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Class of Meromorphic Functions Partially Shared Values with Their Differences or Shifts

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Abstract. For a value $s \in \mathbb{C} \cup \{\infty\}$, two meromorphic functions f and g are said to share the value s, CM , (or IM), provided that $f(z)-s$ and $g(z)-s$ have the same set of zeros, counting multiplicities, (respectively, ignoring multiplicities). We say that a meromorphic function f shares $s \in \hat{\mathbb{S}}$ partially with a meromorphic function g if $E(s,f) \subseteq E(s,g)$. It is easy to see that "partially shared values CM" are more general than "shared values CM". With the help of partially shared values, in this paper, we prove some uniqueness results between a non-constant meromorphic function and its generalized differences or shifts. We exhibit some examples to show that the result of Charak *et al.* [8] is not true for $k = 2$ or $k = 3$. We find some gaps in proof of the result of Lin *et al.* [24]. We not only correct these resuts, but also generalize them in a more convenient way. We give a number of examples to validate certain claims of the main results of this paper and also to show that some of conditions are sharp. Finally, we pose some open questions for further investigation.

1. Introduction

We assume that the reader is familiar with the elementary Nevanlinna theory, for detailed information, we refer the reader [15, 16, 22] and references therein. Meromorphic functions are non-constant unless otherwise specified. For such a function f and $a \in \overline{\mathbb{C}} =: \mathbb{C} \cup \{\infty\}$, each z with $f(z) = a$ will be called a-point of f. We will use here some standard definitions and basic notations from this theory. In particular, $N(r,a;f)$ ($\overline{N}(r,a;f)$) is the counting function (reduced counting function) of a-points of f, $T(r,f)$ is the Nevanlinna characteristic function of f, and

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$S(r,f)$ is any function of smaller order than $T(r,f)$ when $r \to \infty$. We also denote $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$.

For a meromorphic function f, the order $\rho(f)$ and the hyper-order $\rho_2(f)$ of f are defined by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log^+ T(r,f)}{\log r}, \quad \rho_2(f) = \limsup_{r\to\infty} \frac{\log^+ \log^+ T(r,f)}{\log r}.$$

For $a \in \mathbb{C} \cup \{\infty\}$, we also define

$$\Theta(a; f) = 1 - \limsup_{r\to+\infty} \frac{\overline{N}\,(r, 1/(f-a))}{T(r,f)}.$$

We denote by $\mathcal{S}(f)$ the family of all meromorphic functions s for which $T(r,s) = o(T(r,f))$, where $r \to \infty$ outside of a possible exceptional set of finite logarithmic measure. Moreover, we also include all constant functions in $\mathcal{S}(f)$, and let $\hat{\mathcal{S}}(f) = \mathcal{S}(f) \cup \{\infty\}$. For $s \in \hat{\mathcal{S}}(f)$, we say that two meromorphic functions f and g share s CM when $f - s$ and $g - s$ have the same zeros with the same multiplicities. If multiplicities are not taking into account, then we say that f and g share s IM.

The five-point and four-point uniqueness theorems of Nevanlinna [30] are classical uniqueness results in the theory of meromorphic functions. The five-point theorems states that if two meromorphic functions f, g share five distinct values in the extended complex plane IM, then $f \equiv g$. The beauty of this result lies in the fact that there is no counterpart of this result in case of real valued functions. On the other hand, four-point theorem states that if two meromorphic functions f, g share four distinct values in the extended complex plane CM, then $f \equiv T \circ g$, where T is a Möbius transformation.

These results initiated the study of uniqueness of two meromorphic functions f and g. The study of such uniqueness becomes more interesting if the function g has some expressions in terms of f.

The following definition will be used later.

Definition 1.1. Let f and g be two meromorphic functions such that f and g share the value a with weight k where $a \in \mathbb{C} \cup \{\infty\}$. We denote by $\overline{N}_E^{(k+1}\,(r, 1/(f-a))$ the counting function of those a-points of f and g where $p = q \geq k+1$, each point in this counting function is counted only once.

In what follows, let c be a non-zero constant. For a meromorphic function f, let us now denote its shift $I_c f$ and difference operators $\Delta_c f$, respectively, by $I_c f(z) = f(z+c)$ and $\Delta_c f(z) = (I_c - 1)f(z) = f(z+c) - f(z)$.

For finite ordered meromorphic functions, Halburd and Korhonen [17], and independently Chiang and Feng [13], developed parallel difference versions of the famous Nevanlinna theory. As applications of this theory, we refer the reader to such articles as [3, 4, 7, 6, 9, 36]) for set sharing problems, as [1, 27, 32] for finding solutions to the Fermat-type difference equations, as [14] for Nevanlinna theory of the AskeyWilson divided difference operators, and as [28] and references therein for meromorphic solutions to the difference equations of Malmquist type.

Regarding periodicity of meromorphic functions, Heittokangas *et al.* [20, 21] have considered the problem of value sharing between meromorphic functions and their shifts and obtained the following results.

Theorem A.([20]) *Let f be a meromorphic function of finite order, and let $c \in \mathbb{C}^*$. If $f(z)$ and $f(z+c)$ share three distinct periodic functions $s_1, s_2, s_3 \in \hat{\mathbb{S}}(f)$ with period c CM, then $f(z) \equiv f(z+c)$ for all $z \in \mathbb{C}$.*

In 2009, Heittokangas *et al.* [21] improved Theorem A by replacing "sharing three small functions CM" by "$2\,CM + 1\,IM$" as follows.

Theorem B.([21]) *Let f be a meromorphic function of finite order, and let $c \in \mathbb{C}^*$. Let $s_1, s_2, s_3 \in \hat{\mathbb{S}}(f)$ be three distinct periodic function with period c. If $f(z)$ and $f(z+c)$ share $s_1, s_2 \in \hat{\mathbb{S}}(f)$ CM and s_3 IM, then $f(z) \equiv f(z+c)$ for all $z \in \mathbb{C}$.*

In 2014, Halburd *et al.* [19] extended some results in this direction to meromorphic functions f whose hyper-order $\rho_2(f)$ is less than one. One may get much more information (see [1, 4, 11, 12, 13, 18, 19, 20, 21, 25, 26] and the references therein) about the relationship between a meromorphic function $f(z)$ and it shift $f(z+c)$.

