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A Class of Normaloid Weighted Composition Operators on the Fock Space over $\mathbb{C}$

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Abstract. Let ϕ be an entire self map on $\mathbb{C}$ and let ψ be an entire function on $\mathbb{C}$. A weighted composition operator induced by ϕ with weight ψ is given by $C_{\psi,\phi}$. In this paper we investigate under what conditions the weighted composition operators $C_{\psi,\phi}$ on the Fock space over $\mathbb{C}$ induced by ϕ with weight of the form $k_c(\zeta) = e^{\langle \zeta, c\rangle - \frac{|c|^2}{2}}$ is normaloid and essentially normaloid.

1. Introduction

In this paper, we work with a class of weighted composition operators acting on the Fock space $\mathcal{F}^2$, also known as the *Bargmann* space over the complex plane $\mathbb{C}$. This is the Hilbert space of analytical functions $f(\zeta)$ such that $\|f\|^2 = \frac{1}{\pi}\int_{\mathbb{C}} |f(\zeta)|^2 e^{-|\zeta|^2} dA(\zeta), \zeta \in \mathbb{C}$, where dA is the usual Lebesque measure on $\mathbb{C}$. In $\mathcal{F}^2$, the inner product is defined as $\langle f, g\rangle = \frac{1}{\pi}\int_{\mathbb{C}} f(\zeta)\overline{g(\zeta)} e^{-|\zeta|^2} dA(\zeta)$. It is known that $\mathcal{F}^2$ is a reproducing kernel Hilbert space (RKHS) with kernel $K_\eta \zeta = e^{\langle \zeta, \eta\rangle}$ for $\eta, \zeta \in \mathbb{C}$. Let $k_\eta = \frac{K_\eta}{\|K_\eta\|}$ be the normalization of K_η.

A *composition operator* C_ϕ on $\mathcal{F}^2$ is defined as $C_\phi f = f \circ \phi$, where ϕ is an analytical self map on $\mathbb{C}$. For an analytical function ψ, the *weighted composition operator* on $\mathcal{F}^2$ is defined as $C_{\psi,\phi} f = \psi . f \circ \phi$. It is clear that when $\psi \equiv 1$, $C_{\psi,\phi}$ is reduced to C_ϕ. The classical Fock space has been studied by many authors; see, for example [1], [6]

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and [11]. For more on background on composition operators, one recommend the excellent books [2] and [9]. The book [12] is an excellent reference on the Fock space.

For a bounded operator A, we denote

- $\sigma(A)$ the spectrum of A.
- $\sigma_e(A)$ the essential spectrum of A.
- $r_\sigma(A) = \sup\{|\lambda|, \lambda \in \sigma(A)\}$, the spectral radius of $\sigma(A)$.
- $r_{\sigma_e}(A)\}$, the essential spectral radius of $\sigma_e(A)$.
- $W(A)$ the numerical range of A.
- $r_w(A) = \sup\{|\lambda|, \lambda \in W(A)\}$, the numerical radius of $W(A)$.
- $\|A\|_e = \inf\{\|A - B\| : B \text{is compact}\}$, the essential norm.
- $r_\sigma(A) = \lim_{n\to\infty}(\|A^n\|)^{\frac{1}{n}}$
- $r_{\sigma_e}(A) = \lim_{n\to\infty}(\|A^n\|_e)^{\frac{1}{n}}$

Also we have the following definitions for an operator A

- Normaloid if $\|A\| = r_\sigma(A)$
- Essentially Normaloid if $\|A\|_e = r_{\sigma_e}(A)$
- Spectraloid if $r_\sigma(A) = r_w(A)$

It is also well known that every hyponormal operator is normaloid and an operator is normaloid iff $\|A^n\| = \|A\|^n$ for every integer $n \geq 1$. By [[5], Theorem 1.3-2], if $r_w(A) = \|A\|$, then $r_\sigma(A) = \|A\|$. We will also use the fact that unitarily equivalent bounded operators have the same numerical range and norm.

In [10], the author gave an exact characterization for when weighted composition operators on the *classical Hardy space* $\mathcal{H}^2$ are normaloid. Inspired by the article [10], we will investigate under which conditions a class of weighted composition operators on the Fock space $\mathcal{F}^2$ is normaloid and under which it is essentially normaloid.

