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A DECOMPOSITION THEOREM FOR UTUMI AND DUAL-UTUMI MODULES

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Abstract. We show that if M is a Utumi module, in particular if M is quasi-continuous, then $M = Q \oplus K$, where Q is quasi-injective that is both a square-full as well as a dual-square-full module, K is a square-free module, and Q & K are orthogonal. Dually, we also show that if M is a dual-Utumi module whose local summands are summands, in particular if M is quasi-discrete, then $M = P \oplus K$ where P is quasi-projective that is both a square-full as well as a dual-square-full module, K is a dual-square-free module, and P & K are factor-orthogonal.

1. Preliminaries

A module Y is called a square if $Y \cong X \oplus X$ for some module X. A module M is called square-free if it does not contain a non-zero square. A submodule X of a module M is called a square-root in M if $X \oplus X$ embeds in M. The module M is called square-full if every non-zero submodule of M contains a non-zero square-root. A well-known result of Mohamed and Müller, [8, Theorem 2.37], asserts that every quasi-continuous module M has a decomposition $M = M_1 \oplus M_2$, unique up to superspectivity, such that:

(1) M_1 is square-free;
(2) M_2 is square-full and quasi-injective;
(3) M_1 and M_2 are orthogonal.

The notion of square-free was dualized in [1] as follows: a right R-module M is called dual-square-free if M has no proper submodules A and B with $M = A + B$ and $M/A \cong M/B$. Equivalently, [7], if L is a factor module of M such that $L \cong N \oplus N$ for some module N, then $N = 0$. Subsequently, a thorough investigation of dual-square-free modules was carried out in [2].

In [6], the notion of factor-square-full modules was introduced and a dualization of the aforementioned result of Mohamed and Müller was established.

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According to [6], a submodule $Y \subseteq M$ is called dual-square-root if there is an epimorphism $f : M \to (M/Y)^2$, where $(M/Y)^2 := (M/Y) \oplus (M/Y)$. A module M is called factor-square-full if, every proper submodule X of M is contained in a proper dual-square-root Y of M. It was shown in [6, Proposition 3.4 and Theorem 3.7] that every quasi-discrete module M is a direct sum $M_1 \oplus M_2$ of a factor-square-full module M_1 and a dual-square-free module M_2, which are factor orthogonal. Moreover, such a decomposition is unique up to isomorphism and the module M_1 is quasi-projective.

In this paper we show that if M is a Utumi module (U-module, for short), then $M = Q \oplus K$ where Q is quasi-injective that is both a square-full as well as a dual-square-full module, K is a square-free module, and Q and K are orthogonal. In particular, such a decomposition holds for quasi-continuous modules. Dually, we also show that if M is a Dual-Utumi module (DU-module, for short) whose local summands are summands, then $M = P \oplus K$, where P is quasi-projective that is both a square-full as well as a dual-square-full module, K is a dual-square-free module, and P & K are factor-orthogonal. In particular, such a decomposition holds for quasi-discrete modules. Our results may be considered as an improvement of the work on quasi-discrete modules in [6].

Let's recall first some definitions. According to [3], the notion of a U-module was introduced as a non-trivial and simultaneous generalization of quasi-continuous, square-free and automorphism-invariant modules, where a right R-module M is called a U-module if, whenever A and B are submodules of M with $A \cong B$ and $A \cap B = 0$, there exist two summands K and T of M such that $A \subseteq^{ess} K$, $B \subseteq^{ess} T$ and $K \oplus T \subseteq^{\oplus} M$. Dually, in [4], the notion of DU-modules was introduced as a strict and simultaneous generalization of the quasi-discrete, pseudo-discrete and dual-square-free modules. As defined in [4], a right R-module M is called a DU-module if, for any two proper submodules A and B of M with $M/A \cong M/B$ and $A + B = M$, there exist two summands K and L of M such that A lies over K, B lies over L and $K \cap L \subseteq^{\oplus} M$. For the definitions of quasi-continuous, quasi-discrete, discrete, quasi-injective, and quasi-projective, we refer the reader to the textbooks [8] and [9].

Throughout, all rings R are associative with unity and all modules are unitary R-modules. For a module M, we use $rad(M)$, $E(M)$ and $End(M_R)$ to denote the Jacobson radical, the injective hull and the endomorphism ring of M, respectively. If $M = R$, we write $J(R) = rad(R)$. We write $N \subseteq M$ if N is a submodule of M, $N \subseteq^{ess} M$ if N is an essential submodule of M, $N \subseteq^{\oplus} M$ if N is a direct summand of M, and $N \ll M$ if N is a small submodule of M. A submodule N of M is called proper if $N \subsetneq M$. A submodule N of a right R-module M is said to lie over a direct summand of M if there is a decomposition $M = M_1 \oplus M_2$ with $M_1 \subseteq N$ and $N \cap M_2 \ll M$. Furthermore, two right R-modules M and N are called orthogonal, if they do not contain non-zero isomorphic submodules. Dually, M and N are called factor orthogonal if no non-zero factor of M is isomorphic to a factor of N.

2. Results

Lemma 2.1 ([3, Theorem 3.13])**.** *If M is a U-module, then $M = Q \oplus T$, where*

1. *Q is a quasi-injective module*;
2. *$Q = A \oplus B \oplus D$, where $A \cong B$ and D is isomorphic to a direct summand of $A \oplus B$*;
3. *T is a square-free module*;
4. *T is Q-injective, and*
5. *Q and T are orthogonal.*

Recall that a local summand of a module M is a direct sum $L := \oplus_{i \in I} N_i$ of submodules of M such that $\oplus_{i \in F} N_i$ is a summand of M for any finite subset F of I.