In 2016, in this direction, Li and Yi [23] obtained a uniqueness result of meromorphic functions f sharing four values with their shifts $f(z+c)$.

Theorem C.([23]) *Let f be a non-constant meromorphic function of hyper-order $\rho_2(f) < 1$ and $c \in \mathbb{C}^*$. Suppose that f and $f(z+c)$ share 0, 1, η IM, and share ∞ CM, where η is a finite value such that $\eta \neq 0, 1$. Then $f(z) \equiv f(z+c)$ for all $z \in \mathbb{C}$.*

We now recall here the definition of partially shared values by two meromorphic functions f and g.

Definition 1.2.([10]) *Let f and G be non-constant meromorphic functions and $s \in \mathbb{C} \cup \{\infty\}$. Denote the set of all zeros of $f - s$ by $E(s, f)$, where a zero of multiplicity m is counted m times. If $E(s, f) \subset E(s, g)$, then we say that f and g partially share the value s CM. Note that $E(s, f) = E(s, g)$ is equivalent to f and g share the value s CM. Therefore, it is easy to see that the condition "partially shared values CM" is more general than the condition "shared value CM".*

In addition, let $\overline{E}(s, f)$ denote the set of zeros of $f - s$, where a zero is counted only once in the set, and $\overline{E}_{k)}(s, f)$ denote the set of zeros of $f - s$ with multiplicity $l \leq k$, where a zero with multiplicity l is counted only once in the set. The reduced counting function corresponding to to $\overline{E}_{k)}(s, f)$ are denoted by $\overline{N}_{k)}(r, 1/(f - s))$.

Charak *et al.* [8] gave the following definition of partial sharing.

Definition 1.3.([8]) *We say that a meromorphic function f share $s \in \hat{\mathbb{S}}$ partially with a meromorphic function g if $\overline{E}(s, f) \subseteq \overline{E}(s, g)$, where $\overline{E}(s, f)$ is the set of zeros of $f(z) - s(z)$, where each zero is counted only once.*

Let f and g be two non-constant meromorphic functions and $s(z) \in \hat{\mathbb{S}}(f) \cap \hat{\mathbb{S}}(g)$. We denote by $\overline{N}_0(r, s; f, g)$ the counting function of common solutions of $f(z) - s(z) = 0$ and $g(z) - s(z) = 0$, each counted only once. Put

$$\overline{N}_{12}(r, s; f, g) = \overline{N}\left(r, \frac{1}{f - s}\right) + \overline{N}\left(r, \frac{1}{g - s}\right) - 2\overline{N}_0(r, s; f, g).$$

It is easy to see that $\overline{N}_{12}(r, s; f, g)$ denoted the counting function of distinct solutions of the simultaneous equations $f(z) - s(z) = 0$ and $g(z) - s(z) = 0$.

In 2016, *Charak et al.* [8] introduced the above notion of partial sharing of values and applying this notion of sharing, they have obtained the following interesting result.

Theorem D.([8]) *Let f be a non-constant meromorphic function of hyper order $\rho_2(f) < 1$, and $c \in \mathbb{C}^*$. Let $s_1, s_2, s_3, s_4 \in \hat{\mathbb{S}}(f)$ be four distinct periodic functions with period c. If $\delta(s, f) > 0$ for some $s \in \hat{\mathbb{S}}(f)$ and*

$$\overline{E}(s_j, f) \subseteq \overline{E}(s_j, f(z + c)), \quad j=1,\ 2,\ 3,\ 4,$$

then $f(z) = f(z + c)$ for all $z \in \mathbb{C}$.

In 2018, *Lin et al.* [24] investigated further on the result of *Charak et al.* [8] replacing the condition "partially shared value $\overline{E}(s, f) \subseteq \overline{E}(s, f(z + c))$" by the condition "truncated partially shared value $\overline{E}_{k)}(s, f) \subseteq \overline{E}_{k)}(s, f(z + c))$", k is a positive integer. By the following example, *Lin* et. al. [24] have shown that the result of *Charak* et. al. [8] is not be true for $k = 1$ if truncated partially shared values is considered.

Example 1.4.([24]) *Let $f(z) = 2e^z/(e^{2z} + 1)$ and $c = \pi i$, $s_1 = 1$, $s_2 = -1$, $s_3 = 0$, $s_4 = \infty$ and $k = 1$. It is easy to see that $f(z + \pi i) = -2e^z/(e^{2z} + 1)$ and $f(z)$ satisfies all the other conditions of Theorem D, but $f(z) \not\equiv f(z + c)$.*

However, after a careful investigation, we find that Theorem D is not valid in fact for each positive integer k although $f(z)$ and $f(z+c)$ share value $s \in \{s_1, s_2, s_3, s_4\}$ CM. We give here only two examples to show that the result of *Charak et al.* [8] is not true for $k = 2$ and $k = 3$.

Example 1.5. *Let $f(z) = \big(ae^z(e^{2z} + 3)\big)/\big(3e^{2z} + 1\big)$, $c = \pi i$ and $s_1 = a$, $s_2 = -a$, where $a \in \mathbb{C}^*$, $s_3 = 0$, $s_4 = \infty$ and $k_1 = 2 = k_2$. It i easy to see that $f(z + \pi i) = -\big(ae^z(e^{2z} + 3)\big)/\big(3e^{2z} + 1\big)$ and $f(z)$ satisfies all the conditions of Theorem D, but $f(z) \not\equiv f(z + c)$.*

Example 1.6. Let $f(z) = \big(4ae^z(e^{2z} + 1)\big)/\big(e^{4z} + 6e^{2z} + 1\big)$ and $c = \pi i$, $s_1 = a$, $s_2 = -a$, where $a \in \mathbb{C}^*$, $s_3 = 0$, $s_4 = \infty$ and $k_1 = 3 = k_2$. Then clearly $f(z + \pi i) = -\big(4ae^z(e^{2z} + 1)\big)/\big(e^{4z} + 6e^{2z} + 1\big)$ and $f(z)$ satisfies all the conditions of Theorem D, but $f(z) \not\equiv f(z + c)$.

In 2018, *Lin et al.* [24] established the following result considering partially sharing values.

