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RIEMANNIAN AND LORENTZIAN VOLUME COMPARISONS WITH THE BAKRY-EMERY RICCI TENSOR

Jong Ryul Kim

Abstract. The Bishop and Bishop-Gromov volume comparisons with the Bakry-Emery Ricci tensor in a metric measure space are studied by the comparisons of the Jacobi differential equations in a Riemannian and Lorentzian manifold.

1. Introduction

Let M be an $(n+1)$-dimensional complete and simply connected Riemannian manifold with metric g. Given a real valued smooth function f on M and Riemannian volume density $d\mathrm{vol}_g$, a triple $(M, g, e^{-f}d\mathrm{vol}_g)$ is called a metric measure space or a weighted manifold. The Bakry-Emery Ricci tensor Ric_f is defined by

$$\mathrm{Ric}_f = \mathrm{Ric} + \mathrm{Hess} f, \tag{1.1}$$

which becomes the Ricci tensor if f is constant. In connection with the m-Bakry-Emery Ricci tensor defined by

$$\mathrm{Ric}_f^m = \mathrm{Ric} + \mathrm{Hess} f - \frac{1}{m} df \otimes df, \quad \text{for} \quad 0 < m \leq \infty,$$

the Bakry-Emery Ricci tensor $\mathrm{Ric}_f = \mathrm{Ric}_f^\infty$ is also called the ∞-Bakry-Emery Ricci tensor. When $\mathrm{Ric}_f = \lambda g$ for some constant λ, we have a gradient Ricci soliton which is an important topic in Ricci flow. The diffusion operator on a complete metric measure space via Bakry-Emery Ricci tensor has geometrical applications ([1] [12]). The Bishop and Bishop-Gromov volume comparisons with the Bakry-Emery Ricci tensor have been studied in [10], [11]. Let $\bar{M}(k)$ be an $(n+1)$-dimensional Riemannian manifold of constant curvature k. Given a smooth function

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$h(x) = -a \cdot d(x, \bar{p})$ for $\bar{p} \in \bar{M}$, a positive real number a and the Riemannian distance function d with respect to the metric $\bar{g}$ on $\bar{M}(k)$, a quadruple $(\bar{M}(k), \bar{g}, e^{-h} d\mathrm{vol}_{\bar{g}}, \bar{p})$ is called the pointed metric measure space. Wei and Wylie proved mean curvature comparison under the Bakry-Emery Ricci inequality $\mathrm{Ric}_f \geq nk$ together with $f' > -a$ [11]. Using mean curvature comparison, they showed the Bishop-Gromov volume comparison in terms of the weighted volume. And Ruan presented the Bishop volume comparison theorem by using the m-Bakry-Emery Ricci tensor and weighted Laplacian theorem in [10].

In this paper, we use the *f-Jacobi equation* (2.9) and (∞, f)-Raychaud huri equation for a Jacobi tensor along a geodesic introduced for the study of Lorentzian singularity theorems in [2]. Our motivation is that the volume comparisons with the Bakry-Emery Ricci tensor can also be proved by the comparisons of the f-Jacobi differential equations using Lemma 5 obtained by Eschenburg and O'Sullivan in [5]. We show the Bishop and Bishop-Gromov volume comparisons between level hypersurfaces of geodesic balls with the Bakry-Emery Ricci tensor. A Lorentzian volume by the language of K-distance wedge defined in [3] is used to define the Lorentzian weighted volume. The Lorentzian Bishop and Bishop-Gromov volume comparisons with the Bakry-Emery Ricci tensor are similarly stated by using the same methods in Riemannian geometry. From the viewpoint of physics, the Lorentzian f-volume expansion rate between spacelike hypersurfaces under the m-Bakry-Emery Ricci tensor condition was considered in [8]. Our Lorentzian f-volume comparison could also be applied to the relative f-volume comparisons (cf. [7] [8] [9] [13]).

2. Preliminaries

Let H be a hypersurface in an $(n+1)$-dimensional Riemannian manifold M and γ be a unit-speed geodesic orthogonal to H at $\gamma(r)$. For a unit normal vector field N along H with $N_{\gamma(r)} = \gamma'(r)$ and $q \in H$, a mapping $\phi : I \times H \to M$ given by

$$\phi(t, q) = \exp(t - r) N_q,$$

where $t \in I = [r, r_1)$, is called a normal geodesic variation of γ along the hypersurface H [5]. For each fixed $q \in H$, let γ_q be the geodesic given by $\gamma_q(t) = \phi(t, q)$ and define $\phi_t : H \to M$ by $\phi_t(q) = \phi(t, q)$ for $q \in H$. We denote by S_{-N} the shape operator of the hypersurface H. An H-Jacobi tensor along γ is defined by

