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Estimations of Zeros of a Polynomial Using Numerical Radius Inequalities

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Abstract. We present new bounds for the numerical radius of bounded linear operators and 2×2 operator matrices. We apply upper bounds for the numerical radius to the Frobenius companion matrix of a complex monic polynomial to obtain new estimations for the zeros of that polynomial. We also show with numerical examples that our new estimations improve on the existing estimations.

1. Introduction

The purpose of the present article is to present a general method to estimate the zeros of a monic polynomial. The estimation for the zeros of a polynomial have important applications in many areas of sciences such as signal processing, control

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theory, communication theory, coding theory and cryptography. To find the exact zeros of a polynomial of higher order is very difficult and there is no standard method as such. For this reason, the estimation of the disk containing all the zeros of a polynomial is an important area of research. Over the years many mathematicians have developed various tools to estimate the disk that contains all the zeros. We use the numerical radius inequalties of the Frobenius companion matrix associated with a given polynomial to find a disk of smaller radius that contains all the zeros of the polynomial. This is the time to introduce some notations and terminologies to be used in this article.

Let $\mathbb{H}_1$, $\mathbb{H}_2$ be two complex Hilbert spaces with usual inner product $\langle .,. \rangle$ and $B(\mathbb{H}_1, \mathbb{H}_2)$ denote the set of all bounded linear operators from $\mathbb{H}_1$ into $\mathbb{H}_2$. If $\mathbb{H}_1 = \mathbb{H}_2 = \mathbb{H}$ then we write $B(\mathbb{H}_1, \mathbb{H}_2) = B(\mathbb{H})$. For $T \in B(\mathbb{H})$, let $Re(T)$ and $Im(T)$ denote the real part of T and the imaginary part of T, respectively, i.e., $Re(T) = \frac{1}{2}(T + T^*)$ and $Im(T) = \frac{1}{2i}(T - T^*)$. Here T^* denotes the adjoint of T. For $T \in B(\mathbb{H})$, the operator norm $\|T\|$ of T is defined as :

$$\|T\| = \sup\left\{\|Tx\| : x \in \mathbb{H}, \|x\| = 1\right\}.$$

For $T \in B(\mathbb{H})$, the numerical range $W(T)$, numerical radius $w(T)$ and Crawford number $m(T)$ of T are defined as:

$$\begin{aligned} W(T) &= \left\{\langle Tx, x\rangle : x \in \mathbb{H}, \|x\| = 1\right\}, \\ w(T) &= \sup\left\{|\mu| : \mu \in W(T)\right\}, \\ m(T) &= \inf\left\{|\mu| : \mu \in W(T)\right\}. \end{aligned}$$

It is easy to verify that $w(.)$ is a norm on $B(\mathbb{H})$ and equivalent to the operator norm $\|.\|$ satisfying the following inequality

$$\frac{1}{2}\|T\| \leq w(T) \leq \|T\|.$$

Observe that spectrum $\sigma(T)$ of T is contained in the closure of the numerical range $W(T)$ of T, so the spectral radius $r(T)$ of T always satisfies $r(T) \leq w(T)$. Let us consider a monic polynomial of degree n, $p(z) = z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \ldots + a_1 z + a_0$, where the coefficients $a_i \in \mathbb{C}$ for $i = 0, 1, \ldots, n-1$. The Frobenius companion matrix $C(p)$, associated with polynomial $p(z)$, is given by

$$C(p) = \begin{pmatrix} -a_{n-1} & -a_{n-2} & \ldots & -a_1 & -a_0 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix}_{n \times n}.$$

It is easy to verify that all the eigenvalues of $C(p)$ are exactly the zeros of the polynomial $p(z)$. Considering $C(p)$ as a bounded linear operator on $\mathbb{C}^n$, we get

$r(C(p)) \leq w(C(p))$ and so $|\lambda| \leq w(C(p))$, where λ is a zero of $p(z)$. If R is the radius of a disk with center at the origin that contains all the zeros of $p(z)$, then $w(C(p))$ is one such R. Over the years various mathematicians have estimated radius R using various technique. Few of them are listed in below.