Theorem E.([24]) *Let f be a non-constant meromorphic function of hyper-order $\rho_2(f) < 1$ and $c \in \mathbb{C}^*$. Let k_1, k_2 be two positive integers, and let $s_1, s_2 \in \mathcal{S}(f) \cup \{0\}$, and $s_3, s_4 \in \hat{\mathcal{S}}(f)$ be four distinct periodic functions with period c such that f and $f(z+c)$ share s_3, s_4 CM and*

$$\overline{E}_{k_j)}(s_j, f) \subseteq \overline{E}_{k_j)}(s_j, f(z+c)), \;\; j = 1, 2.$$

If $\Theta(0, f) + \Theta(\infty; f) > 2/(k+1)$, where $k = \min\{k_1, k_2\}$, then $f(z) \equiv f(z+c)$ for all $z \in \mathbb{C}$.

As a consequence of Theorem E, *Lin et al.* [24] have obtained the following result.

Theorem F.([24]) *Let f be a non-constant meromorphic function of hyper order $\rho_2(f) < 1$, $\Theta(\infty, f) = 1$ and $c \in \mathbb{C}^*$. Let $s_1, s_2, s_3 \in \mathcal{S}(f)$ be three distinct periodic functions with period c such that $f(z)$ and $f(z+c)$ share s_3 CM and*

$$\overline{E}_{k)}(s_j, f) \subseteq \overline{E}_{k)}(s_j, f(z+c)), \;\; j = 1, 2.$$

If $k \geq 2$, then $f(z) \equiv f(z+c)$ for all $z \in \mathbb{C}$.

The authors have showed that number "$k = 2$" is sharp for the function $f(z) = \sin z$ and $c = \pi$. It is easy to see that $f(z+c)$ and $f(z)$ share the value 0 CM and $\overline{E}_{1)}(1, f(z)) = \overline{E}_{1)}(1, f(z+c)) = \phi$ and $\overline{E}_{1)}(-1, f(z)) = \overline{E}_{1)}(-1, f(z+c)) = \phi$ but $f(z+c) \not\equiv f(z)$. Since Theorem F is true for $k \geq 2$, hence *Lin et al.* [24] investigated further to explore the situation when $k = 1$ and obtained the result.

Theorem G([24]) *Let f be a non-constant meromorphic function of hyper order $\rho_2(f) < 1$, $\Theta(\infty, f) = 1$ and $c \in \mathbb{C}^*$. Let $s_1, s_2, s_3 \in \mathcal{S}(f)$ be three distinct periodic functions with period c such that $f(z)$ and $f(z+c)$ share s_3 CM and*

$$\overline{E}_{1)}(s_j, f) \subseteq \overline{E}_{1)}(s_j, f(z+c)), \;\; j = 1, 2.$$

Then $f(z) \equiv f(z+c)$ or $f(z) \equiv -f(z+c)$ for all $z \in \mathbb{C}$. Moreover, the later occurs only if $s_1 + s_2 = 2s_3$.

Remark 1.7. We find that in the proof of [24, Theorem 1.6], the authors *Lin et al.* made a mistake. In Theorem 1.6, they have obtained $f(z+c) \equiv -f(z)$ as one of the conclusion when $s_1 + s_2 = 2s_3$, where originally it will be $f(z+c) \equiv -f(z) + 2s_3$. One can easily understand it from the following explanation. In [24, Proof of Theorem 1.6, page - 476] the authors have obtained $\alpha = -1$, where α, the way they have defined, finally will be numerically equal with $(f(z+c) - s_3)/(f(z) - s_3) = \alpha$, when $s_1 + s_2 = 2s_3$. Hence after combining, it is easy to see that $(f(z+c) - s_3)(f(z) - s_3) = -1$ and this implies that $f(z+c) \equiv -f(z) + 2s_3$.

In this paper, our aim is to take care of these points. We also want to extend the above results with certain suitable setting. Henceforth, for a meromorphic function f and $c \in \mathbb{C}^*$, we recall here (see [2]) $\mathcal{L}_c(f) := c_1 f(z+c) + c_0 f(z)$, where

$c_1(\neq 0), c_0 \in \mathbb{C}$. Clearly, $\mathcal{L}_c(f)$ is a generalization of shift $f(z+c)$ as well as the difference operator $\Delta_c f$.

In this paper, to give a correct version of the result of Lin *et al.* with a general setting, we are mainly interested to find the affirmative answers of the following questions.

Question 1.8. Is it possible to extend $f(z+c)$ upto $\mathcal{L}_c(f)$, in all the above mentioned results?

Question 1.9. Can we obtained a similar result of Theorem E, replacing the condition $\Theta(0;f)+\Theta(\infty;f) > 2/(k+1)$, where $k=\min\{k_1,k_2\}$ by a more general one?

If answers of the above questions are found to be affirmative, then it is natural to raise the following questions.

Question 1.10. Is the new general condition, so far obtained, sharp?

Question 1.11. Can we find the class of all the meromorphic function which satisfies the difference equation $\mathcal{L}_c(f) \equiv f$?

Answering the above questions is the main objective of this paper. We organize the paper as follows: In Section 2, we state the main results of this paper and exhibit several examples pertinent with the different issues regarding the main results. In Section 3, key lemmas have been stated and proved some of them. Section 4 is devoted specially to prove the main results of this paper. In Section 5, some questions have raised for further investigations on the main results of this paper.

2. Main Results

We prove the following result generalizing that of *Lin et al.* [24].

Theorem 2.1. *Let f be a non-constant meromorphic function of hyper order $\rho_2(f) < 1$ and $c, c_1 \in \mathbb{C}^*$. Let k_1, k_2 be two positive integers, and s_1, $s_2 \in \mathcal{S}\setminus\{0\}$, s_3, $s_4 \in \hat{\mathcal{S}}(f)$ be four distinct periodic functions with period c such that f and $\mathcal{L}_c(f)$ share s_3, s_4 CM and*

$$\overline{E}_{k_j)}(s_j,f) \subseteq \overline{E}_{k_j)}(s_j,\mathcal{L}_c(f)), \quad j=1,2.$$

If

$$\Theta(0;f)+\Theta(\infty;f) > \frac{1}{k_1+1}+\frac{1}{k_2+1},$$

then $\mathcal{L}_c(f) \equiv f$. Furthermore, f assumes the following form

$$f(z) = \left(\frac{1-c_0}{c_1}\right)^{\dfrac{z}{c}} g(z),$$

where $g(z)$ is a meromorphic function such that $g(z+c)=g(z)$, for all $z \in \mathbb{C}$.