2. Preliminary Results

In this section, we list well-known results on weighted composition operators on $\mathcal{F}^2$.

Theorem 2.1.([1], Theorem 1) Suppose $\phi : \mathbb{C} \to \mathbb{C}$ is an analytic function.
(a) If C_ϕ is bounded on $\mathcal{F}^2$ then $\phi(\zeta) = \mu\zeta + \nu$, where $\mu, \nu \in \mathbb{C}$, $|\mu| \leq 1$ and if

$|\mu| = 1$ then $\nu = 0$.
(b) If C_ϕ is compact on $\mathcal{F}^2$ then $\phi(\zeta) = \mu\zeta + \nu$, where $|\mu| < 1$.

By ([1], Theorem 2), the converse of the above theorem is also true.

Theorem 2.2.([[6], Theorem 2.2]) Suppose ψ, ϕ be analytic functions on $\mathbb{C}$ such that ψ is not identically zero. Then $C_{\psi,\phi}$ is bounded iff ψ belongs to $\mathcal{F}^2$, $\phi(\zeta) = \phi(0) + \lambda\zeta$ with $|\lambda| \leq 1$ and $M(\psi, \phi) := sup\{|\psi|^2 e^{(|\phi(\zeta)|^2 - |\zeta|^2)}; \zeta \in \mathbb{C}\} < \infty$.

Theorem 2.3.([[6], Theorem 2.3]) Let ψ, ϕ be entire functions such that ψ is not identically zero. Then the operator $C_{\psi,\phi}$ is a normal bounded operator on $\mathcal{F}^2$ iff one of the following two cases occurs:
a. $\phi(\zeta) = \lambda\zeta + \nu$ with $|\lambda| = 1$ and $\psi = \psi(0)K_{\overline{\lambda}\nu}$. In this case, $C_{\psi,\phi}$ is a constant multiple of a unitary operator.
b. $\phi(\zeta) = \lambda\zeta + \nu$ with $|\lambda| < 1$ and $\psi = \psi(0)K_c$, where $c = \nu\frac{1-\overline{\lambda}}{1-\lambda}$. In this case, $C_{\psi,\phi}$ is unitarily equvalent to $\psi(0)C_{\lambda\zeta}$.

Theorem 2.4.([[6], Theorem 2.4]) If ψ, ϕ be analytic functions on $\mathbb{C}$ such that ψ is not identically zero. Then $C_{\psi,\phi}$ is compact on $\mathcal{F}^2$ if and only if $\phi(\zeta) = \mu\zeta + \nu$ with $|\mu| < 1$ and $\lim_{|\zeta|\to\infty} |\psi|^2 e^{|\phi(\zeta)|^2 - |\zeta|^2} = 0$.

In the following result, the author calculated norm of the composition operators acting on the Fock space over $\mathbb{C}^n$

Theorem 2.5.([[1], Theorem 4]) Suppose that ϕ is a self-map on $\mathbb{C}^n$ such that $\phi(\zeta) = \Sigma\zeta + \Lambda$, where either $\|\Sigma\| < 1$ and Λ is arbitrary, or $\|\Sigma\| = 1$ and $\langle \Sigma\eta, \Lambda \rangle = 0$ whenever $|\Sigma\eta| = |\eta|$. Then on $\mathcal{F}^2(\mathbb{C}^n)$, we have $\|C_\phi\| = e^{\frac{1}{4}(|w_0|^2 - |\Sigma w_0|^2 + |\Lambda|^2}$, where w_0 is any solution to $(I - \Sigma^*\Sigma)w = \Sigma^*\Lambda$.

In one variable setting, when ϕ is of the form $\phi(\zeta) = \mu\zeta + \nu$, the above result is reduced to $\|C_\phi\| = e^{\frac{1}{2}\frac{|\nu|^2}{1-|\mu|^2}}$ where $|\mu| < 1$. In [3], the author extended the norm calculation into *non-Hilbert Fock space* $\mathcal{F}^p(\mathbb{C}^n)$.

