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The Evaluation of the Conditions for the Non-Null Curves to be Inextensible in Lorentzian 6-Space

Muradíye Çímdíker Aslan* and Yasín Ünlütürk
Department of Mathematics, Kirklareli University, Kirklareli 39100, Turkey
e-mail : muradiyecimdiker@klu.edu.tr *and* yasinunluturk@klu.edu.tr

Abstract. In this study, we obtain various conditions for the non-null curve flows to be inextensible in the 6-dimensional Lorentzian space $\mathbb{L}^6$. Then, we find partial differential equations which characterize the family of inextensible non-null curves.

1. Introduction

The concept of elasticity has long been considered in the application of geometry. Since the subject of "elastica" was studied by such greats as Galileo, Bernoulli, and Euler, it has found numerous applications across physics, astronomy and mathematics. One finds in central of works in field theory, nonlinear optics, fluid dynamics, sigma models, relativity, water wave theory, and so on [14, 15, 20].

The concept of elasticity is mainly described by the means of *flow*. Briefly, the flow of a curve or a surface represents the time evolution of these geometric objects. The term "inextensible" is used to indicate a flow curve whose arc-length is preserved in space. If the flow of a curve or a surface is inextensible, then its strain energy is zero [10, 11]. Researching the inextensibility of curves in different spaces is common among topics for geometers [3, 6, 8, 9, 13, 18, 16]. Inextensible flows of planar curves have been researched in detail, and some examples of the latter have been given in [11]. Gürbüz studied the properties of spacelike, timelike and null curve flows to be inextensible in [3]. Körpınar et.al. approached inextensible flows of curves in $\mathbb{E}^3$ by a new method [6]. Öğrenmiş et.al. investigated inextensible curve flows in Galilean space [13]. Yıldız et.al. examined the subject in the n-dimensional Euclidean space $\mathbb{E}^n$ [18].

In the early stages of Einstein's theory, a bridge was built between physics and geometry using the concepts of maps and curves. Null cases were studied to understand general relativity as a dynamical theory of Frenet formalism. In this way black holes were investigated in five and six dimensional spaces by considering

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a timelike curve [4]. In higher dimensional Lorentzian spaces such as Lorentzian 5- space, and Lorentzian 6-space, studying various characterizations of curves have attracted the interest of several researchers [1, 2, 5, 7, 19, 17]. In particular, Yılmaz et.al. determined the Frenet-Serret invariants of non-null curves in Lorentzian 6-space [17].

The motivation of the present work is the goal of characterizing curves in a higher dimensional space. For this purpose, we investigate the properties of non-null curves characterizing inextensibility in Lorentzian 6-space. We give necessary and sufficient conditions for the flow of a family of non-null curves to be inextensible, and also present a system of partial differential equations for such a family of curves in Lorentzian 6-space.

2. Basic Concepts

The Lorentzian space $\mathbb{L}^6$ is a real vector space $\mathbb{R}^6$ with the following metric:

$$g = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2 + dx_6^2, \tag{2.1}$$

where $x = (x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{L}^6$. An arbitrary vector x of $\mathbb{L}^6$ is said to be spacelike if $g(x,x) > 0$ or $x = 0$, timelike if $g(x,x) < 0$ and lightlike or null if $g(x,x) = 0$ and $x \neq 0$. For $x \in \mathbb{L}^6$, the norm x is defined by $\|x\| = (|g(x,x)|)^{\frac{1}{2}}$, and the vector x is called a unit vector if $g(x,x) = \pm 1$. The vectors $x, y \in \mathbb{L}^6$ are said to be orthogonal if the inner product of the vectors x, y are equal to zero. A vector $\alpha(s)$ is called a unit speed curve if its velocity vector satisfies $\|\alpha'\| = (|g(\alpha', \alpha')|)^{\frac{1}{2}} = 1$ [12].