Remark 2.2. The following examples show that the condition

$$\Theta(0;f)+\Theta(\infty;f)>\frac{1}{k_1+1}+\frac{1}{k_2+1}$$

in Theorem 2.1 is sharp.

Example 2.3. Let $f(z)=\big(ae^z(e^{2z}+3)\big)/\big(3e^{2z}+1\big)$, $c=\pi i$ and $s_1=a$, $s_2=-a$, where $a\in\mathbb{C}^*$, $s_3=0$, $s_4=\infty$ and $k_1=2=k_2$. It is easy to see that $\mathcal{L}_{\pi i}(f)=-\big(ae^z(e^{2z}+3)\big)/\big(3e^{2z}+1\big)$, where $c_1=c_0+1$, $c_0,c_1\in\mathbb{C}^*$, and $f(z)$ satisfies all the conditions of Theorem 2.1 and

$$\Theta(0;f)+\Theta(\infty;f)=\frac{2}{3}=\frac{1}{k_1+1}+\frac{1}{k_2+1},$$

where $\Theta(0,f)=1/3=\Theta(\infty,f)$, but $\mathcal{L}_{\pi i}(f)\not\equiv f$.

Example 2.4. Let $f(z)=\big(4ae^z(e^{2z}+1)\big)/\big(e^{4z}+6e^{2z}+1\big)$, $c=\pi i$, $s_1=a$, $s_2=-a$, where $a\in\mathbb{C}^*$, $s_3=0$, $s_4=\infty$ and $k_1=3=k_2$. Then clearly $\mathcal{L}_{\pi i}(f)=-\big(4ae^z(e^{2z}+1)\big)/\big(e^{4z}+6e^{2z}+1\big)$, where $c_1=c_0+1$, $c_0,c_1\in\mathbb{C}^*$, and $f(z)$ satisfies all the conditions of *Theorem 2.1* and

$$\Theta(0;f)+\Theta(\infty;f)=\frac{1}{2}=\frac{1}{k_1+1}+\frac{1}{k_2+1},$$

where $\Theta(0,f)=1/2$, $\Theta(\infty,f)=0$ but we see that $\mathcal{L}_{\pi i}(f)\not\equiv f$.

As the consequence of Theorem 2.1, we obtain the following result.

Theorem 2.5. *Let f be a non-constant meromorphic function of hyper-order $\rho_2(f)<1$, $\Theta(\infty,f)=1$ and $c,c_1\in\mathbb{C}^*$. Let $s_1,s_2,s_3\in\hat{\mathbb{S}}(f)$ be three distinct periodic functions with period c such that f and $\mathcal{L}_c(f)$ share s_3 CM and*

$$\overline{E}_{k_j)}(s_j,f)\subseteq\overline{E}_{k_j)}(s_j,\mathcal{L}_c(f)),\quad j=1,2.$$

If $k_1,k_2\geq 2$, then $\mathcal{L}_c(f)\equiv f$. Furthermore, f assumes the following form

$$f(z)=\left(\frac{1-c_0}{c_1}\right)^{\frac{z}{c}}g(z),$$

where $g(z)$ is a meromorphic function such that $g(z+c)=g(z)$, for all $z\in\mathbb{C}$.

The following example shows that the number $k_1=2=k_2$ is sharp in Theorem 2.5.

Example 2.6. We consider $f(z)=a\cos z$, where $a\in\mathbb{C}^*$, $s_1=a,s_2=-a$ and $s_3=0$. We choose $\mathcal{L}_\pi(f)=c_1f(z+\pi)+c_0f(z)$, where $c_1,\ c_0\in\mathbb{C}^*$ with $c_1=c_0+1$. Clearly f and $\mathcal{L}_\pi(f)$ share s_3 CM, $\Theta(\infty,f)=1$, $\overline{E}_{1)}(a,f)=\phi=\overline{E}_{1)}(a,\mathcal{L}_\pi(f))$ and $\overline{E}_{1)}(-a,f)=\phi=\overline{E}_{1)}(-a,\mathcal{L}_\pi(f))$, but $f(z)\not\equiv\mathcal{L}_\pi(f)$.

Naturally, we are interested to explore the situation when $k_1 = 1 = k_2$ and hence, we obtain the following result.

Theorem 2.7. *Let f be a non-constant meromorphic function of hyper-order $\rho_2(f) < 1$ with $\Theta(\infty, f) = 1$ and $c, c_1 \in \mathbb{C}^*$. Let $s_1, s_2, s_3 \in \hat{\mathbb{S}}(f)$ be three distinct periodic functions with period c such that f and $\mathcal{L}_c(f)$ share s_3 CM and*

$$\overline{E}_{1)}(s_j, f) \subseteq \overline{E}_{1)}(s_j, \mathcal{L}_c(f)), \quad j = 1, 2.$$

Then $\mathcal{L}_c(f) \equiv f$ or $\mathcal{L}_c(f) \equiv -f + 2s_3$. Furthermore,

(i) *If $\mathcal{L}_c(f) \equiv f$, then*

$$f(z) = \left(\frac{1 - c_0}{c_1}\right)^{\frac{z}{c}} g(z).$$

(ii) *If $\mathcal{L}_c(f) \equiv -f + 2s_3$, then*

$$f(z) = \left(\frac{-1 - c_0}{c_1}\right)^{\frac{z}{c}} g(z) + 2s_3, \quad \text{for all } z \in \mathbb{C},$$

where $g(z)$ is a meromorphic function such that $g(z+c) = g(z)$. Moreover, $\mathcal{L}_c(f) \equiv -f + 2s_3$ occurs only if $s_1 + s_2 = 2s_3$.

Remark 2.8. We see that Theorems 2.1, 2.2 and 2.3 directly improved, respectively, Theorems E, F and G.

Remark 2.9. We see from Example 2.3 that, in Theorem 2.3, the possibility $\mathcal{L}_c(f) \equiv -f + 2s_3$ could be occurred.