DEFINITION 2.1. Let γ be a unit-speed geodesic orthogonal to a hypersurface H at $\gamma(r)$ with $N_{\gamma(r)} = \gamma'(r)$. A smooth $(1,1)$ tensor field $A : (\gamma')^{\perp} \to (\gamma')^{\perp}$ associated with ϕ is called an *H-Jacobi tensor* along γ if it satisfies

$$A'' + R(A,\gamma')\gamma' = 0, \quad \mathrm{ker}A \cap \mathrm{ker}A' = \{0\}, \quad A(r) = \mathrm{Id}, \quad A'(r) = S_{-N},$$

where Id is the identity endomorphism of $(\gamma')^{\perp}$. A point $\gamma(t_0)$ for $t_0 \in (r, r_1)$ is called a focal point to H if $\det A(t_0) = 0$.

The shape operator $S_{-\gamma'(t)}$ of each level hypersurface H_t of H associated with ϕ is given by as in [5]

$$A'A^{-1}(t) = S_{-\gamma'(t)} = S_t.$$

We denote by $\theta(t) = \mathrm{tr}S_t$ the *mean curvature* of H_t along $\gamma(t)$.

Put $B = A'A^{-1}$ for an H-Jacobi tensor A along γ, then we have

$$B' = A''A^{-1} - A'A^{-1}A'A^{-1} = -R_{\gamma'} - B \circ B, \tag{2.1}$$

where we put $R(A,\gamma')\gamma' = R_{\gamma'}A$. The mean curvature is also expressed as

$$\theta = \mathrm{tr}(B) = \frac{(\det(A))'}{\det(A)}$$

and the *shear tensor* σ of A along γ is defined by

$$\sigma = B - \frac{\theta}{n}\mathrm{Id}.$$

Note that a variation tensor field A associated with ϕ is a Lagrange tensor (Proposition 1 in [5]). So the vorticity $\frac{1}{2}(B - B^*)$ is zero, where $*$ denotes the adjoint. Taking the trace of (2.1), we get the *Raychaudhuri equation*

$$\theta' + \frac{\theta^2}{n} + \mathrm{Ric}(\gamma',\gamma') + \mathrm{tr}\sigma^2 = 0, \tag{2.2}$$

where $\mathrm{Ric}(\gamma',\gamma') = \sum_{i=1}^{n} g(R(e_i,\gamma')\gamma', e_i)$ for an orthonormal basis $\{e_i\}_{i=1}^{n}$ of $\gamma'^{\perp}$.

Put $x = (\det A)^{\frac{1}{n}}$, then we have

$$x' = \frac{1}{n}x\theta, \quad x'' = \frac{1}{n}(\theta' + \frac{\theta^2}{n})x. \tag{2.3}$$

So we obtain the *Jacobi equation* by (2.2) and (2.3)

$$x'' + \frac{1}{n}(\mathrm{Ric}(\gamma',\gamma') + \mathrm{tr}\sigma^2)x = 0.$$

Definition 2.2. [2] Let A be an H-Jacobi tensor along a geodesic γ. For a smooth function $f : M \to \mathbb{R}$, define $B_f = A'A^{-1} - \frac{1}{n}(f \circ \gamma)'\mathrm{Id}$. The *f-expansion* θ_f, *f-shear tensor* σ_f of A along γ is defined by

$$\theta_f = \mathrm{tr}(B_f), \quad \sigma_f = B_f - \frac{\theta_f}{n}\mathrm{Id},$$

respectively.

For a measure $e^{-f} d\mathrm{vol}_g$, the f-expansion θ_f of an H-Jacobi tensor along a geodesic γ can be also expressed as

$$\theta_f = \frac{(e^{-f} \det A)'}{e^{-f} \det A} = \theta - f'. \tag{2.4}$$

So we have

$$\sigma_f = B_f - \frac{\theta_f}{n}\mathrm{Id} = A'A^{-1} - \frac{f'}{n}\mathrm{Id} - \frac{\theta - f'}{n}\mathrm{Id} = B - \frac{\theta}{n}\mathrm{Id} = \sigma.$$

Note that

$$\mathrm{Hess} f(\gamma', \gamma') = g(D_{\gamma'} \nabla f, \gamma') = \gamma' g(\nabla f, \gamma') = f''.$$

Differentiating (2.4), we have

$$\theta_f' = \theta' - \mathrm{Hess} f(\gamma', \gamma'). \tag{2.5}$$

Using $\sigma_f = \sigma$, (1.1) and (2.5), the Raychaudhuri equation (2.2) is changed to

$$\theta_f' = -(\frac{1}{n}\theta^2 + \mathrm{Ric}(\gamma', \gamma') + \mathrm{tr}\sigma^2) - \mathrm{Hess} f(\gamma', \gamma') = -\frac{1}{n}\theta^2 - \mathrm{Ric}_f(\gamma', \gamma') - \mathrm{tr}\sigma_f^2.$$

