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Ricci-Yamabe Solitons and Gradient Ricci-Yamabe Solitons on Kenmotsu 3-manifolds

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Abstract. The aim of this paper is to characterize a Kenmotsu 3-manifold whose metric is either a Ricci-Yamabe soliton or gradient Ricci-Yamabe soliton. Finally, we verify the obtained results by an example.

1. Introduction

Hamilton introduced Yamabe flow in 1982 [8] at the same time he introduced Ricci flow. Ricci solitons and Yamabe solitons are the limit of the solutions of Ricci flow and Yamabe flow respectively. In dimension $n = 2$ the Yamabe soliton is equivalent to the Ricci soliton. However, in dimension $n > 2$, the Yamabe and Ricci solitons do not agree as the Yamabe soliton preserves the conformal class of the metric but the Ricci soliton does not in general.

Over the past twenty years the theory of geometric flows, such as Ricci flow and Yamabe flow has been the focus of attention of many geometers. Recently, in 2019, Guler and Crasmareanu [7] introduced the study of a new geometric flow which is a scalar combination of Ricci and Yamabe flows under the name Ricci-Yamabe map. This is also called the Ricci-Yamabe flow of type (α, β). The Ricci-Yamabe flow is an evolution equation for the metrics on the Riemannian or semi-Riemannian manifolds defined as [7]

$$\frac{\partial}{\partial t}g(t) = -2\alpha Ric(t) + \beta r(t)g(t), g_0 = g(0). \tag{1.1}$$

A solution to the Ricci-Yamabe flow is called Ricci-Yamabe soliton if it depends only on one parameter group of diffeomorphism and scaling. A Ricci-Yamabe soliton

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on a Riemannian manifold (M, g) consists of data $(g, V, \lambda, \alpha, \beta)$ satisfying

$$\pounds_V g + 2\alpha S + (2\lambda - \beta r)g = 0, \tag{1.2}$$

where $\pounds_V$ is the Lie-derivative, S is the Ricci tensor, r is the scalar curvature and $\lambda, \alpha, \beta \in \mathbb{R}$. If V is the gradient of a smooth function f on M, then the above soliton is called the gradient Ricci-Yamabe soliton and then equation (1.2) reduces to

$$\nabla^2 f + \alpha S = (\lambda - \frac{1}{2}\beta r)g, \tag{1.3}$$

where $\nabla^2 f$ is the Hessian of f.

The Ricci-Yamabe soliton is said to be expanding, steady, or shrinking according to whether λ is negative, zero, or positive. It is called an almost Ricci-Yamabe soliton if α, β and λ are smooth functions on M. A Ricci-Yamabe soliton is said to be a [5]

- Ricci soliton [8] if $\alpha = 1, \beta = 0$.
- Yamabe soliton[9] if $\alpha = 0, \beta = 1$.
- Einstein soliton [2] if $\alpha = 1, \beta = -1$.
- $\rho-$Einstein soliton [3] if $\alpha = 1, \beta = -2\rho$.

When $\alpha \neq 0, 1$, a Ricci-Yamabe soliton is proper. All of these classifications apply to gradient Ricci-Yamabe solitons as well.

The paper is organized as follows. After preliminaries in Section 2, we consider Ricci-Yamabe solitons on Kenmotsu 3-manifolds in Section 3. In Section 4 we study Ricci-Yamabe solitons on Kenmotsu 3-manifolds with η-parallel Ricci tensors. Section 5 is devoted to studying gradient Ricci-Yamabe solitons on Kenmotsu 3-manifolds. Finally, in Section 6 we construct an example of a 3-dimensional Kenmotsu manifold admitting a Ricci-Yamabe soliton.