The following example shows that the restrictions on the growth of f in our above results are necessary and sharp.

Example 2.10. Let $f(z) = e^{g(z)}$, where $g(z)$ is an entire function with $\rho(g) = 1$, and hence for $c_1 = 1/2 = c_0$, it is easy to see that $\mathcal{L}_\pi(z) = \left(e^{2g(z)} + 1\right)/2e^{g(z)}$. We choose $s_1 = 1$, $s_2 = -1$ and $s_3 = \infty$. Clearly $\Theta(\infty, f) = 1$, $\rho_2(f) = 1$, f and $\mathcal{L}_\pi(f)$ share s_3 CM and $\overline{E}_{1)}(1, f) \subseteq \overline{E}_{1)}(1, \mathcal{L}_\pi(z))$ and $\overline{E}_{1)}(-1, f) \subseteq \overline{E}_{1)}(-1, \mathcal{L}_\pi(z))$ but we see that neither $\mathcal{L}_\pi(z) \not\equiv f$ nor $\mathcal{L}_\pi(z) \not\equiv -f + 2s_3$. Also the function has not the specific form.

Remark 2.11. The next example shows that the condition $\Theta(\infty, f) = 1$ in Theorem 2.3 can not be omitted.

Example 2.12. Let $f(z) = 1/\cos z$, $c_1 = 1$, $c_0 = 0$, $s_1 = 1$, $s_2 = -1$ and $s_3 = 0$. Clearly $\Theta(\infty, f) = 0$, f and $\mathcal{L}_{3\pi/2}(f)$ share s_3 CM, $\overline{E}_{1)}(1, f) \subseteq \overline{E}_{1)}(1, \mathcal{L}_{3\pi/2}(z))$ and $\overline{E}_{1)}(-1, f) \subseteq \overline{E}_{1)}(-1, \mathcal{L}_{3\pi/2}(z))$. However, one may observe that neither $\mathcal{L}_{3\pi/2}(z) \not\equiv f$ nor $\mathcal{L}_{3\pi/2}(z) \not\equiv -f + 2s_3$. Also the function has not the specific form.

3. Key Lemmas

In this section, we present some necessary lemmas which will play a key role in proving the main results. Henceforth, for a non-zero complex number c and for integers $n \geq 1$, we define the higher order difference operators $\Delta_c^n f := \Delta_c^{n-1}(\Delta_c f)$.

Lemma 3.1.([34]) *Let $c \in \mathbb{C}$, $n \in \mathbb{N}$, let f be a meromorphic function of finite order. Then any small periodic function $a \equiv a(z) \in \mathcal{S}(f)$*

$$m\left(r, \frac{\Delta_c^n f}{f(z) - a(z)}\right) = S(r, f),$$

where the exponential set associated with $S(r, f)$ is of at most finite logarithmic measure.

Lemma 3.2. ([29, 31]) *If $\mathcal{R}(f)$ is rational in f and has small meromorphic coefficients, then*

$$T(r, \mathcal{R}(f)) = \deg_f(\mathcal{R})T(r, f) + S(r, f).$$

Lemma 3.3.([35]) *Suppose that h is a non-constant entire function such that $f(z) = e^{h(z)}$, then $\rho_2(f) = \rho(h)$.*

In [13, 17], the first difference analogue of the lemma on the logarithmic derivative was proved and for the hyper-order $\rho_2(f) < 1$, the following is the extension, see [19].

Lemma 3.4.([19]) *Let f be a non-constant finite order meromorphic function and $c \in \mathbb{C}$. If c is of finite order, then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = O\left(\frac{\log r}{r} T(r, f)\right)$$

for all r outside of a set E with zero logarithmic density. If the hyper order $\rho_2(f) < 1$, then for each $\epsilon > 0$, we have

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = 0\left(\frac{T(r, f)}{r^{1-\rho_2-\epsilon}}\right)$$

for all r outside of a set of finite logarithmic measure.

Lemma 3.5.([33]) *Let f be a non-constant meromorphic function, $s_j \in \hat{\mathcal{S}}(f)$, $j = 1, 2, ..., q$, $(q \geq 3)$. Then for any positive real number ϵ, we have*

$$(q - 2 - \epsilon)T(r, f) \leq \sum_{j=1}^{q} \overline{N}\left(r, \frac{1}{f - s_j}\right), \ r \notin E,$$

where $E \subset [0,\infty)$ *and satisfies* $\int_E d\log\log r < \infty$.

We now prove the following lemma, a similar proof of this lemma can also be found in [2].

Lemma 3.6. *Let* f *be a non-constant meromorphic function such that*

$$\overline{E}(s_j, f) \subseteq \overline{E}(s_j, c_1 f(z+c) + c_0 f(z)), \quad j = 1, 2,$$

where $s_1, s_2 \in \mathcal{S}(f)$, c, c_0, $c_1(\neq 0) \in \mathbb{C}^*$, *then* f *is not a rational.*

Proof. We wish to prove this lemma by the method of contradiction. Let f be a rational function. Then $f(z) = P(z)/Q(z)$ where P and Q are two polynomials relatively prime to each other and $P(z)Q(z) \not\equiv 0$. Hence

$$E(0,P) \cap E(0,Q) = \phi \tag{3.1}$$

It is easy to see that

$$\begin{aligned} c_1 f(z+c) + c_0 f(z) &= c_1 \frac{P(z+c)}{Q(z+c)} + c_0 \frac{P(z)}{Q(z)} \\ &= \frac{c_1 P(z+c)Q(z) + c_0 P(z)Q(z+c)}{Q(z+c)Q(z)} \\ &= \frac{P_1(z)}{Q_1(z)}, \text{(say)} \end{aligned}$$

where P_1 and Q_1 are two relatively prime polynomials and $P_1(z)Q_1(z) \not\equiv 0$.

Since $\overline{E}(s_1, f) \subseteq \overline{E}(s_1, c_1 f(z+c) + c_0 f(z))$ and f is a rational function, there must exists a polynomial $h(z)$ such that

$$c_1 f(z+c) + c_0 f(z) - s_1 = (f - s_1)h(z),$$

which can be re-written as

$$\frac{c_1 P(z+c)Q(z) + c_0 P(z)Q(z+c)}{Q(z+c)Q(z)} - s_1 \equiv \left(\frac{P(z)}{Q(z)} - s_1\right) h(z). \tag{3.2}$$

We now discuss the following cases:

Case 1. Let $P(z)$ is non-constant.