Lemma 2.2 ([4, Theorem 4.4])**.** *Let M be a DU-module whose local summands are summands. Then $M = Q \oplus P$, where*

1. *Q is a DSF-module*;
2. *$Q = \oplus_{\lambda \in \Lambda} Q_\lambda$, a direct sum of pairwise non-isomorphic indecomposable modules*;
3. *$P = C \oplus A \oplus B$ is a quasi-projective and discrete module with $A \cong B$, and C is isomorphic to a direct summand of $A \oplus B$*;
4. *Q is P-projective*;
5. *P and Q are factor-orthogonal.*

Lemma 2.3. *If $M = A \oplus B \oplus C$ with $A \overset{f}{\cong} B$, and C is isomorphic to a direct summand of $A \oplus B$, then M is both a square-full as well as a dual square-full module.*

Proof. First we show that M is square-full. Let $0 \neq X \subseteq M = (A \oplus B) \oplus C$ and suppose that $Q =: X \cap A \neq 0$. Therefore, $Q \cong f(Q)$ with $Q \cap f(Q) = 0$. This means that Q is a non-zero square root embedded in M. Similarly, if $S = X \cap B \neq 0$, then S is a non-zero square root embedded in M. Now, suppose that $E =: X \cap C \neq 0$, and let $\sigma : C \longrightarrow A \oplus B$ be an embedding. Clearly, $E \cong \sigma(E)$ with $E \cap \sigma(E) = 0$, and so E is a non-zero square root embedded in M. Therefore, it remains to consider the case when $X \cap A = X \cap B = X \cap C = 0$. By [8, Lemma 1.31], X and one of A, B or C have non-zero isomorphic submodules. Without loss of generality, let $X' \subseteq X$ and $A' \subseteq A$ be such that $X' \cong A'$. Inasmuch as $X' \cap A' = 0$, we infer that X' is a square-root in M. This shows that M is a square-full module. Next, we show that M is dual-square-full. Let X be a proper submodule of M. Clearly, we have the following epimorphism:

$$
\begin{aligned}
M &\to A \oplus B \cong B \oplus B \\
&\cong M/(A \oplus C) \oplus M/(A \oplus C) \to M/(A+X+C) \oplus M/(A+X+C).
\end{aligned}
$$

Now, if $Y := A+X+C \neq M$, then Y is a proper factor-square-full submodule containing X. Otherwise, suppose that $Y := A + X + C = M$. In this case

$M/(X+C) \cong A/(A \cap (X+C))$, and we have the following epimorphism:

$$M \to A \oplus B \cong A \oplus A \to A/(A \cap (X+C)) \oplus A/(A \cap (X+C)) \\ \cong M/(X+C) \oplus M/(X+C).$$

Now, if $X + C \neq M$, then $X + C$ is a proper factor-square-full submodule containing X. If $M = X + C$, then by the hypothesis, $C \cong D \subseteq^{\oplus} A \oplus B$ for a submodule $D \subseteq M$, and we have the following epimorphism:

$$M = A \oplus B \oplus C \to D \oplus C \cong C \oplus C \to C/(X \cap C) \oplus C/(X \cap C) \cong M/X \oplus M/X.$$

In this case X is a proper factor-square-full submodule. This shows that M is dual-square-full, completing the proof. □

Now, the next two results are immediate consequences of Lemma 2.1, Lemma 2.2 and Lemma 2.3. Recall first that a module M is said to satisfy the $C1$-condition if every submodule of M is essential in a direct summand. M is said to satisfy the $C3$-condition if the sum of any two summands of M with zero intersection is a summand of M. A module is called *quasi-continuous* if it satisfies both the $C1$- and $C3$-conditions. Moreover, a module M is called *automorphism-invariant* (*auto-invariant*) if it is invariant under any automorphism of its injective hull.

Theorem 2.4. *If M is a U-module, then $M = Q \oplus K$, where Q is quasi-injective that is both a square-full as well as a dual-square-full module, K is a square-free module, and Q and K are orthogonal. In particular, such a decomposition holds for both quasi-continuous and auto-invariant modules.*

A module M is said to satisfy the $D1$-condition if every submodule N of M lies over a direct summand of M. The module M is said to satisfy the $D3$-condition if M_1 and M_2 are direct summands of M, and $M = M_1 + M_2$, then $M_1 \cap M_2$ is a direct summand of M. A module is called *quasi-discrete* if it satisfies both the $D1$- and $D3$-conditions.

Theorem 2.5. *Let M be a DU-module whose local summands are summands. Then $M = P \oplus K$, where P is quasi-projective and discrete that is both a square-full as well as a dual-square-full module, K is a dual-square-free module, and P and K are factor-orthogonal. In particular, such a decomposition holds for quasi-discrete modules.*

A module M is called H-supplemented [6] if, for any submodule $X \subseteq M$, there exist a submodule $Y \subseteq M$ and a decomposition $M = A \oplus B$ such that $X \subseteq Y$, $A \subseteq Y$, $Y/X \ll M/X$ and $Y/A \ll M/A$. If A and B are modules, then A is called radical-B-projective [6] if, for every homomorphism $f : A \to X$ and every epimorphism $g : B \to X$ there exists a homomorphism $h : A \to B$ such that $\mathrm{Im}(f - gh) \ll X$. A module M is called quasi-radical-projective if M is radical-M-projective.

Theorem 2.6. *Let M be an H-supplemented module that satisfies the $D3$-condition, then $M = Q \oplus P$, where Q is a dual-square-free module, P is a quasi-radical-projective module that is both a square-full as well as a dual-square-full module, and P and Q are factor-orthogonal.*

Proof. It follows from Lemma 2.3 and the proof of Proposition 2.16 in [5]. □

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