2. Preliminaries

An almost contact structure [1] on a $(2n + 1)$-dimensional smooth manifold M^{2n+1} is a triplet (ϕ, ξ, η), where ϕ is a $(1, 1)$-type tensor, ξ a global vector field and η a 1-form, such that

$$\phi^2 = -id + \eta \otimes \xi, \eta(\xi) = 1, \tag{2.1}$$

where 'id' denotes the identity mapping and relation (2.1) implies that $\phi(\xi) = 0$, $\eta \circ \phi = 0$ and $\text{rank}(\phi) = 2n$. The almost contact structure induces a natural almost complex structure J on the product manifold $M \times \mathbb{R}$ defined by $J(U, \lambda d/dt) = (\phi U - \lambda\xi, \eta(U)d/dt)$, where U is tangent to M, t the coordinate of $\mathbb{R}$ and λ a smooth function on $M \times \mathbb{R}$. The almost contact structure is said to be normal [12] if the almost complex structure J is integrable or equivalently $[\phi, \phi] + 2d\eta \otimes \xi$ vanishes, where $[\phi, \phi]$ is the Nijenhuis torsion of ϕ. Let g be a compatible Riemannian metric with (ϕ, ξ, η), that is, let

$$g(\phi U, \phi V) = g(U, V) - \eta(U)\eta(V) \tag{2.2}$$

or equivalently, $\Phi(U, V) = g(U, \phi V)$ along with $g(U, \xi) = \eta(U)$ for all $U, V \in \chi(M)$. With this, M is an almost contact metric manifold equipped with an almost contact metric structure (ϕ, ξ, η, g). An almost contact metric manifold is called a Kenmotsu manifold if it satisfies

$$(\nabla_U \phi)V = g(\phi U, V)\xi - \eta(V)\phi U \tag{2.3}$$

for all $U, V \in \chi(M)$, where ∇ is Levi-Civita connection of the Riemannian metric. A Kenmotsu manifold is normal but not Sasakian. Moreover, it is also not compact since from the formula (2.3) we get

$$\nabla_U \xi = U - \eta(U)\xi, \tag{2.4}$$

which gives $div\xi = 2n$. A conformal change g^* of a Riemannian metric g is called a concircular transformation [14] if geodesic circles of g are transformed into geodesic circles of g^*. Here a geodesic circle means a curve whose first curvature is constant and whose second curvature is identically zero. A cosymplectic structure is defined to be a normal almost contact metric structure (ϕ, ξ, η, g) with both the fundamental 2-form Φ and the 1-form η is closed. An almost contact metric structure is cosymplectic if and only if $\nabla\phi = 0$. In [11], Kirichenko obtained the class of Kenmotsu manifolds from cosymplectic manifolds by the canonical concircular transformations. A Kenmotsu manifold is of constant curvature -1 if and only if it is canonically concircular to $C^n \times \mathbb{R}$[11].

For a $(2n+1)$-dimensional Kenmotsu manifold, the following formulas hold:

$$R(U, V)\xi = \eta(U)V - \eta(V)U, \tag{2.5}$$

$$(\nabla_U \eta)V = g(U, V) - \eta(U)\eta(V), \tag{2.6}$$

$$S(\xi, \xi) = g(Q\xi, \xi) = -2n \tag{2.7}$$

for any $U, V \in \chi(M)$, where S is the Ricci tensor and Q is the Ricci operator.

From [4], we know that for a 3-dimensional Kenmotsu manifold

$$\begin{aligned} R(U, V)W \quad = \quad & \frac{r+4}{2}[g(V, W)U - g(U, W)V] \\ & -\frac{r+6}{2}[g(V, W)\eta(U)\xi - g(U, W)\eta(V)\xi \\ & +\eta(V)\eta(W)U - \eta(U)\eta(W)V], \end{aligned} \tag{2.8}$$

$$QU = \frac{1}{2}[(r+2)U - (r+6)\eta(U)\xi], \tag{2.9}$$

$$S(U,V) = \frac{1}{2}[(r+2)g(U,V) - (r+6)\eta(U)\eta(V)], \tag{2.10}$$

where R is the curvature tensor and r is the scalar curvature of the manifold M.

An almost contact metric manifold is said to be η-Einstein if the Ricci tensor S satisfies

$$S(V,W) = ag(V,W) + b\eta(V)\eta(W) \tag{2.11}$$

for any vector field V,W on M and arbitrary functions a, b on M. An η-Einstein manifold with b vanishing and a constant is obviously an Einstein manifold. An η-Einstein manifold is said to be proper η-Einstein if $b \neq 0$.