So we have

$$\theta_f' + \frac{1}{n}\theta^2 + \mathrm{Ric}_f(\gamma', \gamma') + \mathrm{tr}\sigma_f^2 = 0. \tag{2.6}$$

Hence, by inserting (2.4) to the equation (2.6), we get the f-Raychaudhuri equation as in [2]

$$\theta_f' + \frac{1}{n}(\theta_f^2 + 2\theta_f f' + (f')^2) + \mathrm{Ric}_f(\gamma', \gamma') + \mathrm{tr}\sigma_f^2 = 0. \tag{2.7}$$

Now put $x_f = (e^{-f} \det A)^{\frac{1}{n}}$. Then we see

$$x_f' = \frac{1}{n} x_f \theta_f, \quad x_f'' = \frac{1}{n}(\theta_f' + \frac{1}{n}\theta_f^2) x_f. \tag{2.8}$$

So we obtain the *f-Jacobi equation* by (2.7) and (2.8)

$$x_f'' + \frac{1}{n}(\mathrm{Ric}_f(\gamma', \gamma') + \frac{2\theta_f f' + (f')^2}{n} + \mathrm{tr}\sigma_f^2)x_f = 0. \tag{2.9}$$

3. Riemannian volume comparisons with the Bakry-Emery Ricci tensor

Let M be an $(n+1)$-dimensional Riemannian manifold and γ be a unit speed geodesic with $\gamma(0) = p$ and $\gamma'(0) = v$. Let $S(r_0)$ be a sphere of radius r_0 in T_pM. Put $S_p(r_0) = \exp_p S(r_0)$. A Riemannian volume between $S_p(r_0)$ and $S_p(r)$ is defined by

$$V_{r_0}(r) = \int_{r_0}^{r}\int_{S(1)} |\mathrm{det}A| \; dvdt \tag{3.1}$$

for $0 < r_0 < r < \mathrm{inj}_{S(1)}(p)$, where dv is the volume element of $S(1)$ and $\mathrm{inj}_{S(1)}(p) = \inf\{\mathrm{cut}_v(p) | v \in S(1)\}$. The weighted volume between the geodesic ball $S_p(r_0)$ and $S_p(r)$ is defined by

$$V_{r_0}^f(r) = \int_{r_0}^{r}\int_{S(1)} |e^{-f}\mathrm{det}A| \; dvdt. \tag{3.2}$$

Let $\bar{M}(k)$ be an $(n+1)$-dimensional Riemannian manifold of constant curvature k as the model space of volume comparison and $\bar{\gamma}$ be a unit speed geodesic with $\bar{\gamma}(0) = \bar{p}$ and $\bar{\gamma}'(0) = \bar{v}$. For a Jacobi tensor $\bar{A}$ along $\bar{\gamma}$ with $\bar{A}(0) = 0$ and $\bar{A}'(0) = \mathrm{Id}$, the Jacobi equation along a geodesic $\bar{\gamma}$ is given by

$$\bar{x}'' + k\bar{x} = 0.$$

In order to compare volumes, put $S_{\bar{p}}(r_0) = \exp_{\bar{p}} S(r_0)$ and assume a linear isometry

$$\imath : T_{\gamma(r_0)}S_p(r_0) \to T_{\bar{\gamma}(r_0)}S_{\bar{p}}(r_0) \tag{3.3}$$

such that $\imath(\gamma'(r_0)) = \bar{\gamma}'(r_0)$ and $\imath(E_i(r_0)) = \bar{E}_i(r_0)$ for an orthonormal basis $\{e_1, e_2, ..., e_n\}$ of $T_{\gamma(r_0)}S_p(r_0)$ and its parallel basis $\{E_1, E_2, ..., E_n\}$ along γ with $E_i(r_0) = e_i$ for each i, furthermore $S_{\bar{p}}(r_0) = \exp_{\bar{\gamma}(r_0)} \circ \imath \circ \exp^{-1}_{\gamma(r_0)} S_p(r_0)$. The following Lemma is essential for our volume comparisons.

Lemma 3.1. [5] *Suppose $u : \mathbb{R} \to \mathbb{R}$ is smooth. Let x, $\bar{x}$ be a smooth function such that x, $\bar{x}$ is a solution of the differential inequality $x'' + ux \le 0$, $\bar{x}'' + u\bar{x} = 0$, respectively, with $\bar{x}(t_0) = x(t_0)$ and $x'(t_0) \le \bar{x}'(t_0)$. Suppose that x and $\bar{x}$ are both positive in some interval $[t_0, t)$. Let s,*

$\bar{s}$ be the first positive zero of x, $\bar{x}$, respectively. Then $s \le \bar{s}$, $x \le \bar{x}$ on $[t_0, \bar{s}]$ and $\frac{x'}{x} \le \frac{\bar{x}'}{\bar{x}}$ on $[t_0, s]$.