Let $\{V_1(s), V_2(s), V_3(s), V_4(s), V_5(s), V_6(s)\}$ be a moving Frenet-Serret frame along the curve $\alpha(s)$ in $\mathbb{L}^6$. For a non-null unit speed curve $\alpha(s)$, the Frenet-Serret formulae are given as

$$\begin{aligned} \frac{\partial V_1}{\partial s} &= \kappa_1 V_2, \\ \frac{\partial V_i}{\partial s} &= -\varepsilon_{i-2}\varepsilon_{i-1}\kappa_{i-1}V_{i-1} + \kappa_i V_{i+1}, \text{ for } i \in \{2,3,4,5\}, \\ \frac{\partial V_6}{\partial s} &= -\varepsilon_4\varepsilon_5\kappa_5 V_5. \end{aligned} \tag{2.2}$$

Here, $g(V_i, V_j) = \varepsilon_{j-1} = \pm 1$ for $1 \leq j \leq 6$, with respect to the character of the frame vectors. The functions $\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5$ are the Frenet-Serret curvatures of the curve $\alpha(s)$ in $\mathbb{L}^6$ [1, 2, 17].

Let

$$\alpha : [0, l] \times [0, w) \to \mathbb{L}^6$$

be a one parameter family of smooth curves in $\mathbb{L}^6$, where l is the arc-length of the initial curve. Let u be the curve parametrization variable $0 \leq u \leq l$. If the speed curve α is denoted by

$$v = \left(\left|g\left(\tfrac{d\alpha}{du}, \tfrac{d\alpha}{du}\right)\right|\right)^{\frac{1}{2}}, \tag{2.3}$$

then the arc-length of the curve α is

$$s(u)=\int_0^u\left(\left|g\left(\frac{d\alpha}{du},\frac{d\alpha}{du}\right)\right|\right)^{\frac{1}{2}}du=\int_0^u vdu. \tag{2.4}$$

Any flow of the curve α is expressed by

$$\frac{\partial\alpha}{\partial t}=\sum_{i=1}^{n} f_iV_i, \tag{2.5}$$

where f_i denotes the i^{th} scalar speed of the curve α. The arc-length variation is

$$s(u,t)=\int_0^u vdu. \tag{2.6}$$

A curve evolution $\alpha(u,t)$, and its flow $\frac{\partial\alpha}{\partial t}$ is called an inextensible flow [18] if

$$\frac{\partial}{\partial t}\left(\left|g\left(\frac{\partial\alpha}{\partial u},\frac{\partial\alpha}{\partial u}\right)\right|\right)^{\frac{1}{2}}=0. \tag{2.7}$$

3. Main Results

Let α be a family of differentiable non-null curves in Lorentzian 6-space $\mathbb{L}^6$. Define $v=\left(\left|g\left(\frac{d\alpha}{du},\frac{d\alpha}{du}\right)\right|\right)^{\frac{1}{2}}$ and $ds=vdu$. The flows of a non-null curve family α are parametrized by

$$\frac{\partial\alpha}{\partial t}=\sum_{i=1}^{6} f_iV_i, \tag{3.1}$$

where the components f_i are in $\mathbb{L}^6$.

Lemma 3.1. *The flows of the non-null curves family α are inextensible in Lorentzian 6-space $\mathbb{L}^6$, then we have*

$$\frac{\partial v}{\partial t}=\frac{\partial f_1}{\partial u}-\varepsilon_0\varepsilon_1 f_2 v\kappa_1. \tag{3.2}$$

Proof. We have

$$v^2=\left|g\left(\frac{d\alpha}{du},\frac{d\alpha}{du}\right)\right|. \tag{3.3}$$

Differentiating the expression (3.3) with respect to t, then we obtain

$$\begin{aligned} 2v\frac{\partial v}{\partial t} &= \frac{\partial}{\partial t}\varepsilon_0\left(\left|g\left(\frac{d\alpha}{du},\frac{d\alpha}{du}\right)\right|\right) \\ &= 2\varepsilon_0\left[g\left(\frac{d\alpha}{du},\frac{d}{du}\frac{d\alpha}{dt}\right)\right]. \end{aligned} \tag{3.4}$$