Three-dimensional Kenmotsu manifold have been studied in the papers ([4], [13]).

Lemma 2.1.([13]) *On any three-dimensional Kenmotsu manifold* $(M^3, \phi, \xi, \eta, g)$ *we have*

$$\xi r = -2(r+6). \tag{2.12}$$

Lemma 2.2.([4]) *A 3-dimensional Riemannian manifold is a manifold of constant sectional curvature -1 if and only if the scalar curvature* $r = -6$.

Lemma 2.3.(Proposition 8 of ([10]) *Let* M *be an* η*-Einstein Kenmotsu manifold* $S = ag + b\eta \otimes \eta$ *for scalars* a *and* b. *If either* a *or* b *is constant then the manifold becomes an Einstein manifold.*

3. Ricci-Yamabe Solitons on Kenmotsu 3-manifolds

Assume that the Kenmotsu 3-manifold admits a proper Ricci-Yamabe soliton $(g, \xi, \lambda, \alpha, \beta)$. Then the relation (1.2) yields

$$(\pounds_\xi g)(U,V) + 2\alpha S(U,V) + (2\lambda - \beta r)g(U,V) = 0. \tag{3.1}$$

In Kenmotsu 3-manifolds

$$\begin{aligned} (\pounds_\xi g)(U,V) &= g(\nabla_U \xi, V) + g(U, \nabla_V \xi) \\ &= 2[g(U,V) - \eta(U)\eta(V)]. \end{aligned} \tag{3.2}$$

Using (3.2) in (3.1), we get

$$S(U,V) = \frac{1}{\alpha}[(\frac{\beta}{2}r - \lambda - 1)g(U,V) + \eta(U)\eta(V)], \tag{3.3}$$

which implies

$$QU = \frac{1}{\alpha}[(\frac{\beta}{2}r - \lambda - 1)U + \eta(U)\xi]. \tag{3.4}$$

Hence we can state the following theorem using Lemma 2.3 :
Theorem 3.1. *A proper Ricci-Yamabe soliton on a 3-dimensional Kenmotsu manifold is an Einstein manifold.*

4. Ricci-Yamabe Solitons on Kenmotsu 3-manifolds with η-parallel Ricci Tensor

In this section we study Ricci-Yamabe solitons on Kenmotsu 3-manifolds with η-parallel Ricci tensor. A Kenmotsu manifold is said to have η-parallel Ricci tensor if [6]

$$g((\nabla_V Q)U, W) = 0 \tag{4.1}$$

for all smooth vector fields U, V, W.
In Kenmotsu 3-manifolds

$$\begin{aligned}(\nabla_V Q)U &= \nabla_V QU - Q(\nabla_V U)\\ &= \frac{1}{\alpha}[\frac{\beta}{2}(Vr)U + ((\nabla_V\eta)U)\xi + \eta(U)\nabla_V\xi]\\ &= \frac{1}{\alpha}[\frac{\beta}{2}(Vr)U + g(U,V)\xi + \eta(U)V - 2\eta(U)\eta(V)\xi].\end{aligned} \tag{4.2}$$

Using (4.2) in (4.1), we get

$$\frac{1}{\alpha}[\frac{\beta}{2}(Vr)g(U,W) + g(U,V)\eta(W) + g(V,W)\eta(U) - 2\eta(U)\eta(V)\eta(W)] = 0. \tag{4.3}$$

Putting $V = \xi$ in (4.3) and using Lemma 2.1 we obtain

$$\frac{\beta}{\alpha}(r+6)g(U,W) = 0. \tag{4.4}$$

If $\alpha = 1$, then (4.4) implies either $\beta = 0$ or, $r = -6$.