(1) Abdurakhmanov [1] proved that

$$|\lambda| \leq \frac{1}{2}\left(|a_{n-1}| + \cos\frac{\pi}{n} + \sqrt{\left(|a_{n-1}| - \cos\frac{\pi}{n}\right)^2 + \left(1 + \sqrt{\sum_{j=0}^{n-2}|a_j|^2}\right)^2}\right) = R_A.$$

(2) Abu-Omar and Kittaneh [4] proved that

$$|\lambda| \leq \sqrt{\frac{1}{4}(|a_{n-1}|^2 + \alpha)^2 + \alpha + \cos^2\frac{\pi}{n+1}} = R_{AK},$$

where $\alpha = \sqrt{\sum_{j=0}^{n-1}|a_j|^2}$.

(3) Bhunia et. al. [6] proved that

$$|\lambda| \leq |\frac{a_{n-1}}{n}| + \cos\frac{\pi}{n} + \frac{1}{2}[(1+\alpha)^2 + 4\alpha + 4\sqrt{\alpha}(1+\alpha)]^{\frac{1}{4}} = R_{BBP},$$

where

$$\begin{aligned}\alpha_r &= \sum_{k=r}^{n} {}^kC_r\big(-\frac{a_{n-1}}{n}\big)^{k-r} a_k, \quad r = 0, 1, \ldots, n-2, a_n = 1, {}^0C_0 = 1,\\ \alpha &= \sum_{i=0}^{n-2}|\alpha_i|^2.\end{aligned}$$

(4) Cauchy [12] proved that

$$|\lambda| \leq 1 + \max\{|a_0|, |a_1|, \ldots, |a_{n-1}|\} = R_C.$$

(5) Carmichael and Mason [12] proved that

$$|\lambda| \leq (1 + |a_0|^2 + |a_1|^2 + \ldots + |a_{n-1}|^2)^{\frac{1}{2}} = R_{CM}.$$

(6) Fujii and Kubo [10] proved that

$$|\lambda| \leq \cos\frac{\pi}{n+1} + \frac{1}{2}\big[\big(\sum_{j=0}^{n-1}|a_j|^2\big)^{\frac{1}{2}} + |a_{n-1}|\big] = R_{FK}.$$

(7) Kittaneh [16] proved that

$$|\lambda| \leq \frac{1}{2}\left(|a_{n-1}| + 1 + \sqrt{(|a_{n-1}| - 1)^2 + 4\sqrt{\sum_{j=0}^{n-2}|a_j|^2}}\right) = R_{K_1}.$$

(8) Kittaneh [16] proved that

$$|\lambda| \leq \frac{1}{2}\left(|a_{n-1}| + \cos\frac{\pi}{n} + \sqrt{\left(|a_{n-1}| - \cos\frac{\pi}{n}\right)^2 + (|a_{n-2}|+1)^2 + \sum_{j=0}^{n-3}|a_j|^2}\right) = R_{K_2}.$$

(9) Montel [12] proved that

$$|\lambda| \leq \max\left\{1, \sum_{r=o}^{n-1}|a_r|\right\} = R_M.$$

In this article, we obtain some upper bounds for the numerical radius of bounded linear operators and operator matrices. Using these bounds and the bounds obtained in [5, 6, 7, 8, 9], we obtain bounds for the radius of the disk with centre at the origin that contains all the zeros of a complex monic polynomial. Also we show with numerical examples that these bounds obtained here improve on the existing bounds.

2. Estimations for the Numerical Radius

In this section, we obtain upper bounds for the numerical radius of bounded linear operators which will be used to estimate the zeros of a polynomial in the next section. We need the following numerical radius equality [20].