3. Main Results

In this section, we characterize a class of bounded normaloid weighted composition operators induced by the self map ϕ of the form $\phi(\zeta) = \mu\zeta + \nu$ where $\mu, \nu \in \mathbb{C}$ such that $|\mu| < 1$ and the weight function ψ of the form $\psi(\zeta) = k_c\zeta$ for some $c \in \mathbb{C}$. By Theorem 2.1, $|\mu| = 1$, implies $\nu = 0$. This implies $\phi(\zeta) = \mu\zeta$ which induces a normal weighted composition operator on $\mathcal{F}^2$. Moreover every normal operator is normaloid. Therefore we consider the case $|\mu| < 1$.
The following two lemmas are easy to derive.

Lemma 3.1. Let $\psi_1, \psi_2, ...\psi_n$ be analytic functions on $\mathbb{C}$ and $\phi_1, \phi_2, ...\phi_n$ be an analytic self-map on $\mathbb{C}$. If $C_{\psi_1,\phi_1}, C_{\psi_2,\phi_2}, ...C_{\psi_n,\phi_n}$, are bounded operators on $\mathcal{F}^2$, then $C_{\psi_1,\phi_1}C_{\psi_2,\phi_2}...C_{\psi_n,\phi_n} = C_{\psi_1(\psi_2\circ\phi_1)....(\psi_n\circ\phi_{n-1}\circ...\circ\phi_1),\phi_n\circ\phi_{n-1}\circ...\circ\phi_1}$.

Lemma 3.2. Let ψ, ϕ be holomorphic function on $\mathbb{C}$ and $C_{\psi,\phi}$ is a bounded operator on $\mathcal{F}^2$, then for $\zeta \in \mathbb{C}$, then $C^*_{\psi,\phi} K_\zeta = \overline{\psi(\zeta)} K_{\phi(\zeta)}$

In the following result we derive criterion for a composition operator on $\mathcal{F}^2$ to be normaloid.

Theorem 3.3. Suppose ϕ is a holomorphic on $\mathbb{C}$ such that $\phi(\zeta) = \mu\zeta + \nu, |\mu| < 1$. Then C_ϕ is normaloid on $\mathcal{F}^2$ if and only if $\phi(0) = 0$.

Proof. First, assume that $\phi(0) = 0$. This implies $\phi(\zeta) = \mu\zeta$. Thus C_ϕ is an diagonal operator which is normal. Hence C_ϕ is normaloid on $\mathcal{F}^2$.
For the other direction, suppose C_ϕ is normaloid. (i.e) $r_\sigma(C_\phi) = \|C_\phi\|$. By ([1], Theorem 4), we have $\|C_\phi\| = e^{\frac{1}{2}\frac{|\nu|^2}{(1-|\mu|^2)}}$.
In ([4], Proposition 3.3), it is given that $\sigma(C_\phi) = \overline{\{\mu^n; n \in \mathbb{Z}_+\}}$ for $|\mu| < 1$. This gives $r_\sigma(C_\phi) = 1$. Therefore, we have $e^{\frac{1}{2}\frac{|\nu|^2}{(1-|\mu|^2)}} = 1$. This implies $\nu = 0$. Therefore $\phi(0) = 0$. □

Next, we find the criterion for the composition operator on the Fock space to be essesntially normaloid.

Theorem 3.4. Let ϕ be a holomorphic function on $\mathbb{C}$ such that $\phi(\zeta) = \mu\zeta + \nu$ with $|\mu| \leq 1$. Then C_ϕ is essentially normaloid.

Proof. Consider the case $|\mu| = 1$, then by ([1], Theorem 1), we have $\nu = 0$. Therefore $\phi(\zeta) = \mu\zeta$. By ([8], Theorem 2.2), we have $\|C_\phi\|_e = \|C_\phi\| = 1$. In this case, we have $C^k_\phi = C_{\phi^k} = C_{\mu^k\zeta}$ with $|\mu^k| = 1$. This implies $\|C_{\phi^k}\|_e = \|C_{\phi^k}\|_e = 1$. Therefore $r_{\sigma_e}(C_\phi) = \lim_{k\to\infty} \|C^k_\phi\|_e^{\frac{1}{k}} = 1$. Hence C_ϕ is essentially normaloid.
On the other hand, if $|\mu| \neq 1$, then $|\mu| < 1$. This implies C_ϕ is compact ([1], Theorem 2). This implies $\|C_\phi\|_e = 0$. Moreover, $C^k_\phi = C_{\phi^k} = C_{\mu^k\zeta+\nu(\mu^{k-1}+\mu^{k-2}+\ldots+1)}$. Since $|\mu^k| < 1$, we have C_{ϕ^k} is compact. Therefore $\|C^k_\phi\|_e = 0$. Thus $r_{\sigma_e}(C_\phi) = \lim_{k\to\infty} \|C^k_\phi\|_e^{\frac{1}{k}} = 0$. Hence C_ϕ is essentially normaloid. □