Case I: If $\beta = 0$ and $\alpha = 1$, then Ricci-Yamabe soliton becomes Ricci soliton.
Case II: If $r = -6$, then from Lemma 2.2, the manifold becomes a manifold of constant sectional curvature -1.
Hence we conclude the following:
Theorem 4.1. *If a 3-dimensional Kenmotsu manifold admits a Ricci-Yamabe soliton with η-parallel Ricci tensor, then either it is a Ricci soliton or it is a manifold of constant sectional curvature -1, provided $\alpha = 1$.*

5. Gradient Ricci-Yamabe Solitons on Kenmotsu 3-manifolds

Suppose a Kenmotsu 3-manifold admits the gradient Ricci-Yamabe soliton. Then from equation (1.3), we get

$$\nabla_U Df = (\lambda - \frac{1}{2}\beta r)U - \alpha QU. \tag{5.1}$$

Differentiating (5.1) covariantly along any vector field V, we get

$$\nabla_V \nabla_U Df = (\lambda - \frac{1}{2}\beta r)\nabla_V U - \frac{\beta}{2}(Vr)U - \alpha \nabla_V QU. \tag{5.2}$$

Interchanging U and V in the above equation, we infer

$$\nabla_U \nabla_V Df = (\lambda - \frac{1}{2}\beta r)\nabla_U V - \frac{\beta}{2}(Ur)V - \alpha \nabla_U QV. \tag{5.3}$$

Hence from the above equations, we get

$$R(U,V)Df = \frac{\beta}{2}[(Vr)U - (Ur)V] - \alpha[(\nabla_U Q)V - (\nabla_V Q)U]. \tag{5.4}$$

Now, in 3-dimension Kenmotsu manifolds

$$\begin{aligned}(\nabla_U Q)V - (\nabla_V Q)U &= \frac{1}{2}[(Ur)V - (Ur)\eta(V)\xi - (Vr)U + (Vr)\eta(U)\xi] \\ &\quad -(\frac{r}{2} + 3)[\eta(Y)X - \eta(X)Y].\end{aligned} \tag{5.5}$$

Using (5.5) in (5.4), we get

$$\begin{aligned}R(U,V)Df &= \frac{\beta}{2}[(Vr)U - (Ur)V] \\ &\quad -\frac{\alpha}{2}[(Ur)V - (Ur)\eta(V)\xi - (Vr)U + (Vr)\eta(U)\xi \\ &\quad -(r+6)\{\eta(V)U - \eta(U)V\}].\end{aligned} \tag{5.6}$$

Contracting (5.6) and using Lemma 2.1, we get

$$S(V, Df) = (\beta + \frac{\alpha}{2})(Vr). \tag{5.7}$$

Again, replacing U by Df in (2.10), we get

$$S(V, Df) = (\frac{r}{2} + 1)(Vf) - (\frac{r}{2} + 3)\eta(V)(\xi f). \tag{5.8}$$

In view of (5.7) and (5.8) we infer

$$(\frac{r}{2} + 1)(Vf) - (\frac{r}{2} + 3)\eta(V)(\xi f) = (\beta + \frac{\alpha}{2})(Vr). \tag{5.9}$$

Putting $V = \xi$ in the above equation, we get

$$\xi f = (r+6)(\beta + \frac{\alpha}{2}). \tag{5.10}$$

Taking inner product of (5.7) with the vector field ξ, we get

$$\eta(V)(Uf) - \eta(U)(Vf) = \frac{\beta}{2}[(Vr)\eta(U) - (Ur)\eta(V)]. \tag{5.11}$$

Putting $U = \xi$ in (5.11) and using Lemma 2.1, we get

$$Vf = -\frac{\beta}{2}(Vr) + \frac{\alpha}{2}(r+6)\eta(V). \tag{5.12}$$

Using (5.10) and (5.12) in (5.9), we obtain

$$(\frac{\beta}{2}r + 3\beta + \alpha)[(Vr) + 2(r+6)\eta(V)] = 0. \tag{5.13}$$

Hence above equation implies either, $\frac{\beta}{2}r + 3\beta + \alpha = 0$ or, $Vr + 2(r+6)\eta(V) = 0$.

Case I: If $\frac{\beta}{2}r + 3\beta + \alpha = 0$, then r is constant.