Lemma 2.1. *Let $T \in B(\mathbb{H})$ and $H_\theta = Re(e^{i\theta}T)$, where $\theta \in \mathbb{R}$. Then, $w(T) = \sup_{\theta\in\mathbb{R}} \|H_\theta\|$.*

First we obtain the following refinement of [13, Th. 3.6].

Theorem 2.2. *Let $T \in B(\mathbb{H})$. Then, $w^2(T) \leq w(T^2) + \min\left\{\|Re(T)\|^2, \|Im(T)\|^2\right\}$.*

Proof. Let $H_\theta = Re(e^{i\theta}T)$, where $\theta \in \mathbb{R}$. Then

$$\begin{aligned}
4H_\theta^2 &= e^{2i\theta}T^2 + e^{-2i\theta}T^{*2} + TT^* + T^*T \\
\Rightarrow 4H_\theta^2 &= (e^{2i\theta}-1)T^2 + (e^{-2i\theta}-1)T^{*2} + T^2 + T^{*2} + TT^* + T^*T \\
\Rightarrow H_\theta^2 &= \frac{1}{2}Re\left\{(e^{2i\theta}-1)T^2\right\} + (Re(T))^2 \\
\Rightarrow H_\theta^2 &= \sin\theta\, Re\left\{e^{i(\theta+\frac{\pi}{2})}T^2\right\} + (Re(T))^2 \\
\Rightarrow \|H_\theta\|^2 &\leq \|Re\left\{e^{i(\theta+\frac{\pi}{2})}T^2\right\}\| + \|Re(T)\|^2 \\
&\leq w(T^2) + \|Re(T)\|^2, \quad \text{using Lemma 2.1.}
\end{aligned}$$

Taking supremum over $\theta \in \mathbb{R}$ and then using Lemma 2.1 we get, $w^2(T) \leq w(T^2) + \|Re(T)\|^2$. Applying similar argument we can prove that $w^2(T) \leq w(T^2) + \|Im(T)\|^2$. This completes the proof of the theorem. □

Remark 2.3. (i) We would like to remark that Theorem 2.2 stronger than the inequality [13, Th. 3.6], stated as $w^2(T) \le w(T^2) + 2\min\left\{\|Re(T)\|^2, \|Im(T)\|^2\right\}$.
(ii) It follows from Theorem 2.2 that if $T^2 = 0$ then $w(T) \le \|Re(T)\|$ and $w(T) \le \|Im(T)\|$. From [5, Th. 3.3], it follows that for any $T \in B(\mathbb{H})$, $\|Re(T)\|^2 + m^2(Im(T)) \le w^2(T)$, $\|Im(T)\|^2 + m^2(Re(T)) \le w^2(T)$. So, if $T^2 = 0$ then $w(T) = \|Re(T)\| = \|Im(T)\|$ and $m(Re(T)) = m(Im(T)) = 0$. Also we have from Theorem 2.2 and [5, Th. 3.3] that for any $T \in B(\mathbb{H})$, $m(Re(T)) \le \sqrt{w(T^2)}$ and $m(Im(T)) \le \sqrt{w(T^2)}$.

Next we obtain an upper bound for the numerical radius of 2×2 operator matrices.

Theorem 2.4. *Let* $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, *where* $A \in B(\mathbb{H}_1), B \in B(\mathbb{H}_2, \mathbb{H}_1), C \in B(\mathbb{H}_1, \mathbb{H}_2), D \in B(\mathbb{H}_2)$. *Then,*

$$w(T) \le \frac{1}{2}\left[w(A) + w(D) + \sqrt{(w(A) - w(D))^2 + \|B\|^2 + \|C\|^2 + 2w(CB)}\right].$$

