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Characterization of Pseudo n-Jordan Homomorphisms

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Abstract. In this paper, among other things, we show that under special hypotheses every [pseudo] $(n+1)$-Jordan homomorphism is a [pseudo] n-Jordan homomorphism and vice versa.

1. Introduction

Let A and B be algebras, B be a right [left] A-module and let $\varphi : A \longrightarrow B$ be a linear map. Then φ is called a *pseudo n-Jordan homomorphism* if there exists an element $w \in A$ such that for all $a \in A$,

$$\varphi(a^n w) = \varphi(a)^n \cdot w, \qquad [\varphi(a^n w) = w \cdot \varphi(a)^n].$$

The element w is called Jordan coefficient of φ. This concept was introduced and studied by Ebadian et al., in [4] and some interesting results related to these maps are given in [9]. If $n = 2$, then φ is called simply a pseudo Jordan homomorphism.

Let A and B be Banach algebras and $\varphi : A \longrightarrow B$ be a linear map. Then φ is called an *n-Jordan homomorphism* if $\varphi(a^n) = \varphi(a)^n$, for all $a \in A$. This notion was introduced by Herstein in [7]. Also φ is called an *n-homomorphism* if $\varphi(\prod_{i=1}^n a_i) = \prod_{i=1}^n \varphi(a_i)$, for every $a_i \in A$, where $1 \leq i \leq n$. The concept of an n-homomorphism was studied for complex algebras in [6].

For the case $n = 2$, this concepts coincides the classical definitions of Jordan homomorphism and homomorphism, respectively.

Clearly, every n-homomorphism is an n-Jordan homomorphism, but in general the converse is false. There are plenty of known examples of n-Jordan homomorphism which are not homomorphism. For example, it is proved in [8] that some Jordan homomorphism on the polynomial rings can not be homomorphism.

The following result is due to Zelazko [11], concerning the characterization of Jordan homomorphisms.

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Theorem 1.1. *Each Jordan homomorphism φ from Banach algebra A into a semisimple commutative Banach algebra B is a homomorphism.*

This result has been proved by the author in [12] for 3-Jordan homomorphism with the extra condition that the Banach algebra A is unital, and then it is extended for all $n \in \mathbb{N}$ in [1]. For nonunital Banach algebra A, Bodaghi and İnceboz in [3], extended Theorem 1.1 for $n \in \{3, 4\}$ by considering an extra condition on the mapping $\varphi : A \longrightarrow B$ as

$$\varphi(a^2 b) = \varphi(ba^2), \quad a, b \in A.$$

Also based on the property of the Vandermonde matrix, they proved in [2] that every n-Jordan homomorphism between two commutative Banach algebras is an n-homomorphism where n is an arbitrary and fixed positive integer.

Obviously, every n-Jordan homomorphism from unital Banach algebra A into B which is unitary Banach A-module is a pseudo n-Jordan homomorphism.

Example 1.2. Let

$$A = \left\{ \begin{bmatrix} 0 & x & a & b \\ 0 & 0 & y & c \\ 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 \end{bmatrix} : \quad x, y, z, a, b, c \in \mathbb{R} \right\},$$

and define $\varphi : A \longrightarrow A$ via

$$\varphi\left(\begin{bmatrix} 0 & x & a & b \\ 0 & 0 & y & c \\ 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & x & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then, $\varphi(u^n) = \varphi(u)^n$ for all $u \in A$ and for $n \geq 4$. Therefore, φ is an n-Jordan homomorphism, but $\varphi(u^3) \neq \varphi(u)^3$, for all $u \in A$, where $x, y, z \neq 0$. Hence, φ is not 3-Jordan homomorphism. Set

$$w = \begin{bmatrix} 0 & \alpha & s & t \\ 0 & 0 & \beta & r \\ 0 & 0 & 0 & \gamma \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where $\gamma \neq 0$. Then φ is a pseudo 3-Jordan homomorphism with the Jordan coefficient w, but it is not a pseudo Jordan homomorphism.

Example 1.3. Let

$$A = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : \ a, b \in \mathbb{R} \right\},$$

and let $\varphi, \psi : A \longrightarrow A$ be a linear map defined by

$$\varphi\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \right) = \begin{bmatrix} -a & 0 \\ 0 & -b \end{bmatrix}, \quad \psi\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \right) = \begin{bmatrix} -b & 0 \\ 0 & -a \end{bmatrix}.$$

Then φ is a 3-Jordan homomorphism, but it is not 4-Jordan homomorphism. Also, ψ is a pseudo Jordan homomorphism with the Jordan coefficient $w = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, but it is not a pseudo 3-Jordan homomorphism.