Then by the *Fundamental Theorem of Algebra*, there exists $z_0 \in \mathbb{C}$ such that $P(z_0) = 0$. Then it follows from (3.2) that

$$c_1 \frac{P(z_0+c)}{Q(z_0+c)} \equiv (1 - h(z_0))s_1^0, \tag{3.3}$$

where $s_1^0 = s_1(z_0)$.

Subcase 1.1. Let $z_0 \in \mathbb{C}$ be such that $s_1(z_0) = 0$.

Then from (3.3), it is easy to see that $P(z_0 + c) = 0$. Then we can deduce from (3.1) that $P(z_0 + mc) = 0$ for all positive integer m. However, this is impossible, and hence we conclude that the polynomial $P(z)$ is a non-zero constant.

Subcase 1.2. Let $z_0 \in \mathbb{C}$ be such that $s_1(z_0) \neq 0$.

Then from (3.3), we obtain

$$P(z_0 + c) \equiv \frac{s_1^0}{c_1}(1 - h(z_0))Q(z_0 + c).$$

Next proceeding exactly same way as done in above, we obtain

$$P(z_0 + mc) \equiv \frac{s_1^0}{c_1}(1 - h(z_0))Q(z_0 + mc). \tag{3.4}$$

In view of (3.3) and (3.4), a simple computation shows that

$$\frac{P(z_0 + c)}{Q(z_0 + c)} = \frac{P(z_0 + mc)}{Q(z_0 + mc)} \text{ for all positive integers } m,$$

which contradicts the fact that $E(0, P) \cap E(0, Q) = \phi$.

Therefore, it is easy to see that $f(z)$ takes the form $f(z) = \eta/Q(z)$, where $P(z) = \eta = \text{constant } (\neq 0)$.

Case 2. Let $Q(z)$ be non-zero constant.

Now

$$c_1 f(z + c) + c_0 f(z) \quad = \quad \frac{c_1\eta\, Q(z) + c_0\eta\, Q(z + c)}{Q(z + c)Q(z)}. \tag{3.5}$$

Since $E(s_2, f) = E(s_2, c_1 f(z+c)+c_0 f(z))$, hence there must exists a polynomial $h_1(z)$ such that $c_1 f(z + c) + c_0 f(z) - s_2 = (f - s_2)h_1(z)$, which can be written as

$$c_1\, Q(z) + c_0 Q(z + c) \equiv \frac{\eta - s_2 Q(z)}{d} h_1(z)Q(z + c). \tag{3.6}$$

Since $Q(z)$, and hence $Q(z + c)$ are a non-constant polynomials, hence by the *Fundamental Theorem of Algebra*, there must exist z_0 and z_1 such that $Q(z_0) = 0 = Q(z_1 + c)$.

Subcase 2.1. When $Q(z_0) = 0$, then from (3.6), we see that $h_1(z_0) = -c_0/\eta$, which is absurd.

Subcase 2.2. When $Q(z_1 + c) = 0$, then from (3.6), we get $Q(z_1) = 0$, which is not possible.

Therefore, we conclude that $Q(z)$ is a non-zero constant, say η_2. Thus we have $f(z) = \eta/\eta_2$, a constant, which is a contradiction. This completes the proof. $\square$

Lemma 3.7.([19]) *Let $T : [0, +\infty] \to [0, +\infty]$ be a non-decreasing continuous function, and let $s \in (0, +\infty)$. If the hyper-order of T is strictly less than one, i.e.,*

$$\limsup_{r\to+\infty} \frac{\log^+ \log^+ T(r)}{\log r} = \rho_2 < 1,$$

then

$$T(r+s) = T(r) + o\left(\frac{T(r)}{r^{1-\rho_2-\epsilon}}\right),$$

where $\epsilon > 0$ and $r \to \infty$, outside of a set of finite logarithmic measure.

4. Proof of the Main Results

In this section, we give the proofs of our main results.

Proof of Theorem 1.1. First of all we suppose that $s_j \in \mathbb{C}$, $j = 1, 2, 3, 4$. By the assumption of the theorem, $f(z)$ and $\mathcal{L}_c(f) = c_1 f(z+c) + c_0 f(z)$ share s_3, s_4 CM, hence we must have

$$\frac{(f - s_3)(\mathcal{L}_c(f) - s_4)}{(f - s_4)(\mathcal{L}_c(f) - s_3)} = e^{h(z)}, \tag{4.1}$$

where $h(z)$ is an entire function with $\rho(h) < 1$ by Lemma 3.3. In view of Lemma 3.4, we obtain

$$T\left(r, e^h\right) = S(r, f).$$

Next we suppose that $z_0 \in \overline{E}_{k)}(s_1, f) \cup \overline{E}_{k)}(s_2, f)$. Then from (4.1), one may easily deduce that $e^{h(z_0)} = 1$. For the sake of convenience, we set $\gamma := e^{h(z)}$ and

$$S(r) := S(r, \mathcal{L}(f)) = S(r, f).$$

We now split the problem into two cases.

Case 1. Let $e^{h(z)} \neq 1$.

