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ON MINIMAL SURFACES WITH GAUSSIAN CURVATURE OF BIANCHI SURFACE TYPE

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Abstract. We consider the local uniqueness of a catenoid under the condition for the Gaussian curvature analogous to Bianchi surfaces. More precisely, if a nonplanar minimal surface in $\mathbb{R}^3$ has the Gaussian curvature $K = -\frac{1}{(U(u)+V(v))^2}$ for any functions $U(u)$ and $V(v)$ with respect to a line of curvature coordinate system (u, v), then it is part of a catenoid. To do this, we use the relation between a conformal line of curvature coordinate system and a Chebyshev coordinate system.

1. Introduction

In 1878, Chebyshev considered a two-parameter family of curves on a surface in $\mathbb{R}^3$ such that every quadrilaterals made by these curves have opposite sides of equal length. It is called a *Chebyshev net*. If the coordinate curves form this net, then the first fundamental form is given by

$$ds^2 = du^2 + 2\cos\phi \; dudv + dv^2,$$

where (u, v) is a local coordinate system and ϕ is the angle between parameter curves. He also showed

$$\phi_{uv} + K \sin\phi = 0, \tag{1.1}$$

where K is the Gaussian curvature of a surface. The equation (1.1) is known as the sine-Gordon equation if the Gaussian curvature is a negative constant (see [7]).

Hilbert proved that, for a surface in $\mathbb{R}^3$ with constant negative Gaussian curvature, the asymptotic curves form a Chebyshev net and the angle between the asymptotic directions satisfies the sine-Gordon equation (1.1) in 1900. Let Σ be a spacelike (resp. timelike) surface of constant

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negative (resp. positive) Gaussian curvature in a 3-dimensional pseudo-Riemannian manifold of constant curvature. Chern [1] proved that there is a local coordinate system (u, v) such that each u- and v- parameter curve is an asymptotic curve of Σ. It is called a *Chebyshev coordinate system.* Moreover, the angle between the asymptotic directions satisfies the sine-Gordon (resp. sinh-Gordon) equation relative to the Chebyshev coordinate system. This result was extended to a surface with negative Gaussian curvature as follows.

THEOREM ([6], [4]). Let Σ be a surface in $\mathbb{R}^3$ with negative Gaussian curvature K and $p \in \Sigma$. Then there is a local coordinate system (u, v) in a neighborhood D of p such that each u- and v- parameter curve is an asymptotic curve of Σ. More precisely, if $K = -\frac{1}{\rho^2}$ for a positive function ρ on D, then two fundamental forms are given by the following formulas.

$$\begin{aligned} \mathrm{I} &= A^2 du^2 + 2AB\cos\phi \; dudv + B^2 dv^2, \\ \mathrm{II} &= \frac{2AB\sin\phi}{\rho} \; dudv, \end{aligned}$$

where ϕ is the angle between two asymptotic curves, we call it the *Chebyshev angle.* The integrability conditions are also given by

$$\phi_{uv} + \left(\frac{\rho_u b}{2\rho a}\sin\phi\right)_u + \left(\frac{\rho_v a}{2\rho b}\sin\phi\right)_v - ab\sin\phi = 0, \qquad \text{(Gauss Eq.)}$$

$$\begin{cases} a_v + \frac{\rho_v}{2\rho}a - \frac{\rho_u}{2\rho}b\cos\phi = 0; \\ b_u + \frac{\rho_u}{2\rho}b - \frac{\rho_v}{2\rho}a\cos\phi = 0, \end{cases} \qquad \text{(Codazzi Eq.)}$$

where $a = \frac{A}{\rho}$, $b = \frac{B}{\rho}$.

If Σ is minimal in $\mathbb{R}^3$, then $K \le 0$ and $\phi = \frac{\pi}{2}$ except a planar point. Therefore the Codazzi equation becomes simpler so that $a = \alpha(u)\rho^{-\frac{1}{2}}$ and $b = \beta(v)\rho^{-\frac{1}{2}}$, where α and β are functions in u and v only, respectively. It yields $A\rho^{-\frac{1}{2}} = \alpha(u)$, $B\rho^{-\frac{1}{2}} = \beta(u)$. This implies that we can change coordinates such that the following holds.

THEOREM ([4]). Let Σ be a minimal surface in $\mathbb{R}^3$ and $p \in \Sigma$ such that $K(p) \neq 0$. Then there is a local coordinate system (u, v) in a neighborhood D of p such that each u- and v- parameter curve is an asymptotic curve of Σ. Fundamental forms are given by

$$\mathrm{I} = e^{2\varphi}\left(du^2 + dv^2\right), \ \mathrm{II} = 2dudv, \tag{1.2}$$

where φ is a smooth function on D satisfying $\triangle_0\varphi = -Ke^{2\varphi}$.