Proof. Put $h = \frac{x}{\bar{x}}$ and $g = h'\bar{x}^2 = x'\bar{x} - x\bar{x}'$. If $\bar{x}(t_0) = x(t_0)$ and $x'(t_0) \le \bar{x}'(t_0)$, then

$$g(t_0) = x'(t_0)\bar{x}(t_0) - x(t_0)\bar{x}'(t_0) = (x'(t_0) - \bar{x}'(t_0))\bar{x}(t_0) \le 0$$

and $g' = x''\bar{x} - x\bar{x}'' = (x'' + ux)\bar{x} \le 0$. So $g \le 0$, hence $h' \le 0$. Since $h(t_0) = 1$, we see $h \le 1$. Therefore $x \le \bar{x}$. It follows from

$$\frac{x'}{x} - \frac{\bar{x}'}{\bar{x}} = \frac{x'\bar{x} - x\bar{x}'}{x\bar{x}},$$

$g(t_0) \le 0$, $g' \le 0$ that $\frac{x'}{x} \le \frac{\bar{x}'}{\bar{x}}$. □

We denote by $\bar{V}^f_{r_0}(r)$, $\bar{V}_{r_0}(r)$ the f-volume, volume between level hypersurfaces in Riemannian manifold $\bar{M}(k)$ of constant curvature k respectively (3.2), (3.1). Under the above notations, we have

Proposition 3.2. *Let $\bar{M}(k)$ be an $(n+1)$-dimensional Riemannian manifold of constant curvature $k \ge 0$. Assume that $e^{-f(r_0)}\det A(r_0) = \det\bar{A}(r_0)$ and $\theta_f(r_0) \le \bar{\theta}(r_0)$ for a complete metric measure space $(M, g, e^{-f}d\mathrm{vol}_g)$. If $Ric_f(\gamma', \gamma') \ge nk$, $\theta_f(r_0) \le 0$ and $f' \le 0$, then we get*

$$V^f_{r_0}(r) \le \bar{V}_{r_0}(r), \quad \frac{V^f_{r_0}(R)}{\bar{V}_{r_0}(R)} \le \frac{V^f_{r_0}(r)}{\bar{V}_{r_0}(r)}$$

for $0 < r_0 < r < R < \mathrm{inj}_{S(1)}(p)$. Equality holds if and only if each level hypersurface $S_p(t) = \exp_p S(t)$ is isometric to $S_{\bar{p}}(t) = \exp_{\bar{p}} S(t)$ for $r_0 \le t < r$ and $f = 0$.

Proof. Assume that $e^{-f(r_0)}\det A(r_0) = \det\bar{A}(r_0)$ and $\theta_f(r_0) \le \bar{\theta}(r_0)$ with the linear isometry (3.3) $\imath : T_{\gamma(r_0)}S_p(r_0) \to T_{\bar{\gamma}(r_0)}S_{\bar{p}}(r_0)$ such that $S_{\bar{p}}(r_0) = \exp_{\bar{\gamma}(r_0)} \circ \imath \circ \exp^{-1}_{\gamma(r_0)} S_p(r_0)$. From (2.6) and $\mathrm{Ric}_f(\gamma', \gamma') \ge nk$, it follows that

$$\theta'_f = -(\frac{1}{n}\theta^2 + \mathrm{Ric}_f(\gamma', \gamma') + \mathrm{tr}\sigma_f^2) \le -\mathrm{Ric}_f(\gamma', \gamma') \le -nk.$$

Integration gives

$$\int_{r_0}^{t} \theta'_f\, ds \le -\int_{r_0}^{t} nk\, ds$$

$$\theta_f(t) \le -nk(t - r_0) + \theta_f(r_0).$$

Thus we get $\theta_f(t) \le 0$ under the assumption $\theta_f(r_0) \le 0$. Furthermore we have $\theta_f(t)f'(t) \ge 0$, since we assume $f'(t) \le 0$. From the f-Jacobi equation (2.9) along a geodesic γ, it follows that

$$\frac{x_f''(t)}{x_f(t)} \le -\frac{1}{n}\mathrm{Ric}_f(\gamma'(t),\gamma'(t)) - \frac{2}{n^2}\theta_f(t)f'(t) \le -\frac{1}{n}\mathrm{Ric}_f(\gamma'(t),\gamma'(t)),$$

since $\mathrm{tr}\sigma_f^2 \ge 0$ and $\theta_f(t)f'(t) \ge 0$. Thus we get

$$\bar{x}'' + k\bar{x} = 0, \quad \frac{x_f''}{x_f} \le -\frac{1}{n}\mathrm{Ric}_f(\gamma',\gamma') \le -k. \tag{3.4}$$

The inequality (3.4) under $e^{-f(r_0)}\mathrm{det}A(r_0) = \mathrm{det}\bar{A}(r_0)$ and $\theta_f(r_0) \le \bar{\theta}(r_0)$ implies $x_f \le \bar{x}$ by Lemma 3.1 . So we obtain the volume inequality $V_{r_0}^f(r) \le \bar{V}_{r_0}(r)$.