Proof. Abu-Omar and Kittaneh in [2, Cor. 2] proved that

$$w(T) \le \frac{1}{2}\left[w(A) + w(D) + \sqrt{(w(A) - w(D))^2 + 4w^2(T_0)}\right].$$

where $T_0 = \begin{pmatrix} O & B \\ C & O \end{pmatrix}$. We proved in [6, Th. 2.5] that

$$w^4\begin{pmatrix} O & B \\ C & O \end{pmatrix} \le \frac{1}{16}\|S\|^2 + \frac{1}{4}w^2(CB) + \frac{1}{8}w(CBS + SCB),$$

where $S = |B|^2 + |C^*|^2$. Our required bound follows from these above two bounds, using the facts that $w(CBS + SCB) \le 2w(CB)\|S\|$, (see [9, Remark 5]) and $\|S\| \le \|B\|^2 + \|C\|^2$. $\square$

Remark 2.5. Paul and Bag in [18, Th. 2.1, (i)] proved that

$$w(T) \le \frac{1}{2}\Big[w(A) + w(D) + \sqrt{(w(A) - w(D))^2 + (\|B\| + \|C\|)^2}\Big].$$

Clearly, it is weaker than the inequality obtained in Theorem 2.4.

Next we give an another upper bound for the numerical radius of 2×2 operator matrices.

Theorem 2.6. *Let* $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, *where* $A \in B(\mathbb{H}_1), B \in B(\mathbb{H}_2, \mathbb{H}_1), C \in B(\mathbb{H}_1, \mathbb{H}_2), D \in B(\mathbb{H}_2)$. *Then,*

$$w(T) \le \frac{1}{2}\left[w(A) + w(D) + \sqrt{(w(A) - w(D))^2 + 4\alpha_1^2}\right],$$

where

$$\alpha_1 = \left[\frac{1}{8}\max\left\{\left(\|B\|^2+\|C\|^2\right)^2+4w^2\left(BC\right),\left(\|B\|^2+\|C\|^2\right)^2+4w^2\left(CB\right)\right\}\right]^{\frac{1}{4}}.$$

Proof. This inequality follows from the two inequalities proved in [2, Cor. 2] and [8, Th. 2.6], respectively, stated below:

$$w(T) \leq \frac{1}{2}\left[w(A)+w(D)+\sqrt{(w(A)-w(D))^2+4w^2(T_0)}\right], T_0 = \begin{pmatrix} O & B \\ C & O \end{pmatrix}.$$

and

$$w^4\begin{pmatrix} O & B \\ C & O \end{pmatrix} \leq \frac{1}{8}\max\left\{\|BB^*+C^*C\|^2+4w^2(BC), \|B^*B+CC^*\|^2+4w^2(CB)\right\}.$$

□

Next we obtain some upper bounds for the numerical radius of bounded linear operators defined on $\mathbb{H}$. We need the Aluthge transform of an operator T. For $T \in B(\mathbb{H})$, the Aluthge transform [14] of T, denoted as $\widetilde{T}$, is defined as $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$, where $|T| = (T^*T)^{\frac{1}{2}}$ and U is the partial isometry associated with the polar decomposition of T, i.e., $T = U|T|$. It follows easily from the definition of $\widetilde{T}$ that $\|\widetilde{T}\| \leq \|T\|$ and $r(\widetilde{T}) = r(T)$, also $w(\widetilde{T}) \leq w(T)$ (see [14]). For definition and more information about the Aluthge transform we refer the reader to [20] and references therein.

Theorem 2.7. *Let $T \in B(\mathbb{H})$. Then,* $w^2(T) \leq \frac{1}{4}\|T\|\|T^2\|^{\frac{1}{2}} + \frac{1}{4}\|T^2\| + \frac{1}{2}\|T\|^2$.

Proof. The proof follows from the observations $w(\widetilde{T}^2) \leq \|T\|^2$, $\|\widetilde{T}\| \leq \|T^2\|^{\frac{1}{2}}$, $\|T^*T + TT^*\| \leq \|T^2\| + \|T\|^2$. and the inequality [7, Th. 4],

$$w^2(T) \leq \frac{1}{4}w(\widetilde{T}^2) + \frac{1}{4}\|T\|\|\widetilde{T}\| + \frac{1}{4}\|T^*T + TT^*\|.$$

□

We end this section with the following inequalities, the proof of which follows from [7, Th. 2] as well as [3, Cor. 2.6].