If we differentiate the flows of α with respect to u, then we arrive at

$$\frac{d}{du}\frac{d\alpha}{dt} = \frac{d}{du}\sum_{i=1}^{6} f_i V_i = \sum_{i=1}^{6}\left(\frac{\partial f_i}{\partial u}V_i + f_i\frac{\partial V_i}{\partial u}\right), \tag{3.5}$$

Substituting the expression (3.5) into the expression (3.4) and using the expression (2.2), then we get

$$\begin{aligned} 2v\frac{\partial v}{\partial t} &= 2\varepsilon_0\left[g\left(vV_1, \left(\frac{\partial f_1}{\partial u}V_1 + vf_2(-\varepsilon_0\varepsilon_1\kappa_1 V_1 + \kappa_2 V_3)\right)\right)\right] \\ &= 2\varepsilon_0\left[v\frac{\partial f_1}{\partial u}\varepsilon_0 - v^2 f_2\varepsilon_0^2\varepsilon_1\kappa_1\right]. \end{aligned} \tag{3.6}$$

□

Rearranging the expression (3.6), the proof of Lemma 3.1. is completed.

Theorem 3.2. *The flows of the non-null curves family α are inextensible in Lorentzian 6-space $\mathbb{L}^6$ if and only if*

$$\frac{\partial f_1}{\partial s} = \varepsilon_0\varepsilon_1\kappa_1 f_2. \tag{3.7}$$

Proof. We know that

$$\frac{\partial}{\partial t}s(u,t) = \int_0^u \frac{\partial v}{\partial t}du = 0, \tag{3.8}$$

$u \in [0, l]$.
From the expression (3.8), we reach

$$\frac{\partial}{\partial t}s(u,t) = \int_0^u \left(\frac{\partial f_1}{\partial u} - \varepsilon_0\varepsilon_1 f_2 v\kappa_1\right)du = 0. \tag{3.9}$$

Hence, the proof is completed. □

Lemma 3.3. *Let $\{V_1, V_2, V_3, V_4, V_5, V_6\}$ be a moving Frenet-Serret frame along the non-null curves family α in Lorentzian 6-space $\mathbb{L}^6$. Then the derivatives of the moving Frenet-Serret frame with respect to t are:*

$$\frac{\partial V_1}{\partial t} = \sum_{i=2}^{5} V_i\left[\frac{\partial f_i}{\partial s} + f_{i-1}\kappa_{i-1} - \varepsilon_{i-1}\varepsilon_i\kappa_i f_{i+1}\right] + V_6\left[\frac{\partial f_6}{\partial s} + f_5\kappa_5\right], \tag{3.10}$$

and for $i = 2 \dots 5$,

$$\begin{aligned} \frac{\partial V_i}{\partial t} &= -\varepsilon_0 \varepsilon_{i-1} \left[\frac{\partial f_i}{\partial s} + f_{i-1}\kappa_{i-1} - \varepsilon_{i-1}\varepsilon_i \kappa_i f_{i+1} \right] V_1 \\ &+ \sum_{j=2}^{6} \left[g\left(\frac{\partial V_i}{\partial t}, V_j\right) V_j \right] - g\left(\frac{\partial V_i}{\partial t}, V_i\right) V_i, \\ \frac{\partial V_6}{\partial t} &= -\varepsilon_0 \varepsilon_5 \left[\frac{\partial f_6}{\partial s} + f_5 \kappa_5 \right] V_1 + \sum_{j=2}^{6} \left[g\left(\frac{\partial V_i}{\partial t}, V_j\right) V_j \right] - g\left(\frac{\partial V_i}{\partial t}, V_i\right) V_i, \end{aligned} \tag{3.11}$$

Proof. To calculate $\frac{\partial V_1}{\partial t}$, we need to differentiate the flows of α with respect to t. When $i \in \{2, \dots, 6\}$, for $\frac{\partial V_i}{\partial t}$ we use

$$g(V_1, V_i) = 0. \tag{3.12}$$

Differentiating the expression (3.12) with respect to t, the following expression is found:

$$g\left(\frac{\partial V_1}{\partial t}, V_i\right) + g\left(V_1, \frac{\partial V_i}{\partial t}\right) = 0. \tag{3.13}$$