If the equality $V_{r_0}^f(r) = \bar{V}_{r_0}(r)$ holds, then we have $e^{-f}\mathrm{det}A = \mathrm{det}\bar{A}$. So $\bar{\theta} = \theta_f$. Recall that the Raychaudhuri equation in $\bar{M}(k)$ is

$$\bar{\theta}' + \frac{1}{n}\bar{\theta}^2 + nk = 0. \tag{3.5}$$

Since $\bar{\theta} = \theta_f$, we get by subtracting (3.5) from (2.7)

$$nk - \mathrm{Ric}_f(\gamma',\gamma') = \frac{2\theta_f f'}{n} + \frac{(f')^2}{n} + \mathrm{tr}\sigma_f^2.$$

By the assumption of $\mathrm{Ric}_f(\gamma',\gamma') \ge nk$, we get $\sigma_f = \sigma = 0$ and $f' = 0$. Therefore we obtain $\bar{\theta} = \theta_f = \theta$. Then we have $\bar{B} = B$ and we get $R_{\gamma'} = k\mathrm{Id}$ from (2.1). The conclusion follows from Theorem 4 in [6].

By Lemma 3.1, we get $\theta_f(t) \le \bar{\theta}(t)$ for $r_0 \le t < R$. The Bishop-Gromov volume comparison theorem follows from [4] (cf. [3]) under the assumptions $\theta_f(r_0) \le \bar{\theta}(r_0)$ and $e^{-f(r_0)}\mathrm{det}A(r_0) = \mathrm{det}\bar{A}(r_0)$. If the equality of the Bishop-Gromov comparison holds, then we get $x_f = \bar{x}$ [4] (cf. [3]), that is $e^{-f}\mathrm{det}A = \mathrm{det}\bar{A}$. The above arguments lead to the conclusion. □

Recall that given a smooth function $h(x) = -a \cdot d(x,\bar{p})$ for a positive real number a and $\bar{p} \in \bar{M}(k)$, where $\bar{M}(k)$ is an $(n+1)$-dimensional Riemannian manifold of constant curvature k and d is the Riemannian distance function in $\bar{M}$, a quadruple $(\bar{M}(k), \bar{g}, e^{-h}d\mathrm{vol}_{\bar{g}}, \bar{p})$ is called the pointed metric measure space. For the measure $e^{-h}d\mathrm{vol}_{\bar{g}}$, the h-expansion is given by

$$\theta_h(t) = \frac{(e^{-h(t)}\det \bar{A}(t))'}{e^{-h(t)}\det \bar{A}(t)} = \bar{\theta}(t) + a,$$

where $\bar{\theta}(t)$ is the mean curvature of each level hypersurface $S_{\bar{p}}(t)$ along a geodesic $\bar{\gamma}$ orthogonal to a hypersurface $S_{\bar{p}}(r_0)$.

THEOREM 3.3. *Let $(\bar{M}(k), \bar{g}, e^{-h}d\mathrm{vol}_{\bar{g}}, \bar{p})$ be the pointed metric measure space for $k \geq 0$. For a complete metric measure space $(M, g, e^{-f}d\mathrm{vol}_g)$, assume that $\theta_f(r_0) \leq \theta_h(r_0)$ and $e^{-f(r_0)}\det A(r_0) = e^{-h(r_0)}\det \bar{A}(r_0)$. If $Ric_f(\gamma', \gamma') \geq nk$, $\theta_f(r_0) \leq a$ and $f' \leq -a$, then we get*

$$V^f_{r_0}(r) \leq V^h_{r_0}(r), \quad \frac{V^f_{r_0}(R)}{V^h_{r_0}(R)} \leq \frac{V^f_{r_0}(r)}{V^h_{r_0}(r)}$$

for $0 < r_0 < r < R < inj_{S(1)}(p)$. Equality holds if and only if each level hypersurface $S_p(t)$ is isometric to $S_{\bar{p}}(t)$ for $r_0 \leq t < r$ and $f = h$.