Theorem 2.8. *Let $T \in B(\mathbb{H})$. Then,*

$$(i) \quad w^2(T) \leq \frac{1}{2}\|T\|\|T^2\|^{\frac{1}{2}} + \frac{1}{4}\|T^2\| + \frac{1}{4}\|T\|^2$$

and

$$(ii) \quad w^2(T) \leq \frac{3}{4}\|T^2\| + \frac{1}{4}\|T\|^2.$$

Proof. The proof of (i) follows from [7, Th. 2], i.e., $w^2(T) \leq \frac{1}{2}\|T\|\|\widetilde{T}\| + \frac{1}{4}\|T^*T + TT^*\|$, and the inequalities $\|\widetilde{T}\| \leq \|T^2\|^{\frac{1}{2}}$ and $\|T^*T + TT^*\| \leq \|T^2\| + \|T\|^2$. The proof of (i) also follows from [3, Cor. 2.6], i.e., $w(T) \leq \frac{1}{2}\sqrt{\|T\|^2 + \|T^2\| + 2w(T^2)}$. The proof of (ii) follows from [3, Cor. 2.6]. □

3. Estimations for Zeros of a Polynomial

Consider a monic polynomial of degree n, $p(z) = z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \ldots + a_1 z + a_0$, where the coefficients $a_i \in \mathbb{C}$ for $i = 0, 1, \ldots, n-1$. Let R denote radius of a disk with center at the origin that contains all the zeros of $p(z)$. If λ is a zero of $p(z)$, equivalently, if λ is an eigen value of the Frobenius companion matrix $C(p)$ (as described in the introduction), then $\mid \lambda \mid \leq R$. Our goal in this section is to obtain smaller possible values of R. To do so we need the following two well known results on the numerical radius equality.

Lemma 3.1. ([11, pp. 8-9]) *If* $L_n = \begin{pmatrix} 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix}_{n \times n}$ *, then* $w(L_n) = \cos\frac{\pi}{n+1}$.

Lemma 3.2. ([10]) *If* $x_i \in \mathbb{C}$ *for each* $i = 1, 2, \ldots, n$, *then*

$$w\begin{pmatrix} x_1 & x_2 & \ldots & x_n \\ 0 & 0 & \ldots & 0 \\ 0 & & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix} = \frac{1}{2}\left(|x_1| + \sqrt{\sum_{r=1}^{n} |x_r|^2}\right).$$

By using Lemmas 3.1 and 3.2, we obtain some new bounds for the zeros of $p(z)$. First using Theorem 2.4, we prove the following theorem.

Theorem 3.3. *Let* λ *be a zero of* $p(z)$. *Then,*

$$|\lambda| \leq w(C(p)) \leq \frac{1}{2}\left[|a_{n-1}| + cos\frac{\pi}{n} + \sqrt{(|a_{n-1}| - cos\frac{\pi}{n})^2 + \sum_{r=0}^{n-2} |a_r|^2 + 1 + \alpha}\right] = R_1,$$

where $\alpha = |a_{n-2}| + \sqrt{\sum_{r=0}^{n-2} |a_r|^2}$.

Proof. Let $C(p) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $A = (a_{n-1})_{1\times 1}$, $C = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{n-1\times 1}$,
$B = (-a_{n-2} - a_{n-3} \ldots - a_1 - a_0)_{1\times n-1}$ and $D = L_{n-1}$. Then using Lemmas 3.1 and 3.2 in Theorem 2.4 we get,

$$w(C(p)) \leq \frac{1}{2}\left[|a_{n-1}| + cos\frac{\pi}{n} + \sqrt{(|a_{n-1}| - cos\frac{\pi}{n})^2 + \sum_{r=0}^{n-2}|a_r|^2 + 1 + \alpha}\right],$$

where $\alpha = |a_{n-2}| + \sqrt{\sum_{r=0}^{n-2}|a_r|^2}$. This completes the proof. □

Next using Theorem 2.6, we prove the following theorem.