A simple computation shows that that

$$\overline{N}_{k_1)}\left(r, \frac{1}{f - s_1}\right) \leq N\left(r, \frac{1}{\gamma - 1}\right) \leq T(r, \gamma) + O(1) \leq S(r) \tag{4.2}$$

and

$$\overline{N}_{k_2)}\left(r, \frac{1}{f - s_2}\right) \leq N\left(r, \frac{1}{\gamma - 1}\right) \leq T(r, \gamma) + O(1) \leq S(r). \tag{4.3}$$

Without loss of generality, we may assume that s_3, $s_4 \in \mathcal{S}(f) \setminus \{0\}$. By Lemma 3.5, for

$$\epsilon \in \left(0, \frac{1}{3}\left(\Theta(0; f) + \Theta(\infty; f)\right) - \frac{1}{k_1 + 1} - \frac{1}{k_2 + 1}\right),$$

we obtain

$$(4.4)\quad (4-\epsilon)T(r,f) \le \overline{N}(r,f) + \overline{N}\left(r,\frac{1}{f}\right) + \sum_{j=1}^{4}\overline{N}\left(r,\frac{1}{f-s_j}\right) + S(r,f).$$

With the help of (4.2) and (4.3), it follows from (4.4) that

$$(2-\epsilon)T(r,f) \le \overline{N}(r,f) + \overline{N}\left(r,\frac{1}{f}\right) + \sum_{j=1}^{2}\overline{N}_{(k_j+1}\left(r,\frac{1}{f-s_j}\right) + S(r,f)$$

which gives

$$\Theta(0;f) + \Theta(\infty;f) \le \frac{1}{k_1+1} + \frac{1}{k_2+1}$$

and this contradicts

$$\Theta(0;f) + \Theta(\infty;f) > \frac{1}{k_1+1} + \frac{1}{k_2+1}.$$

Case 2. Therefore, we have $e^{h(z)} \equiv 1$ and hence

$$\frac{(f-s_3)(\mathcal{L}_c(f)-s_4)}{(f-s_4)(\mathcal{L}_c(f)-s_3)} = 1.$$

On simplification, it is easy to obtain $\mathcal{L}_c(f) \equiv f(z)$, for all $z \in \mathbb{C}$.

We are now to find the class of all the meromorphic functions satisfying the difference equation $\mathcal{L}_c(f) \equiv f$. By assumption of the result, and using Lemma 3.6, it is easy to see that f is not a rational function. Therefore $f(z)$ must be a transcendental meromorphic function.

We also see that $f(z)$ and $f(z+c)$ are related by

$$(4.5)\quad f(z+c) = \left(\frac{1-c_0}{c_1}\right) f(z).$$

Let $f_1(z)$ and $f_2(z)$ be two solutions of (4.5) (see [2] for more details). Then it is easy to see that

$$(4.6)\quad f_1(z+c) = \left(\frac{1-c_0}{c_1}\right) f_1(z)$$

$$(4.7)\quad f_2(z+c) = \left(\frac{1-c_0}{c_1}\right) f_2(z).$$

We set $h(z) := f_1(z)/f_2(z)$. Then in view of (4.6) and (4.7), we obtain

$$h(z+c) = \frac{f_1(z+c)}{f_2(z+c)} = \frac{\frac{1-c_0}{c_1}f_1(z)}{\frac{1-c_0}{c_1}f_2(z)} = \frac{f_1(z)}{f_2(z)} = h(z),$$

for all $z \in \mathbb{C}$. Therefore, it is easy to verify that

$$f_2(z) = \left(\frac{1-c_0}{c_1}\right)^{\frac{z}{c}} g_2(z),$$

where $g_2(z)$ is a meromorphic function with $g_2(z+c) = g_2(z)$, is a solution of (4.5). Hence, it is also easy to verify that $f_1(z) = f_2(z)h(z)$, a solution of (4.5). Thus the linear combination

$$\begin{aligned} a_1 f_1(z) + a_2 f_2(z) &= \left(\frac{1-c_0}{c_1}\right)^{\frac{z}{c}} (a_1 h(z) + a_2)\, g_2(z) \\ &= \left(\frac{1-c_0}{c_1}\right)^{\frac{z}{c}} \sigma(z), \end{aligned}$$

where $\sigma(z) = (a_1 h(z) + a_2)\, g_2(z)$ is such that $\sigma(z+c) = \sigma(z)$, for all $z \in \mathbb{C}$, is the general solution of (4.5). Hence, the precise form of the function f is the following

$$f(z) = \left(\frac{1-c_0}{c_1}\right)^{\frac{z}{c}} g(z),$$

where $g(z)$ is a meromorphic function with $g(z+c) = g(z)$, for all $z \in \mathbb{C}$.

This completes the proof. □

Proof of Theorem 2.3. Let us suppose that $g(z)$ is the canonical product of the poles of f. Then by Lemma 3.4, we obtain

$$m\left(r, \frac{g(z+c)}{g(z)}\right) = S(r,f). \tag{4.8}$$

Since $\Theta(\infty; f) = 1$, hence it is easy to see that

$$\limsup_{r\to+\infty} \frac{\overline{N}(r,f)}{T(r,f)} = 0.$$

Therefore, it follows from (4.8) that

$$T\left(r, \frac{g(z+c)}{g(z)}\right) = S(r,f). \tag{4.9}$$

Since f and $\mathcal{L}_c(f)$ share s_3 CM, by Lemma 3.3, we obtain

$$\frac{\mathcal{L}_c(f) - s_3}{f - s_3} = e^{\mathcal{H}(z)} \frac{g(z)}{g(z+c)}, \tag{4.10}$$

where $\mathcal{H}(z)$ is an entire function with $\rho(\mathcal{H}) < 1$. By Lemma 3.4, we also obtain

$$T\left(r, e^{\mathcal{H}(z)}\frac{g(z)}{g(z+c)}\right) = S(r,f). \tag{4.11}$$

Therefore, by Lemma 3.2 and (4.11), a simple computation shows that $T(r, \mathcal{L}_c(f)) = T(r,f) + S(r,f)$. For the sake convenience, we set

$$\beta := e^{\mathcal{H}(z)}\frac{g(z)}{g(z+c)} \text{ and } S(r) := S(r, \mathcal{L}_c(f)) = S(r,f).$$

If $\mathcal{L}_c(f) \not\equiv f(z)$. i.e., if $\beta \neq 1$, then with the help of (4.10) and from the assumption, we obtain

$$\overline{N}_{1)}\left(r, \frac{1}{f-s_1}\right) \leq N\left(r, \frac{1}{\beta - 1}\right) \leq T(r,\beta) + O(1) = S(r). \tag{4.12}$$

and

$$\overline{N}_{1)}\left(r, \frac{1}{f-s_2}\right) \leq N\left(r, \frac{1}{\beta - 1}\right) \leq T(r,\beta) + O(1) = S(r). \tag{4.13}$$