Proof. Put $h_{\bar{p}}(x) = -a \cdot d(x, \bar{p})$, $h_p(x) = -a \cdot d(x, p)$ along a unit speed geodesic $\bar{\gamma}$, γ in $\bar{M}$, M, respectively. Then we have

$$h_{\bar{p}}(t) = -a \cdot d(\bar{p}, t) = -at = -a \cdot d(p, t) = h_p(t).$$

For a smooth function $\tilde{f} = f - h_p$ on M, we get $\mathrm{Ric}_{\tilde{f}}(\gamma', \gamma') = \mathrm{Ric}_f(\gamma', \gamma') \geq nk$, since $h_p'' = 0$. We apply Proposition 3.2 for a complete metric measure space $(M, g, e^{-\tilde{f}}d\mathrm{vol}_g)$ under the assumptions $\theta_{f-h_p}(r_0) \leq 0$ and $(f - h_p)' \leq 0$ which are equivalent to $\theta_f(r_0) \leq a$ and $f' \leq h_p' = -a$. With these initial conditions, we have by Lemma 3.1

$$x_{f-h_p} \leq \bar{x}, \quad \theta_{f-h_p} \leq \bar{\theta}. \tag{3.6}$$

The inequality (3.6) leads to

$$x_{f-h_{\bar{p}}} = (e^{-(f-h_{\bar{p}})}\det A)^{\frac{1}{n}} \leq \bar{x}.$$

Hence we get

$$x_f = (e^{-f}\det A)^{\frac{1}{n}} \leq (e^{-h_{\bar{p}}})^{\frac{1}{n}}(\det \bar{A})^{\frac{1}{n}} = (e^{-h_{\bar{p}}}\det \bar{A})^{\frac{1}{n}} = x_h. \tag{3.7}$$

In the same way, we see that

$$\theta_{f-h_p} = \theta_{f-h_{\bar{p}}} = \theta - (f' + a) \leq \bar{\theta}$$

implies

$$\theta_f = \theta - f' \leq \bar{\theta} + a = \bar{\theta} - h'_{\bar{p}} = \theta_h. \tag{3.8}$$

Therefore the volume inequality follows from (3.7) and (3.8). The volume equality holds if and only if each level hypersurface $S_p(t)$ is isometric to $S_{\bar{p}}(t)$ for $r_0 \leq t < r$ and $f = h$ by the same arguments of Proposition 3.2 with $\tilde{f} = f - h_p$. $\square$

Recall that we denote by $V^f_{r_0}(r)$, $V_{r_0}(r)$ the f-volume, volume between level hypersurfaces, respectively (3.2), (3.1). We Put $V^f(r) = V^f_0(r)$ and $V(r) = V_{r_0}(r)$.

Proposition 3.4. *Let $\bar{M}(k)$ be an $(n+1)$-dimensional Riemannian manifold of constant curvature $k \geq 0$. For a complete metric measure space $(M, g, e^{-f}\mathrm{dvol}_g)$, assume that $\theta_f(r_0) \leq \bar{\theta}(r_0)$ and $e^{-f(r_0)}\det A(r_0) = \det\bar{A}(r_0)$. If $Ric_f(\gamma', \gamma') \geq nk$, $\theta_f(r_0) \geq 0$ and $f' \geq 0$, then we get*

$$V^f(r_0) \leq \bar{V}(r_0), \qquad \frac{V^f(R)}{\bar{V}(R)} \leq \frac{V^f(r)}{\bar{V}(r)}$$

where $r < R \leq r_0$. Equality holds if and only if each level hypersurface $S_p(t)$ is isometric to $S_{\bar{p}}(t)$ for $0 < t < r_0$ with $f = 0$.

Proof. Consider a Jacobi tensor A along a geodesic $\beta(t) = \gamma(r_0 - t)$ such that $A'A^{-1} = S_{-\beta'(t)}$. Then $\theta_f(t)$ along $\beta(t)$ for $0 \leq t \leq r_0$ satisfies (3.4). Integration gives

$$\int_t^{r_0} \theta'_f\, ds \leq -\int_t^{r_0} nk\, ds$$

$$\theta_f(r_0) - \theta_f(t) \leq -nk(r_0 - t).$$

Hence we get

$$nk(r_0 - t) + \theta_f(r_0) \leq \theta_f(t).$$

Therefore if $\theta_f(r_0) \geq 0$, then $\theta_f(t) \geq 0$. Under the assumption of $f' \geq 0$, we get $\theta_f(t)f'(t) \geq 0$. Hence the conclusions follow from Proposition 3.2 as t approaches zero. $\square$

Apply Theorem 3.3 for a smooth function $\tilde{f} = f - h_p$ on M and $\theta_{\tilde{f}}$ along the geodesic $\beta(t) = \gamma(r_0 - t)$ for $0 \leq t \leq r_0$. Then we get by Proposition 3.4

Theorem 3.5. *Let $\bar{M}(k)$ be an $(n+1)$-dimensional Riemannian manifold of constant curvature $k \geq 0$. For a complete metric measure space $(M, g, e^{-f}\mathrm{dvol}_g)$, assume that $\theta_f(r_0) \leq \theta_h(r_0)$ and $e^{-f(r_0)}\det A(r_0) =$*

$e^{-h(r_0)}\det\bar{A}(r_0)$. *If* $Ric_f(\gamma',\gamma') \geq nk$, $\theta_f(r_0) \geq a$ *and* $f' \geq -a$, *then we get*

$$V^f(r_0) \leq V^h(r_0), \quad \frac{V^f(R)}{V^h(R)} \leq \frac{V^f(r)}{V^h(r)}$$

for $0 < r < R < r_0 < inj_{S(1)}(p)$. *Equality holds if and only if each level hypersurface* $S_p(t)$ *is isometric to* $S_{\bar{p}}(t)$ *for* $0 < t < r_0$ *with* $f = h$.