By using this Chebyshev coordinate system, Fujioka [4] obtained the following uniqueness result.

THEOREM 1.1 ([4]). *A Bianchi surface with constant Chebyshev angle is a piece of a right helicoid.*

Here, a Bianchi surface is a surface with negative Gaussian curvature satisfying

$$((-K)^{-\frac{1}{2}})_{uv} = 0$$

with respect to Chebyshev coordinate system. Equivalently, for any functions $U(u)$ and $V(v)$,

$$K = -\frac{1}{(U(u) + V(v))^2}.$$

It is mainly studied related to integrable systems. After, Riveros and Corro [2] showed that a surface with negative Gaussian curvature and constant Chebyshev angle $\varphi \neq \frac{\pi}{2}$ with respect to a Chebyshev coordinate system has the same geodesic curvatures of asymptotic lines upto sign at each point. They [3] also considered the converse problems: If Σ is a minimal surface whose asymptotic lines (resp. lines of curvature) have the same geodesic curvatures upto sign at each point, then it is a catenoid (resp. a helicoid). Lee [5] generalized it as follows: The ratio of the geodesic curvatures of the lines of curvature on a minimal surface is constant if and only if it is one of the associated family of the catenoid to the helicoid.

In this paper, we consider the local uniqueness of a catenoid under the condition for the Gaussian curvature analogous to Bianchi surfaces. To get a result, we use the relation between a conformal line of curvature coordinate system and a Chebyshev coordinate system.

THEOREM (see Theorem 2.2). Let Σ be a nonplanar minimal surface in $\mathbb{R}^3$. If the Gaussian curvature of Σ is given by, for any functions $U(u)$ and $V(v)$,

$$K = -\frac{1}{(U(u) + V(v))^2}$$

with respect to a line of curvature coordinate system (u, v), then Σ is part of a catenoid.

2. Uniqueness of a catenoid

Let Σ be a minimal surface and $X : D \to \mathbb{R}^3$ be an immersion of Σ. For any nonplanar point $p \in \Sigma$, there is a local coordinate system (u, v) in a neighborhood of p, which is both conformal and a line of curvature coordinate system of Σ. The first and the second fundamental forms are given by

$$\mathrm{I} = e^{2\varphi}\left(du^2 + dv^2\right),$$
$$\mathrm{II} = du^2 - dv^2,$$

with the following relation

$$\triangle_0 \varphi = -Ke^{2\varphi},$$

where $\triangle_0$ is the flat Laplacian. We call it a *conformal line of curvature coordinate system.* Define a unit normal vector field ν to Σ by $\nu = \frac{X_u \times X_v}{|X_u \times X_v|}$. With the frame $\{X_u, X_v, \nu\}$ on Σ, the Gauss-Weingarten equations can be written as

$$\begin{cases} X_{uu} &= \varphi_u X_u - \varphi_v X_v + \nu \\ X_{uv} &= \varphi_v X_u + \varphi_u X_v \\ X_{vv} &= -\varphi_u X_u + \varphi_v X_v - \nu \\ \nu_u &= -e^{-2\varphi} X_u \\ \nu_v &= e^{-2\varphi} X_v. \end{cases}$$

For a minimal surface Σ in $\mathbb{R}^3$, there is a relation between a conformal line of curvature coordinate system of Σ and a Chebyshev coordinate system of the conjugate surface Σ^* of Σ.

Proposition 2.1. *Let $X : D \to \mathbb{R}^3$ be an immersion of a minimal surface Σ and $p \in \Sigma$ be a nonplanar point. Let (u, v) be a conformal line of curvature coordinate system defined on a neighborhood of p, then it is a Chebyshev coordinate system of Σ^* satisfying* (1.2)*, and vice versa. Moreover, u-parameter (resp. v-parameter) curves of Σ correspond to v-parameter (resp. u-parameter) curves of Σ^*.*

Proof. A minimal immersion X in $\mathbb{R}^3$ is harmonic, that is, $\triangle_\Sigma X = \vec{0}$. Since the metric is conformal, $\triangle_0 X = \vec{0}$ with the flat Laplacian. There is a harmonic conjugate X^* of X defined on D satisfying Cauchy-Riemann equations so that

$$\begin{cases} X_u = X_v^* \\ X_v = -X_u^*. \end{cases} \tag{2.1}$$

Let $z = u + iv$ be the canonical complex coordinate on $D \subset \mathbb{R}^2 \simeq \mathbb{C}$. From the Weierstrass representation formula, there is a meromorphic function g and a holomorphic function f defined on D such that the immersion X can be expressed by

$$X = \mathrm{Re} \int \left(\frac{1}{2} f(1 - g^2), \frac{i}{2} f(1 + g^2), fg \right) dz.$$