In the next theorem, we will derive conditions by which the weighted composition operator on the Fock space $\mathcal{F}^2$ induced by the symbol ϕ and the weight function of the form $\psi(\zeta) = k_c(\zeta)$ for some $c \in \mathbb{C}$.
For $p \in \mathbb{C}$, denote $\phi_p(\zeta) = \zeta - p$, $\Phi_p(\zeta) = \phi_p \circ \phi \circ \phi_{-p}$ and $\Psi_p(\zeta) = k_{-p}(\psi \circ \phi_{-p})(k_p \circ \phi \circ \phi_{-p})$.

Lemma 3.5.([11], Proportion 2.3) For $p \in \mathbb{C}, \phi_p(\zeta) = \zeta - p, C_{k_p,\phi_p}$ is unitary.

Theorem 3.6. Let ϕ be an analytic function on $\mathbb{C}$ such that $\phi(\zeta) = \mu\zeta + \nu$ with $|\mu| < 1$ and $\phi(p) = p$ for some $p \in \mathbb{C}$ and $\psi(\zeta) = k_c(\zeta)$ for $c \in \mathbb{C}$. Then the bounded weighted composition operator $C_{\psi,\phi}$ is normaloid.

Proof. Consider

$$C_{k_{-p},\phi_{-p}}C_{\psi,\phi}C_{k_p,\phi_p} = C_{k_{-p}.(\psi\circ\phi_{-p}).(k_p\circ\phi\circ\phi_{-p}),\phi_p\circ\phi\circ\phi_{-p}} = C_{\Psi_p,\Phi_p} \tag{3.1}$$

Hence $C_{\psi,\phi}$ and C_{Ψ_p,Φ_p} are unitarily equivalent operators on the Fock space $\mathcal{F}^2$.

Since $\phi(p) = p$, we have

$$\begin{aligned} \Phi_p(\zeta) &= \phi_p\circ\phi\circ\phi_{-p}(\zeta) \\ &= \phi_p(\phi(\phi_{-p}))(\zeta) \\ &= \phi_p(\phi(\zeta+p)) \\ &= \phi_p(\mu(\zeta+p)+\nu) \\ &= \mu\zeta+\mu p+\nu-p \\ &= \mu\zeta+\phi(p)-p \\ &= \mu\zeta \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} \Psi_p(\zeta) &= k_{-p}(\psi\circ\phi_{-p})(k_p\circ\phi\circ\phi_{-p})(\zeta) \\ &= k_{-p}(\zeta)(\psi(\phi_{-p}(\zeta))(k_p(\phi(\phi_{-p}(\zeta))) \\ &= e^{-\zeta\overline{p}}\psi(\zeta+p)k_p(\phi(\zeta+p)) \\ &= e^{-\zeta\overline{p}-\frac{|p|^2}{2}}e^{(\zeta+p)\overline{c}-\frac{|c|^2}{2}}e^{(\mu(\zeta+p)+\nu)\overline{p}-\frac{|p|^2}{2}} \\ &= e^{-\zeta\overline{p}+(\zeta+p)\overline{c}-\frac{|c|^2}{2}+(\mu(\zeta+p)+\nu)\overline{p}-|p|^2} \end{aligned} \tag{3.3}$$

From $C^*_{\Psi_p,\Phi_p}K_0 = \overline{\Psi_p(0)}K_0$, we have $\|(C_{\Psi_p,\Psi_p} - \Psi_p(0))K_0\| = \|(C_{\Psi_p,\Psi_p} - \Psi_p(0))^*K_0\| = 0$. Since $K_0 = 1$, we get $(C_{\Psi_p,\Phi_p} - \Psi_p(0))K_0 = 0$. This implies $\Psi_p(\zeta) = \Psi_p(0)$.