4. Lorentzian volume comparisons with the Bakry-Emery Ricci tensor

Riemannina volume comparisons with the Bakry-Emery Ricci tensor can be applied very similarly in a Lorentzian manifold by using the K-*distance wedge* in [3]. We introduce it here for reader's convenience. Let M be an $(n+1)$-dimensional globally hyperbolic space-time and γ be a unit speed timelike geodesic with $\gamma(0) = p$ and $\gamma'(0) = v$. Take an orthonormal basis $\{e_1, ..., e_n, \gamma'(0)\}$ of T_pM and let E_i be the parallel field along γ such that $E_i(0) = e_i$ for each i. Consider a geodesic variation

$$\alpha(t,s) = \exp_p(t(v + sE_i))$$

along γ, then we have a Jacobi field $J_i(t) = \alpha_*|_{(t,0)}(\frac{d}{ds}) = (d\exp_p)_{tv}te_i$ such that $J_i(0) = 0$ and $J_i'(0) = e_i$ for each i. Let A be a Jacobi tensor along γ with the initial conditions $A(0) = 0$ and $A(0)' = \mathrm{Id}$, then we obtain

$$t^n|\det\alpha_*|_{(t,0)}| = ||J_1(t) \wedge J_2(t) \wedge \cdots \wedge J_n(t)|| = |\det A|.$$

Let $\mathrm{Fut}(T_pM)$ be the set of all future directed timelike vectors $v \in T_pM$ such that $\exp_p(v)$ is defined and put $H(r_0) = \{v \in \mathrm{Fut}(T_pM) | g(v,v) = -r_0^2\}$. Let $\bar{K}$ be a compact subset of $H(1)$. Define the K-distance wedge $B_p^K(r)$ as

$$B_p^K(r) = \{\exp_p(tv) \mid v \in \bar{K}, \; 0 \leq t \leq r\}$$

and put $V^K(r) = \mathrm{Vol}(B_p^K(r))$. Let du, dv be the volume element of $\mathrm{Fut}(T_pM)$, $\bar{K}$, respectively. The Lorentzian volume element is given by $du = t^n dvdt$ (Lemma 4.2 [3]) and

$$V(r) = V^K(r) = \int_0^r \int_{\bar{K}} |\det A| \; dvdt.$$

for $0 < r < \mathrm{inj}_{\bar{K}}(p)$, where $\mathrm{inj}_{\bar{K}}(p) = \inf\{\mathrm{cut}_v(p) | v \in \bar{K}\}$. Using the comparison of the Jacobi differential equation (Lemma 3.1), the Lorentzian version of the Bishop and Bishop-Gromov comparison theorems under $\mathrm{Ric}(\gamma', \gamma') \geq nk$ are obtained in [3].

Now we show the Lorentzian version of the Bishop and Bishop-Gromov comparisons between level hypersurfaces with the Bakry-Emery Ricci tensor. Let M be an $(n+1)$-dimensional globally hyperbolic space-time and γ be a unit speed timelike geodesic with $\gamma(0) = p$ and $\gamma'(0) = v$. Take a compact subset $H^*(r_0)$ of $H(r_0)$ and put $H^*_{r_0} = \exp_p H^*(r_0)$. The Lorentzian f-volume between level hypersurfaces $H^*_{r_0}$ and H^*_r is defined by

$$V^f_{r_0}(r) = \int_{r_0}^{r} \int_{\bar{K}} |e^{-f} \mathrm{det} A| \; dv dt$$

for $0 < r_0 < r < \mathrm{inj}_{\bar{K}}(p)$.