Theorem 3.4. *Let λ be a zero of $p(z)$. Then,*

$$|\lambda| \leq w(C(p)) \leq \frac{1}{2}\left[|a_{n-1}| + \cos\frac{\pi}{n} + \sqrt{\left(|a_{n-1}| - \cos\frac{\pi}{n}\right)^2 + 4\alpha^2}\right] = R_2,$$

where

$$\begin{aligned} \alpha &= \left[\frac{1}{8}\max\left\{(\beta+1)^2 + 4|a_{n-2}|^2, (\beta+1)^2 + \delta^2\right\}\right]^{\frac{1}{4}}, \\ \beta &= \sum_{r=0}^{n-2}|a_r|^2, \\ \delta &= |a_{n-2}| + \sqrt{\sum_{r=0}^{n-2}|a_r|^2}. \end{aligned}$$

Proof. We consider $C(p) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where A, B, C and D are same as in the proof of Theorem 3.3. Then using Lemmas 3.1 and 3.2 in Theorem 2.6 we have the desired bound. □

In the following example we show with a numerical example that our estimations in Theorems 3.3 and 3.4 are better than the existing estimations.

Example 3.5. We consider a polynomial $p(z) = z^5 + 4z^4 + z^3 + z^2 + z + 1$. Then the upper bounds for the zeros of this polynomial $p(z)$ estimated by different mathematicians are as shown in the following table.

R_{FK}	5.1020
R_{K_1}	4.5615
R_{AK}	4.8131
R_A	4.5943
R_{BBP}	7.2809.

But, Theorem 3.3 gives $R_1 = 4.5365$ and Theorem 3.4 gives $R_2 = 4.5509$. Therefore, for this polynomial $p(z)$, our obtain bounds in Theorems 3.3 and 3.4 are better than the above mentioned bounds.

We next obtain an estimation of radius R and for that we need the following numerical radius inequality [9, Cor. 3].

Lemma 3.6. *Let* $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, *where* $A \in B(\mathbb{H}_1), B \in B(\mathbb{H}_2, \mathbb{H}_1), C \in B(\mathbb{H}_1, \mathbb{H}_2), D \in B(\mathbb{H}_2)$. *Then,*

$$w(T) \leq \sqrt{w^2(A) + \frac{1}{2}\|B\| \left(w(A) + \frac{1}{2}\|B\|\right)} + \sqrt{w^2(D) + \frac{1}{2}\|C\| \left(w(D) + \frac{1}{2}\|C\|\right)}.$$

Theorem 3.7. *Let* λ *be a zero of* $p(z)$. *Then,*

$$|\lambda| \leq w(C(p)) \leq \sqrt{|a_{n-1}|^2 + \frac{1}{2}\alpha \left(|a_{n-1}| + \frac{1}{2}\alpha\right)} + \sqrt{\cos^2 \frac{\pi}{n} + \frac{1}{2}\left(\cos\frac{\pi}{n} + \frac{1}{2}\right)} = R_3,$$

where $\alpha = \sqrt{\sum_{r=0}^{n-2} |a_r|^2}$.

Proof. The proof follows from Lemmas 3.6 and 3.1, and using the similar arguments as in the proof of Theorem 3.3. □

The next example highlights that the above estimation is better than the existing ones.

Example 3.8. We consider a polynomial $p(z) = z^5 + z^3 + z + 2$. Then the upper bounds for the zeros of this polynomial $p(z)$ estimated by different mathematicians are as shown in the following table.

R_C	3
R_M	4
R_{CM}	2.6457

But our bound in Theorem 3.7 gives $R_3 = 2.3688$. Therefore, for this polynomial $p(z)$, the estimation in Theorem 3.7 is better than the existing estimations mentioned above.

We need the following lemma [9, Cor. 4] to prove the next theorem.