We mention that for all $n \in \mathbb{N}$, φ is a $(2n+1)$-Jordan homomorphism, but it is not a $(2n)$-Jordan homomorphism. Similarly, ψ is a pseudo $(2n)$-Jordan homomorphism, but it is not a pseudo $(2n+1)$-Jordan homomorphism.

By Examples 1.2 and 1.3, we see that neither [pseudo] n-Jordan homomorphisms are necessarily [pseudo] $(n+1)$-Jordan homomorphisms nor [pseudo] $(n+1)$-Jordan homomorphisms are automatically [pseudo] n-Jordan homomorphisms.

However, each Jordan homomorphism is an n-Jordan homomorphism [10], but the same is false for $n \geq 3$. That is, in general every n-Jordan homomorphism is not m-Jordan homomorphism, where $m > n \geq 3$. Now the following questions can be raised.

Under which conditions between Banach algebras is any n-Jordan homomorphism automatically an $(n+1)$-Jordan homomorphism and vice versa?

Moreover, when the same is true for pseudo n-Jordan homomorphisms? In this paper, under some conditions, we characterize this fact by proving that every [pseudo] $(n+1)$-Jordan homomorphism, [pseudo] n-Jordan homomorphism and [pseudo] Jordan homomorphism are equivalent.

2. Pseudo n-Jordan Homomorphisms

The next result is [4, Theorem 2.3], concerning the characterization of pseudo n-Jordan homomorphisms.

Theorem 2.1. *Let A and B be Banach algebras, A be unital and B be a right A-module. Let $\varphi : A \longrightarrow B$ be a continuous pseudo n-Jordan homomorphism with a Jordan coefficient w. If $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in A$ with $ab = w$, then φ is a pseudo $(n+1)$-Jordan homomorphism and $\varphi(aw) = \varphi(a)\varphi(w)$.*

Unfortunately, there is an error in the proof of Theorem 2.1. Indeed, the first summation $\sum_{n=1}^{\infty} \lambda^n \varphi(a^n w)$ in line 8 of the proof must be $\varphi(e_A) \sum_{n=1}^{\infty} \lambda^n \varphi(a^n w)$, and hence [4, Corollary 2.4, Corollary 2.5] are incorrect. This error resolved by Ebadian et al., in [5] with the extra conditions that the Banach algebra B is unital, and $\varphi(e_A) = e_B$, i.e., φ is unital.

Next we improve this result as follows.

Theorem 2.2. *Let A and B be Banach algebras, A be unital and B be a right A-module. Let $\varphi : A \longrightarrow B$ be a continuous pseudo n-Jordan homomorphism with a Jordan coefficient w. If $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in A$ with $ab = w$, then φ is a pseudo $(n+1)$-Jordan homomorphism which multiplied by $\varphi(w)$.*

Proof. Let $a \in A$ be arbitrary. For $\lambda \in \mathbb{C}$, with $|\lambda| < 1/\|a\|$, $e_A - \lambda a$ is invertible and $(e_A - \lambda a)^{-1} = \sum_{n=0}^{\infty} \lambda^n a^n$. Then

$$\begin{aligned}
\varphi(w) &= \varphi((e_A - \lambda a)(e_A - \lambda a)^{-1} w) \\
&= \varphi(e_A - \lambda a)\varphi((e_A - \lambda a)^{-1} w) \\
&= (\varphi(e_A) - \lambda\varphi(a))\varphi\left(\sum_{n=0}^{\infty} \lambda^n a^n w\right) \\
&= \varphi(e_A)\varphi(w) + \varphi(e_A)\varphi\left(\sum_{n=1}^{\infty} \lambda^n a^n w\right) - \lambda\varphi(a)\varphi\left(\sum_{n=0}^{\infty} \lambda^n a^n w\right) \\
&= \varphi(w) + \varphi(e_A)\sum_{n=1}^{\infty} \lambda^n \varphi(a^n w) - \lambda\varphi(a)\sum_{n=0}^{\infty} \lambda^n \varphi(a^n w).
\end{aligned}$$

Hence,

$$\varphi(e_A)\sum_{n=1}^{\infty} \lambda^n \varphi(a^n w) - \lambda\varphi(a)\sum_{n=0}^{\infty} \lambda^n \varphi(a^n w) = 0. \tag{2.1}$$

Multiplying $\varphi(w)$ from the left in (2.1) and using $\varphi(w) = \varphi(w)\varphi(e_A)$, we get

$$\varphi(w)\sum_{n=0}^{\infty} \lambda^{n+1}\varphi(a^{n+1} w) - \varphi(w)\sum_{n=0}^{\infty} \lambda^{n+1}\varphi(a)\varphi(a^n w) = 0.$$