By Lemma 3.7, and using (4.12) and (4.13), by a simple computation, we obtain

$$\overline{N}_{1)}\left(r, \frac{1}{\mathcal{L}_c(f)-s_1}\right) \leq \overline{N}_{1)}\left(r, \frac{1}{f-s_1}\right) + S(r) = S(r). \tag{4.14}$$

and

$$\overline{N}_{1)}\left(r, \frac{1}{\mathcal{L}_c(f)-s_2}\right) \leq \overline{N}_{1)}\left(r, \frac{1}{f-s_1}\right) + S(r) = S(r). \tag{4.15}$$

On the other hand, it follows from (4.10) that

$$\begin{aligned} \mathcal{L}_c(f) - s_1 &= (s_3 - s_1) + \beta\,(f - s_3) \\ &= \beta\left(f - \frac{s_1 + (\beta-1)s_3}{\beta}\right). \end{aligned} \tag{4.16}$$

Similarly, we obtain

$$\mathcal{L}_c(f) - s_2 = \beta\left(f - \frac{s_2 + (\beta-1)s_3}{\beta}\right). \tag{4.17}$$

It is easy to see that

$$N\left(r, \frac{1}{\mathcal{L}_c(f)-s_1}\right) = N\left(r, \frac{1}{f - \frac{s_1+(\beta-1)s_3}{\beta}}\right) + S(r). \tag{4.18}$$

and

$$N\left(r, \frac{1}{\mathcal{L}_c(f) - s_2}\right) = N\left(r, \frac{1}{f - \frac{s_2 + (\beta-1)s_3}{\beta}}\right) + S(r). \tag{4.19}$$

Now our aim is to deal with the following three cases.

Case 1. Suppose that $(((\beta - 1)s_3 + s_1)/\beta) \neq s_2$.

Since $(((\beta - 1)s_3 + s_1)/\beta) \neq s_1$ and $\Theta(\infty; f) = 1$, hence by Lemma 3.5 for $\epsilon \in (0, 1/2)$, it follows from (4.10), (4.12), (4.13), (4.14) and (4.18) that

$$\begin{aligned}
&(2-\epsilon)T(r, f) \\
\leq\ & \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f - s_2}\right) + \overline{N}\left(r, \frac{1}{f - \frac{(\beta-1)s_3 + s_1}{\beta}}\right) \\
\leq\ & \overline{N}_{(2}\left(r, \frac{1}{f - s_1}\right) + \overline{N}_{(2}\left(r, \frac{1}{f - s_2}\right) + \overline{N}_{(2}\left(r, \frac{1}{\mathcal{L}_c(f) - s_1}\right) \\
\leq\ & \frac{1}{2}T(r, f) + \frac{1}{2}T(r, f) + \frac{1}{2}T(r, f) + S(r) \\
=\ & \frac{3}{2}T(r, f) + S(r, f),
\end{aligned} \tag{4.20}$$

which is a contradiction.

Case 2. Suppose that $(((\beta - 1)s_3 + s_2)/\beta) \neq s_1$.

Since $(((\beta - 1)s_3 + s_2)/\beta) \neq s_2$ and $\Theta(\infty; f) = 1$, hence by applying the same argument as in Case 1, we arrive at a contradiction.

Therefore, we must have $\mathcal{L}_c(f) \equiv f$, and hence following the proof of Theorem 2.1, we obtain the precise form of the function.

Case 3. Suppose that

$$\frac{(\beta - 1)s_3 + s_2}{\beta} = s_1$$

and

$$\frac{(\beta - 1)s_3 + s_1}{\beta} = s_2.$$

An elementary calculation shows that $\beta = -1$, so that $2s_3 = s_1 + s_2$. Therefore, from (4.10), we have $\mathcal{L}_c(f) \equiv -f(z) + 2s_3$ and by the same argument used in the previous cases, it is not hard to show that $f(z)$ will take the form

$$f(z) = \left(\frac{-1 - c_0}{c_1}\right)^{\frac{z}{c}} g(z) + 2s_3, \quad \text{for all } z \in \mathbb{C},$$

where $g(z)$ is a meromorphic function with period c. This completes the proof. $\square$

5. Concluding Remarks and Open Question

Let us suppose that $\mathcal{L}_c(f) \equiv f$, where f is a non-constant meromorphic functions. Since f can not be rational function (see [2] for detailed information), hence f must be transcendental and hence $f(z)$ takes the precise form

$$f(z) = \left(\frac{1-c_0}{c_1}\right)^{\dfrac{z}{c}} g(z),$$

where $g(z)$ is a meromorphic periodic function c. We can write $f(z) = \alpha^{\frac{z}{c}} g(z)$, where α is a root of the equation $c_1 z + c_0 = 1$.

For more generalization of $\mathcal{L}_c(f)$, we define $\mathcal{L}_c^n(f) := c_n f(z+nc) + \cdots + c_1 f(z+c) + c_0 f(z)$ (see [5] for details), where $c_n (\neq 0), c_1, c_0 \in \mathbb{C}$. For particular values of the constants $c_j = (-1)^{n-j}\binom{n}{j}$ for $j = 0, 1, \ldots, n$, it is easy to see that $\mathcal{L}_c^n(f) = \Delta_c^n(f)$.

One can verify that $f(z) = 2^{\frac{z}{c}} g(z)$, where g is a meromorphic function of period c, solves the difference equation $\Delta_c^n(f) \equiv f$. We are mainly interested to find the precise form of the function f when it solves the difference equation $\mathcal{L}_c^n(f) \equiv f$. However, regarding the complete solution, we conjecture the following.

Conjecture 5.1. *Let f be a meromorphic function such that $\mathcal{L}_c^n(f) \equiv f$, then f assumes the form*

$$f(z) = \alpha_n^{z/c} g_n(z) + \cdots + \alpha_1^{z/c} g_1(z),$$

where g_j $(j = 1, 2, \ldots, n)$ are meromorphic functions of period c, and α_j $(j = 1, 2, \ldots, n)$ are roots of the equation $c_n z^n + \cdots + c_1 z + c_0 = 1$.

Based on the above discussions, we also pose the following question for future investigations on the main results of the paper.

Question 5.2. *Keeping all other conditions intact, for a meromorphic function f, is it possible to get a corresponding result of Theorems 2.1, 2.2 and 2.3 for $\mathcal{L}_c^n(f)$?*

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