Lemma 3.9. *Let* $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, *where* $A \in B(\mathbb{H}_1), B \in B(\mathbb{H}_2, \mathbb{H}_1), C \in B(\mathbb{H}_1, \mathbb{H}_2), D \in B(\mathbb{H}_2)$. *Then,*

$$w(T) \leq \sqrt{2w^2(A) + \frac{1}{2}\left(\|A^*B\| + \|B\|^2\right)} + \sqrt{2w^2(D) + \frac{1}{2}\left(\|D^*C\| + \|C\|^2\right)}.$$

Theorem 3.10. *Let* λ *be a zero of* $p(z)$. *Then,*

$$|\lambda| \leq w(C(p)) \leq \sqrt{2|a_{n-1}|^2 + \frac{1}{2}\left(\alpha + \beta^2\right)} + \sqrt{2\cos^2\frac{\pi}{n} + \frac{1}{2}} = R_4,$$

where $\alpha = \sqrt{\sum_{r=0}^{n-2} |a_r a_{n-1}|^2}$ *and* $\beta = \sqrt{\sum_{r=0}^{n-2} |a_r|^2}$.

Proof. The proof follows from Lemma 3.9, by using Lemma 3.1 and the similar arguments as in the proof of Theorem 3.3. □

As before we provide an example.

Example 3.11. We consider a polynomial $p(z) = z^5 + z^3 + z + 5$. Then the upper bounds for the zeros of this polynomial $p(z)$ estimated by different mathematicians are as shown in the following table.

R_C	6
R_M	7
R_{CM}	5.2915

But our bound in Theorem 3.10 gives $R_4 = 5.0192$. Therefore, for this polynomial $p(z)$, the estimation in Theorem 3.10 is better than the existing estimations mentioned in this example.

We state an upper bound for the numerical radius of 2×2 operator matrices [15, Cor. 3.4] and using it we prove our next theorem.

Lemma 3.12. *Let* $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, *where* $A \in B(\mathbb{H}_1), B \in B(\mathbb{H}_2, \mathbb{H}_1), C \in B(\mathbb{H}_1, \mathbb{H}_2), D \in B(\mathbb{H}_2)$. *Then,*

$$w\begin{pmatrix} A & B \\ C & D \end{pmatrix} \leq \frac{1}{2}\left[w(A) + w(D) + \sqrt{w^2(A) + \|B\|^2} + \sqrt{w^2(D) + \|C\|^2}\right].$$

Theorem 3.13. *Let λ be a zero of $p(z)$. Then,*

$$|\lambda| \leq w(C(p)) \leq \frac{1}{2}\left[|a_{n-1}| + \cos\frac{\pi}{n} + \sqrt{|a_{n-1}|^2 + \alpha} + \sqrt{\cos^2\frac{\pi}{n} + 1}\right] = R_5,$$

where $\alpha = \sqrt{\sum_{r=0}^{n-2} |a_r|^2}$.

Proof. The proof follows from Lemma 3.12, by using Lemma 3.1 and the similar arguments as in the proof of Theorem 3.3. □

Again we give an example to show that the estimation is better than the existing ones.

Example 3.14. We consider the same polynomial $p(z)$ in Remark 3.8, i.e., $p(z) = z^5+z^3+z+2$. Then the upper bounds for the zeros of this polynomial $p(z)$ estimated by different mathematicians are as shown in the following table.

R_C	3
R_M	4
R_{CM}	2.6457
R_{FK}	2.0907
R_A	2.1760
R_{K_1}	2.1430
R_{K_2}	1.9580
R_{AK}	2.1678

But our bound in Theorem 3.13 gives $R_5 = 1.8301$. Therefore for this polynomial $p(z)$, the estimation in Theorem 3.13 is better than all the existing estimations mentioned in this example.

Next we give the following two lemmas which can be found in [17, pp. 335-336] and [15, Th. 2.1], respectively.