Substituting the expression (3.10) into the expression (3.13) gives the following result

$$g\left(\frac{\partial V_1}{\partial t}, V_i\right) = \varepsilon_1 \left[\frac{\partial f_i}{\partial s} + f_{i-1}\kappa_{i-1} - \varepsilon_{i-1}\varepsilon_i \kappa_i f_{i+1} \right]. \tag{3.14}$$

□

Theorem 3.4. *Necessary and sufficient conditions for the non-null curves flow to be inextensible are the following system of the partial differential equations*

$$\begin{aligned} \frac{\partial \kappa_1}{\partial t} &= \frac{\partial^2 f_2}{\partial s^2} + \varepsilon_0\varepsilon_1\kappa_1^2 f_2 + \frac{\partial \kappa_1}{\partial s} f_1 - \varepsilon_1\varepsilon_2 \frac{\partial \kappa_2}{\partial s} f_3 - 2\varepsilon_1\varepsilon_2\kappa_2 \frac{\partial f_3}{\partial s} \\ &\quad -\varepsilon_1\varepsilon_2\kappa_2^2 f_2 + \varepsilon_1\varepsilon_3\kappa_2\kappa_3 f_4, \\ \frac{\partial \kappa_2}{\partial t} &= \frac{\partial}{\partial s} g\left(\frac{\partial V_2}{\partial t}, V_3\right) - \varepsilon_2\varepsilon_3\kappa_3 g\left(\frac{\partial V_2}{\partial t}, V_4\right) + \varepsilon_0\varepsilon_1\kappa_1 \frac{\partial f_3}{\partial s} \\ &\quad +\varepsilon_0\varepsilon_1\kappa_1\kappa_2 f_2 - \varepsilon_0\varepsilon_1\varepsilon_2\varepsilon_3\kappa_1\kappa_3 f_4, \\ \frac{\partial \kappa_3}{\partial t} &= \frac{\partial}{\partial s} g\left(\frac{\partial V_3}{\partial t}, V_4\right) - \varepsilon_3\varepsilon_4\kappa_4 g\left(\frac{\partial V_3}{\partial t}, V_5\right) + \varepsilon_1\varepsilon_2\kappa_2 g\left(\frac{\partial V_2}{\partial t}, V_4\right), \\ \frac{\partial \kappa_4}{\partial t} &= \frac{\partial}{\partial s} g\left(\frac{\partial V_4}{\partial t}, V_5\right) - \varepsilon_4\varepsilon_5\kappa_5 g\left(\frac{\partial V_4}{\partial t}, V_6\right) + \varepsilon_2\varepsilon_3\kappa_3 g\left(\frac{\partial V_3}{\partial t}, V_5\right), \\ \frac{\partial \kappa_5}{\partial t} &= \frac{\partial}{\partial s} g\left(\frac{\partial V_5}{\partial t}, V_6\right) + \varepsilon_3\varepsilon_4\kappa_4 g\left(\frac{\partial V_4}{\partial t}, V_6\right). \end{aligned} \tag{3.15}$$

Proof. Differentiating $\frac{\partial V_1}{\partial t}$ with respect to s and using the expression (2.2) for $i=3$ and (3.7), we find

$$\begin{aligned} \frac{\partial}{\partial s}\frac{\partial V_1}{\partial t} &= V_2\Big[\frac{\partial^2 f_2}{\partial s^2}+\frac{\partial f_1}{\partial s}\kappa_1+f_1\frac{\partial \kappa_1}{\partial s}-\varepsilon_1\varepsilon_2 f_3\frac{\partial \kappa_2}{\partial s} \\ &\quad -\varepsilon_1\varepsilon_2\frac{\partial f_3}{\partial s}\kappa_2-\varepsilon_1\varepsilon_2\frac{\partial f_3}{\partial s}\kappa_2-\varepsilon_1\varepsilon_2\kappa_2^2 f_2+\varepsilon_1\varepsilon_2^2\varepsilon_3\kappa_2\kappa_3 f_4\Big] \end{aligned} \tag{3.16}$$