Thus,

$$\varphi(w)\sum_{n=0}^{\infty} \lambda^{n+1}[\varphi(a^{n+1} w) - \varphi(a)\varphi(a^n w)] = 0,$$

for all scalars $\lambda \in \mathbb{C}$, with $|\lambda| < 1/\|a\|$. Therefore $\varphi(w)\varphi(a^{n+1}w) = \varphi(w)\varphi(a)\varphi(a^n w)$ for $n = 0, 1, 2, \cdots$. Since φ is a pseudo n-Jordan homomorphism, we obtain

$$\varphi(w)\varphi(a)\varphi(a^n w) = \varphi(w)\varphi(a)\varphi(a)^n \cdot w = \varphi(w)\varphi(a)^{n+1} \cdot w.$$

Consequently, $\varphi(w)\varphi(a^{n+1}w) = \varphi(w)\varphi(a)^{n+1} \cdot w$, for all $a \in A$. This finishes the proof. □

We say that $w \in A$ is a left (right) separating point of Banach A-module M if the condition $wx = 0$ ($xw = 0$) for $x \in M$ implies that $x = 0$.

As a consequence of Theorem 2.2, we have the next results.

Corollary 2.3. *With the same hypotheses as in Theorem 2.2, if $\varphi(w)$ is a left separating point of B, then φ is a pseudo $(n+1)$-Jordan homomorphism and $\varphi(aw) = \varphi(a)\varphi(w)$.*

Corollary 2.4. *With the same hypotheses as in Theorem 2.2, if B is unital and $\varphi(w) = e_B$, then φ is a pseudo $(n+1)$-Jordan homomorphism and $\varphi(aw) = \varphi(a)\varphi(w)$.*

Now we give an examples which provided that the condition $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in A$ with $ab = w$, in Corollary 2.3 and Corollary 2.4 are essentiall.

Example 2.5 (i) Let A, ψ and w be as in Example 1.3. Set

$$a = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then $ab = w$, but $\psi(ab) \neq \psi(a)\psi(b)$. On the other hand, $\psi(w) = w$ is a left separating point of A and ψ is a pseudo Jordan homomorphism with a Jordan coefficient w, but it is not a pseudo 3-Jordan homomorphism.

(ii) Suppose that

$$u = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then $\psi(u) = e_A$ and ψ is a pseudo 3-Jordan homomorphism with u as a Jordan coefficient, but it is not a pseudo 4-Jordan homomorphism, because the condition $\psi(ab) = \psi(a)\psi(b)$ for all $a, b \in A$ with $ab = u$ is not holds.

Corollary 2.6. *Let A and B be unital Banach algebras and B be a right A-module. Suppose that $\varphi : A \longrightarrow B$ is a continuous unital n-Jordan homomorphism. If $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in A$ with $ab = e_A$, then φ is an $(n+1)$-Jordan homomorphism.*

Lemma 2.7.[10, Lemma 6.3.2] *Every Jordan homomorphism φ between Banach algebras A and B is an n-Jordan homomorphism, for $n \geq 2$.*

The next result is [13, Theorem 2.7], which has been proved for $n = 2, 3$, and it was claimed that the result can be established for $n \geq 4$ by a similar discussion. Recently, in [14, Theorem 2.11] the author presented a short proof for the general case $n \in \mathbb{N}$.

Theorem 2.8. *Every unital $(n+1)$-Jordan homomorphism $\varphi : A \longrightarrow B$ is an n-Jordan homomorphism.*

Combing Lemma 2.7, Theorem 2.8 and [13, Corollary 2.8], we get the following result.

Corollary 2.9. *Let A and B be unital Banach algebras and let $\varphi : A \longrightarrow B$ be a unital linear map. Then the following conditions are equivalent.*

(i) *φ is a Jordan homomorphism.*

(ii) *φ is an n-Jordan homomorphism.*

(iii) *φ is an $(n+1)$-Jordan homomorphism.*

The following result is an analogues of Theorem 2.8 for pseudo n-Jordan homomorphisms.

Theorem 2.10.([9, Theorem 3.4]) *Let A and B be unital Banach algebras, and B be a right A-module. Then every unital pseudo $(n+1)$-Jordan homomorphism $\varphi : A \longrightarrow B$ with a Jordan coefficient w is a pseudo n-Jordan homomorphism.*

Theorem 2.11. *Let A and B be two unital Banach algebras, and let B be a right A-module. Let $\varphi : A \longrightarrow B$ be a unital linear map and w be a right separating point of B, then the following conditions are equivalent.*