Lemma 3.15. $\|C(p)\| = \left[\frac{1}{2}\left(1 + \sum_{r=0}^{n-1} |a_r|^2 + \sqrt{\left(1 + \sum_{r=0}^{n-1} |a_r|^2\right)^2 - 4|a_0|^2}\right)\right]^{\frac{1}{2}}$.

Lemma 3.16. $\|C^2(p)\| \leq \left(1 + \sum_{r=0}^{n-1} \left(|a_r|^2 + |b_r|^2\right)\right)^{\frac{1}{2}}$, *where* $b_r = a_{n-1}a_r - a_{r-1}$ *for each* $r = 0, 1, \ldots, n-1$ *with* $a_{-1} = 0$.

By using Theorem 2.7, we prove the following theorem.

Theorem 3.17. *Let λ be a zero of $p(z)$. Then,*

$$|\lambda| \leq w(C(p)) \leq \left(\frac{1}{4}\beta\sqrt{\alpha} + \frac{1}{4}\alpha + \frac{1}{2}\beta^2\right)^{\frac{1}{2}} = R_6,$$

where

$$\begin{aligned}
\alpha &= \left[1 + \sum_{r=0}^{n-1} \left(|a_r|^2 + |b_r|^2\right)\right]^{\frac{1}{2}}, \\
\beta &= \left[\frac{1}{2}\left(1 + \sum_{r=0}^{n-1} |a_r|^2 + \sqrt{\left(1 + \sum_{r=0}^{n-1} |a_r|^2\right)^2 - 4|a_0|^2}\right)\right]^{\frac{1}{2}}, \\
b_r &= a_{n-1}a_r - a_{r-1} \quad \text{for each} \quad r = 0, 1, \ldots, n-1 \quad \text{and} \quad a_{-1} = 0.
\end{aligned}$$

Proof. Taking $T = C(p)$ in Theorem 2.7 and using Lemmas 3.15 and 3.16, we get the required bound. □

Next using Theorem 2.8(i), we prove the following theorem.

Theorem 3.18. *Let λ be a zero of $p(z)$. Then,*

$$|\lambda| \leq w(C(p)) \leq \left(\frac{1}{2}\beta\sqrt{\alpha} + \frac{1}{4}\alpha + \frac{1}{4}\beta^2\right)^{\frac{1}{2}} = R_7,$$

where α and β are same as in Theorem 3.17.

Proof. Taking $T = C(p)$ in Theorem 2.8(i), and using Lemmas 3.15 and 3.16 we get the desired bound. □

Our last theorem in this section is the following one.

Theorem 3.19. *Let λ be a zero of $p(z)$. Then,*

$$|\lambda| \leq w(C(p)) \leq \left[\frac{3}{4}\alpha + \frac{1}{4}\beta^2\right]^{\frac{1}{2}} = R_8,$$

where α and β are same as in Theorem 3.17.

Proof. Taking $T = C(p)$ in Theorem 2.8(ii), and using Lemmas 3.15 and 3.16 we get the required bound for zeros of $p(z)$. □

We illustrate with a numerical example to show that the bounds for the zeros of a polynomial obtained by us in Theorems 3.17, 3.18 and 3.19 are better than the existing bounds.

Example 3.20. We consider a polynomial $p(z) = z^5 + 2z^4 + z^3 + z^2 + z + 1$. Then the upper bounds for the zeros of this polynomial $p(z)$ estimated by different mathematicians are as shown in the following table.

R_A	3.0183
R_{CM}	3.0000
R_C	3.0000
R_{FK}	3.2802
R_{K_1}	3.0000
R_{K_2}	2.8552
R_{AK}	3.0670

But for the polynomial $p(z) = z^5 + 2z^4 + z^3 + z^2 + z + 1$, we have $R_6 = 2.7129$, $R_7 = 2.6086$ and $R_8 = 2.4437$. This shows that for this example, our bounds obtained in Theorems 3.17, 3.18 and 3.19 are better than all the estimations mentioned above.

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