On the other hand, we can define a family of isometric minimal surfaces $X^\theta : D \to \mathbb{R}^3$, $0 \leq \theta < 2\pi$, by

$$X^\theta = \mathrm{Re} \left\{ e^{i\theta} \int \left(\frac{1}{2} f(1 - g^2), \frac{i}{2} f(1 + g^2), fg \right) dz \right\},$$

which is called the associated family of X. When $\theta = \frac{\pi}{2}$, $X^{\frac{\pi}{2}}$ is said to be the *conjugate surface* of X. Note that g represents the stereographic projection of the Gauss map of X. The Gauss map is preserved because the Weierstrass data (g^θ, f^θ) of X^θ is given by $g^\theta = g$ and $f^\theta = e^{i\theta} f$. In other words, the unit normal vector field ν^θ of X^θ is the same as that of X for all $\theta \in [0, 2\pi)$.

From the definition of X^θ, we see that $-X^{\frac{\pi}{2}}$ is the harmonic conjugate of X in the sense of (2.1), and hence

$$X^* = -X^{\frac{\pi}{2}},$$

which is a minimal immersion. Since $X^{\frac{\pi}{2}}$ is isometric to X, the first fundamental form of X^* is given by

$$\mathrm{I}^* = e^{2\varphi} \left(du^2 + dv^2 \right). \tag{2.2}$$

Note that the unit normal vector field ν^* is preserved: $\nu^* = \nu$. Now we calculate the Gauss-Weingarten equations of X^* as follows.

$$\begin{cases} X^*_{uu} &= -X_{uv} &= -\varphi_v X_u - \varphi_u X_v &= \varphi_u X^*_u - \varphi_v X^*_v \\ X^*_{uv} &= -X_{vv} &= \varphi_u X_u - \varphi_v X_v + \nu &= \varphi_v X^*_u + \varphi_u X^*_v + \nu^* \\ X^*_{vv} &= X_{uv} &= \varphi_v X_u + \varphi_u X_v &= -\varphi_u X^*_u + \varphi_v X^*_v \\ \nu^*_u &= \nu_u &= -e^{-2\varphi} X_u &= -e^{-2\varphi} X^*_v \\ \nu^*_v &= \nu_v &= e^{-2\varphi} X_v &= -e^{-2\varphi} X^*_u. \end{cases} \tag{2.3}$$

From (2.3), the second fundamental form of X^* is obtained as

$$\mathrm{II}^* = 2dudv, \tag{2.4}$$

and therefore each u- and v- parameter curve in X^* is an asymptotic curve. The equation (2.2) and (2.4) imply that (u, v) is a Chebyshev coordinate system of a minimal surface X^* satisfying (1.2). In the same manner, one can prove that if (u, v) is a Chebyshev coordinate system of X satisfying (1.2), then it is a conformal line of curvature coordinate system of X^*. In any case, u-parameter (resp. v-parameter) curves of X correspond to v-parameter (resp. u-parameter) curves of X^* by (2.1). □

Remark that a line of curvature (resp. an asymptotic curve) of a minimal surface maps to an asymptotic curve (resp. a line of curvature) of the conjugate surface via the conjugate correspondence.

We get a uniqueness theorem for a catenoid in $\mathbb{R}^3$, that can be thought as a parallel result to Theorem 1.1.

THEOREM 2.2. *Let Σ be a nonplanar minimal surface in $\mathbb{R}^3$. If the Gaussian curvature of Σ is given by, for any functions $U(u)$ and $V(v)$,*

$$K = -\frac{1}{(U(u) + V(v))^2} \tag{2.5}$$

with respect to a line of curvature coordinate system (u, v), then Σ is part of a catenoid.

REMARK. Here, we consider a local problem. Note that a planar point of Σ is isolated unless Σ is a plane.

Proof. Let $p \in \Sigma$ be a nonplanar point such that the condition (2.5) is satisfied in an open neighborhood D of p in Σ. Let (u, v) be a line of curvature coordinate system in D. We know that there is a conformal line of curvature coordinate system $(\tilde{u}, \tilde{v})$ in a sufficiently small neighborhood of p. Because both are line of curvature coordinate systems, rearranging the variables, we may assume that $\tilde{u}$ (resp. $\tilde{v}$) is a function of u (resp. v), and vice versa.

Without loss of generality, we may assume that (2.5) holds with respect to a conformal line of curvature coordinate system (u, v). It follows from Proposition 2.1 that the conjugate minimal surface Σ^* satisfies (2.5) with respect to a Chebyshev coordinate system, i.e. Σ^* is a Bianchi surface with constant Chebyshev angle. By Theorem 1.1 and the maximum principle, Σ^* is part of a helicoid. Thus Σ is part of a catenoid since Σ is isometric to $(\Sigma^*)^*$. □

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