(i) *φ is a pseudo Jordan homomorphism.*

(ii) *φ is a pseudo n-Jordan homomorphism.*

(iii) *φ is a pseudo $(n+1)$-Jordan homomorphism.*

Proof. $(iii) \Longrightarrow (ii)$ and $(ii) \Longrightarrow (i)$ follows from Theorem 2.10. $(i) \Longrightarrow (iii)$ Assume that φ is a pseudo Jordan homomorphism, then

$$\varphi(a^2 w) = \varphi(a)^2 \cdot w, \quad a \in A. \tag{2}$$

Replacing a by $a + e_A$ we get $\varphi(aw) = \varphi(a) \cdot w$, for all $a \in A$. Thus,

$$\varphi(a^2 w) = \varphi(a^2) \cdot w. \tag{3}$$

It follows from (2) and (3) that $\big(\varphi(a^2) - \varphi(a)^2\big) \cdot w = 0$. As w is a right separating point of B, we get $\varphi(a^2) = \varphi(a)^2$, and hence φ is a Jordan homomorphism. From Lemma 2.7, we conclude that φ is an $(n+1)$-Jordan homomorphism. Thus,

$$\varphi(a^{n+1} w) = \varphi(a^{n+1}) \cdot w = \varphi(a)^{n+1} \cdot w.$$

Consequently, φ is a pseudo $(n+1)$-Jordan homomorphism. □

We mention that the continuity of φ in [4, Proposition 2.11] is extra and must be omitted. Also by applying [2, Theorem 2.2], we obtain the following extension of [4, Proposition 2.11].

Theorem 2.12. *Let A and B be commutative algebras and B be a right A-module. Let $\varphi : A \longrightarrow B$ be a pseudo n-Jordan homomorphism with a Jordan coefficient w such that w is a right separating point of B. If $\varphi(aw) = \varphi(a) \cdot w$ for each $a \in A$, then φ is an n-Jordan homomorphism, and therefore, it is an n-homomorphism.*

Proposition 2.13 *Let A and B be two unital Banach algebras, and B be a right A-module. Suppose that $\varphi : A \longrightarrow B$ is a unital pseudo n-Jordan homomorphism with a Jordan coefficient w. Then for all $a \in A$ and $1 \leq k \leq n-1$,*

$$\varphi(a^k w) = \varphi(a)^k \cdot w.$$

Proof. Let $\lambda \in \mathbb{C}$ be arbitrary. By the assumption we have

$$\varphi((a + \lambda e_A)^n w) = \varphi(a + \lambda e_A)^n \cdot w, \tag{4}$$

for all $a \in A$. It follows from the equality (4) that

$$\sum_{k=1}^{n-1} \lambda^{n-k} \binom{n}{k} [\varphi(a^k w) - \varphi(a)^k \cdot w] = 0,$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. Hence we have $[\varphi(a^k w) - \varphi(a)^k \cdot w] = 0$ for all $a \in A$. Thus, $\varphi(a^k w) = \varphi(a)^k \cdot w$ for all $1 \leq k \leq n-1$. In particular, $\varphi(aw) = \varphi(a) \cdot w$. □

Corollary 2.14. *Let A and B be unital Banach algebras and B be a right A-bimodule. Suppose that $\varphi : A \longrightarrow B$ is a unital pseudo n-Jordan homomorphism with a Jordan coefficient w such that w is a right separating point of B. If*

(i) *A and B are commutative, or*

(ii) *B is semisimple and commutative,*

then φ is an n-homomorphism.

Proof. If A and B are commutative, then the result follows from Theorem 2.12 and 2.13. Assume that (ii) holds. Then similar to the proof of Theorem 2.11 we conclude that φ is a Jordan homomorphism. Therefore, φ is a homomorphism by Theorem 1.1 and hence it is an n-homomorphism. □

The product of two pseudo n-Jordan homomorphisms is not a pseudo n-Jordan homomorphism, in general. For example, let

$$A = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : \ a, b \in \mathbb{R} \right\}, \qquad B = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : \ a, b, c \in \mathbb{R} \right\}$$

and define $\varphi, \psi : A \longrightarrow B$ by

$$\varphi\left(\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} a & -b \\ 0 & 0 \end{bmatrix}, \quad \psi\left(\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}.$$

Then it is routine to check that φ and ψ are pseudo n-Jordan homomorphism with the Jordan coefficient $w = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$, while $h : A \longrightarrow B$ via $h(x) = \varphi(x)\psi(x)$ is not a pseudo n-Jordan homomorphism with the Jordan coefficient w.

However, if $\varphi, \psi : A \longrightarrow A$ are pseudo n-Jordan homomorphism with the Jordan coefficient w, A is commutative and w is an idempotent in A, then $h : A \longrightarrow A$ defined by $h(x) = \varphi(x)\psi(x)$ is a pseudo n-Jordan homomorphism with the Jordan coefficient w.

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