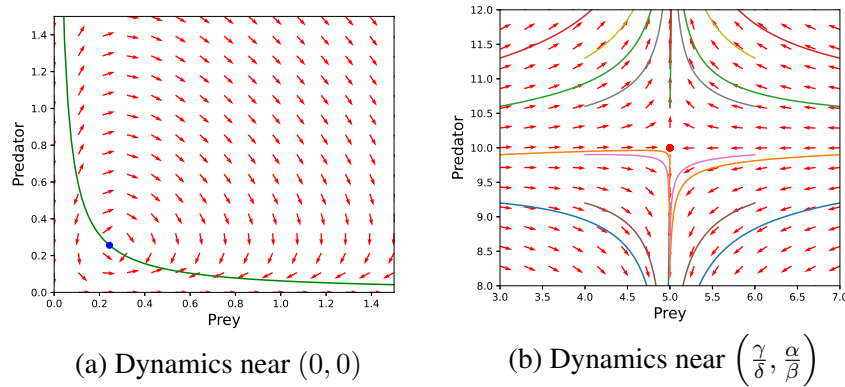


FIGURE 6. Phase portrait near the non-singular fixed points when $\epsilon = 4$ and $\lambda = 4$, with $\alpha = 0.1$, $\beta = 0.01$, $\delta = 0.02$ and $\gamma = 0.1$. The arrows describing the vector field are represented by unit vectors to identifying the directions of tangent vectors.

the emergence of the additional singular fixed point and the singular curve exerts significant influences on the dynamics observed at the existing non-trivial fixed point at $\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)$, and creates several interesting phenomena, as shown Figs. 6 and 7. This remains an interesting subject for further works.

5. CONCLUSION

In this paper, we have analyzed the effect of the additional terms in the modified prey-predator model given in Eq. (3.1) by analyzing the first integral of the system. We find that the first integral, which always exists in the classical model, exists only on a specific straight line on the $\epsilon\lambda$ -plane in the modified model. By considering stability, a variety of cases for the fixed points have been investigated.

As a future work, it is of interest to consider the behavior of the system that is not on the straight line $\lambda = \frac{\delta}{\beta}\epsilon$; these behaviors can be inferred using the Lyapunov function.

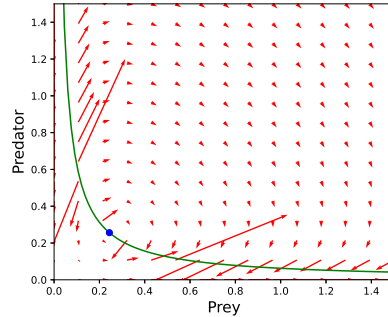


FIGURE 7. Real phase portrait near the singular curve: $y = \frac{1}{\epsilon\lambda} \frac{1}{x}$, for $\epsilon = 4$ and $\lambda = 4$, with $\alpha = 0.1$, $\beta = 0.01$, $\delta = 0.02$, $\gamma = 0.1$. The arrows denote the tangent vectors with the real magnitudes given by the considering vector field.

In the modified model, many interesting phenomena were found; this work has opened the path for further research with the possibility of discovering novel behavior of the modified system. Furthermore, motivated by recent research [10, 11, 12], stochastic differential equations can be added to this model, which can explain the stochastic influences of environment. By applying the stochastic differential equations to our model, we will be able to model environmental fluctuations and/or noise that cannot be modeled in a deterministic model.

6. ACKNOWLEDGMENTS

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