Case II: If $Vr = -2(r+6)\eta(V)$. Using this in (5.12), we get

$$Df = (\beta + \frac{\alpha}{2})(r+6)\xi, \tag{5.14}$$

which implies

$$\nabla_\xi Df = -2(\beta + \frac{\alpha}{2})(r+6)\xi. \tag{5.15}$$

Using the above equation in (5.1), we get

$$(\frac{3}{2}\beta + \alpha)r = -(12\beta + 8\alpha + \lambda), \tag{5.16}$$

which implies r is constant.
Thus, we state the following:
Theorem 5.1. *If the metric of a Kenmotsu 3-manifold M is a gradient Ricci-Yamabe soliton, then the scalar curvature is constant.* If we take $\beta = 1$ and $\alpha = 0$, then both cases implies $r = -6$. Then from Lemma 2.2, we say that it is a manifold of constant sectional curvature -1.

6. Example

We consider the 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3, z \neq 0\}$, where (x, y, z) are the standard coordinate of $\mathbb{R}^3$. Let $\{e_1, e_2, e_3\}$ be a linearly independent global frame on M given by

$$e_1 = z\frac{\partial}{\partial x}, e_2 = z\frac{\partial}{\partial y}, e_3 = -z\frac{\partial}{\partial z}.$$

Let g be the Riemannian metric defined by

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1,$$

$$g(e_1, e_2) = g(e_1, e_3) = g(e_2, e_3) = 0.$$

Let η be the 1-form defined by $\eta(W) = g(W, e_3)$, for all $W \in \chi(M)$ and ϕ be the $(1, 1)$-tensor defined by

$$\phi e_1 = -e_2, \phi e_2 = e_1, \phi e_3 = 0.$$

Then using the linearity of ϕ and g, we get

$$\phi^2 W = -W + \eta(W)e_3, \eta(e_3) = 1,$$

$$g(\phi V, \phi W) = g(V, W) - \eta(V)\eta(W)$$

for any $V, W \in \chi(M)$.

Then for $e_3 = \xi$, the structure (ϕ, ξ, η, g) defines an almost contact metric structure on M.

Let ∇ be the Levi-Civita connection with respect to the metric g. Then we have

$$[e_1, e_2] = 0, [e_2, e_3] = e_2, [e_1, e_3] = e_1.$$

The Riemannian connection ∇ of the metric g and using Koszul's formula , we have

$$\nabla_{e_1} e_1 = -e_3, \nabla_{e_1} e_2 = 0, \nabla_{e_1} e_3 = e_1,$$

$$\nabla_{e_2} e_1 = 0, \nabla_{e_2} e_2 = -e_3, \nabla_{e_2} e_3 = e_2,$$

$$\nabla_{e_3} e_1 = 0, \nabla_{e_3} e_2 = 0, \nabla_{e_3} e_3 = 0.$$

For the above we see that $\nabla_W \xi = W - \eta(W)\xi$ for all $W \in \chi(M)$. Hence the manifold is a Kenmotsu manifold.

Now, we have

$$R(U, V)W = \nabla_U \nabla_V W - \nabla_V \nabla_U W - \nabla_{[U,V]} W. \tag{6.1}$$

With the help of the above results and using (6.1), we obtain

$$R(e_1, e_2)e_3 = 0, R(e_1, e_2)e_2 = -e_1, R(e_1, e_2)e_1 = e_2,$$

$$R(e_2, e_3)e_3 = -e_2, R(e_3, e_2)e_2 = -e_3, R(e_3, e_1)e_1 = -e_3,$$
$$R(e_3, e_1)e_3 = e_1.$$

From the above, we can easily calculate the non-vanishing components of the Ricci tensor S as follows:

$$S(e_1, e_1) = S(e_2, e_2) = S(e_3, e_3) = -2.$$

Hence from the above, we get

$$r = S(e_1, e_1) + S(e_2, e_2) + S(e_3, e_3) = -6,$$

where r is the scalar curvature. From (3.3) we obtain

$$S(e_1, e_1) = S(e_2, e_2) = S(e_3, e_3) = -\frac{1}{\alpha}(3\beta + \lambda).$$

Therefore $\lambda = 2\alpha - 3\beta$. Hence it is Ricci-Yamabe soliton on Kenmotsu 3-manifolds.

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