If we differentiate $\frac{\partial V_1}{\partial s}$ with respect to t, then we get

$$\frac{\partial}{\partial t}\frac{\partial V_1}{\partial s}=\frac{\partial \kappa_1}{\partial t}V_2+\kappa_1\frac{\partial V_2}{\partial t}. \tag{3.17}$$

Equalizing the expressions (3.16) and (3.17), the expression $\frac{\partial \kappa_1}{\partial t}$ is obtained as in (3.15). Differentiating $\frac{\partial V_2}{\partial t}$ with respect to s and using the expressions (2.2) for $i=4$ and (3.10), then we reach

$$\begin{aligned} \frac{\partial}{\partial s}\frac{\partial V_2}{\partial t} &= V_3\Big[\frac{\partial}{\partial s}g\left(\frac{\partial V_2}{\partial t},V_3\right)-\varepsilon_2\varepsilon_3\kappa_3 g\left(\frac{\partial V_2}{\partial t},V_4\right) \\ &\quad +\varepsilon_0\varepsilon_1\kappa_1\frac{\partial f_2}{\partial s}+\varepsilon_0\varepsilon_1\kappa_1\kappa_2 f_2-\varepsilon_0\varepsilon_1\varepsilon_2\varepsilon_3\kappa_1\kappa_3 f_4\Big]. \end{aligned} \tag{3.18}$$

From the expression (2.2) for $i=2$, then we have

$$\frac{\partial}{\partial t}\frac{\partial V_2}{\partial s}=-\varepsilon_0\varepsilon_1\frac{\partial \kappa_1}{\partial t}V_1-\varepsilon_0\varepsilon_1\kappa_1\frac{\partial V_1}{\partial t}+\frac{\partial \kappa_2}{\partial t}V_3+\kappa_2\frac{\partial V_3}{\partial t}. \tag{3.19}$$

Using the expressions (3.18) and (3.19), we obtain the expression for $\frac{\partial \kappa_2}{\partial t}$ in (3.15).

To calculate $\frac{\partial \kappa_3}{\partial t}$, we proceed as for $\frac{\partial \kappa_2}{\partial t}$ with only the obvious changes to the indices. Differentiating $\frac{\partial V_4}{\partial t}$ with respect to s, we get the following:

$$\frac{\partial}{\partial s}\frac{\partial V_4}{\partial t}=V_4\left[\frac{\partial}{\partial s}g\left(\frac{\partial V_4}{\partial t},V_5\right)-\varepsilon_4\varepsilon_5\kappa_5 g\left(\frac{\partial V_4}{\partial t},V_6\right)+\varepsilon_2\varepsilon_3\kappa_3 g\left(\frac{\partial V_3}{\partial t},V_5\right)\right]. \tag{3.20}$$

By the expression (2.2) for $i=4$, then we have

$$\frac{\partial}{\partial t}\frac{\partial V_4}{\partial s}=-\varepsilon_2\varepsilon_3\frac{\partial \kappa_3}{\partial t}V_3-\varepsilon_2\varepsilon_3\kappa_3\frac{\partial V_3}{\partial t}+\frac{\partial \kappa_4}{\partial t}V_5+\kappa_4\frac{\partial V_5}{\partial t}. \tag{3.21}$$

Equalizing the expressions (3.20) and (3.21), then gives $\frac{\partial \kappa_4}{\partial t}$. Using $\frac{\partial}{\partial t}\frac{\partial V_5}{\partial s}=\frac{\partial}{\partial s}\frac{\partial V_5}{\partial t}$, then we compute $\frac{\partial \kappa_5}{\partial t}$. □

4. Conclusion

In the present work, we investigated the circumstances of non-null curves flows to be inextensible in Lorentzian 6-space $\mathbb{L}^6$. As an open problem, null curves can also be characterized by researchers.

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