Let $\bar{M}(-k)$ be an $(n+1)$-dimensional space-time of constant curvature $-k (k > 0)$ and $\bar{\gamma}$ be a unit speed timelike geodesic with $\bar{\gamma}(0) = \bar{p}$ and $\bar{\gamma}'(0) = \bar{v}$. For a Jacobi tensor $\bar{A}$ along $\bar{\gamma}$ with $\bar{A}(0) = 0$ and $\bar{A}'(0) = \mathrm{Id}$, the Jacobi equation along a geodesic $\bar{\gamma}$ is given by

$$\bar{x}'' + k\bar{x} = 0$$

as in [6]. For the Lorentzian distance function d, consider the pointed metric measure space $(\bar{M}(k), \bar{g}, e^{-h} \mathrm{dvol}_{\bar{g}}, \bar{p})$, where $h : \exp_{\bar{p}}(\mathrm{Fut}(T_{\bar{p}}M)) \to [0, \infty)$ and $h(x) = -a \cdot d(x, \bar{p})$. Put $\bar{H}^*_{r_0} = \exp_{\bar{p}} H^*(r_0)$ for a compact subset $H^*(r_0)$ of $H(r_0) \subset T_{\bar{p}}\bar{M}$. We assume a linear isometry $\imath : T_{\gamma(r_0)} H^*_{r_0} \to T_{\bar{\gamma}(r_0)} \bar{H}^*_{r_0}$ such that $\imath(\gamma'(r_0)) = \bar{\gamma}'(r_0)$ and $\imath(E_i(r_0)) = \bar{E}_i(r_0)$ for an orthonormal basis $\{e_1, e_2, ..., e_n\}$ of $T_{\gamma(r_0)} H^*_{r_0}$ and its parallel basis $\{E_1, E_2, ..., E_n\}$ along γ with $E_i(r_0) = e_i$ for each i, furthermore $\bar{H}^*_{r_0} = \exp_{\bar{\gamma}(r_0)} \circ \imath \circ \exp^{-1}_{\gamma(r_0)} H^*_{r_0}$.

All Riemannian volume comparisons with Bakry-Emery Ricci tensor obtained in the previous section can be stated similarly in a Lorentzian manifold. A Lorentzian manifold M is assumed to be globally hyperbolic so that the Lorentzian distance function is finite valued and continuous (cf. [3]).

THEOREM 4.1. *Let $(M, g, e^{-f} \mathrm{dvol}_g)$ be a metric measure space for a globally hyperbolic space-time M and γ be a unit speed timelike geodesic orthogonal to a spacelike hypersurface $H^*_{r_0}$. Let $(\bar{M}(-k), \bar{g}, e^{-h} \mathrm{dvol}_{\bar{g}}, \bar{p})$ be the pointed metric measure space for a space-time of constant curvature $-k (k > 0)$ and $\bar{\gamma}$ be a unit speed timelike geodesic orthogonal to a spacelike hypersurface $\bar{H}^*_{r_0}$. Assume that $\theta_f(r_0) \leq \theta_h(r_0)$ and $e^{-f(r_0)} \mathrm{det} A(r_0) = e^{-h(r_0)} \mathrm{det} \bar{A}(r_0)$. If $Ric_f(\gamma', \gamma') \geq nk$, $\theta_f(r_0) \leq a$ and*

$f' \leq h' = -a$, then we get $\theta_f \leq \theta + a = \theta_h$ and

$$V^f_{r_0}(r) \leq V^h_{r_0}(r), \quad \frac{V^f_{r_0}(R)}{V^h_{r_0}(R)} \leq \frac{V^f_{r_0}(r)}{V^h_{r_0}(r)},$$

where $R(> r)$ is less than the minimum of the focal values of $\bar{H}^*_{r_0}$ and $H^*_{r_0}$. Equality holds if and only if each level hypersurface H^*_t is isometric to $\bar{H}^*_t$ for $r_0 \leq t < R$ and $f = h$.

Theorem 4.2. *Let $(M, g, e^{-f}\text{dvol}_g)$ be a metric measure space for a globally hyperbolic space-time M and γ be a unit speed timelike geodesic orthogonal to a spacelike hypersurface $H^*_{r_0}$. Let $(\bar{M}(-k), \bar{g}, e^{-h}\text{dvol}_{\bar{g}}, \bar{p})$ be the pointed metric measure space for a space-time of constant curvature $-k(k > 0)$ and $\bar{\gamma}$ be a unit speed timelike geodesic orthogonal to a spacelike hypersurface $\bar{H}^*_{r_0}$. Assume that $\theta_f(r_0) \leq \theta_h(r_0)$ and $e^{-f(r_0)}\text{det}A(r_0) = e^{-h(r_0)}\text{det}\bar{A}(r_0)$. If $Ric_f(\gamma', \gamma') \geq nk$, $\theta_f(r_0) \geq a$ and $f' \geq -a$, then we get $\theta_f \leq \bar{\theta} + a = \theta_h$ and*

$$V^f(r_0) \leq V^h(r_0), \quad \frac{V^f(R)}{V^h(R)} \leq \frac{V^f(r)}{V^h(r)},$$

where $r < R \leq r_0$. Equality holds if and only if each level hypersurface H^*_t is isometric to $\bar{H}^*_t$ for $0 < t < r_0$.

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Department of Mathematics
Kunsan National University
Kunsan, 573-701, Republic of Korea
E-mail: kimjr0@kunsan.ac.kr