Thus from (1.3), we have $\Psi_p(\zeta) = \Psi_p(0) = e^{p\overline{c}-\frac{|c|^2}{2}+|p|^2(\mu-1)+\nu\overline{p}}$ which is a constant. Denote $s = e^{p\overline{c}-\frac{|c|^2}{2}+|p|^2(\mu-1)+\nu\overline{p}}$.

Therefore, $C_{\Psi_p,\Phi_p} = sC_{\mu\zeta}$ with $|\mu| < 1$

In ([7], Proposition 2.2), author derived the numerical range of composition operator $C_{\mu\zeta}$ where $|\mu| < 1$ acting on Hardy space $\mathcal{H}^2$ is a closed polygonal region, whose vertices form a finite subset $\{\mu^n|n \geq 1\}$. It is clear from the proof that this result is also true for the Fock space $\mathcal{F}^2$.

It follows from ([7], Proposition 2.2), the numerical range of $C_{\Psi_p,\Phi_p} = sC_{\mu\zeta}$ is $\overline{\{s\mu^n|n \geq 0\}}$.

Thus $r_w(C_{\Psi_p,\Phi_p}) = |s|$.

On the other hand, by ([1], Theorem 4), $\|C_{\Psi_p,\Phi_p}\| = \|sC_{\mu\zeta}\| = |s|$

Thus $r_w(C_{\Psi_p,\Phi_p}) = \|C_{\Psi_p,\Phi_p}\|$. By [[5], Theorem 1.3-2], C_{Ψ_p,Φ_p} is normaloid. Using the fact that unitarily equivalent bounded operators have same numerical range and norm, we get the desired result. □

Theorem 3.7. Let ϕ be an analytic function on $\mathbb{C}$ such that $\phi(\zeta) = \mu\zeta + \nu$ with $|\mu| < 1$ and $\phi(p) = p$ for some $p \in \mathbb{C}$ and $\psi(\zeta) = k_c(\zeta)$ for $c \in \mathbb{C}$. Then the bounded weighted composition operator $C_{\psi,\phi}$ is essentially normaloid.

Proof. Since $|\mu| < 1$, by ([6], Theorem 2.4), $C_{\psi,\phi}$ is compact. This implies $\|C_{\psi,\phi}\|_e = 0$. On the other hand, we know that $C_{\psi,\phi}$ is unitarily equivalent to C_{Ψ_p,Φ_p}, where $\Psi_p(\zeta) = \Psi_p(0)$, which is a constant. Since $|\mu| < 1$, $C_{\Psi_p,\Psi_p} = \Psi_p(0).C_{\mu\zeta}$ is compact. Therefore $\|C_{\Psi_p,\Phi_p}\|_e = 0$. Since unitarily equivalent operators have same essential spectrum, we have $r_{\sigma_e}(C_{\psi,\phi}) = r_{\sigma_e}(C_{\Psi_p,\Phi_p})$. Also $\|C^k_{\Psi_p,\Phi_p}\|_e = \|\Psi_p(0)^k.C_{\Psi_p^k}\|_e = 0$. Thus $r_{\sigma_e}(C_{\psi,\phi}) = 0$. Hence $C_{\psi,\phi}$ is essentially normaloid. □

Corollary 3.8. Let ϕ be an analytic self map on $\mathbb{C}$ such that $\phi(\zeta) = \mu\zeta + \nu$ with $|\mu| < 1$ and $\psi(\zeta) = k_c(\zeta)$. Then C_ϕ is normaloid implies $C_{\psi,\phi}$ is essentially normaloid.

Proof. By Theorem 3.3, we have $\nu = 0$. This implies $C_{\psi,\phi} = C_{\psi,\mu\zeta}$. By ([6], Theorem 2.4), $C_{\psi,\mu\zeta}$ is a compact operator on the Fock space $\mathcal{F}^2$. Therefore $\|C_{\psi,\mu\zeta}\|_e = 0$. On the other hand, by Lemma 3.1, we have $C^k_{\psi,\mu\zeta} = C_{\psi(\zeta).(\psi(\mu\zeta)).(\psi(\mu^2\zeta))\ldots(\psi(\mu^k\zeta)),\mu^k\zeta}$ with $|\mu| < 1$ and $|\mu^k| < 1$ implies $C^k_{\psi,\mu\zeta}$ is compact.
Thus $r_{\sigma_e}(C_{\psi,\phi}) = \lim_{k\to\infty}(\|C^k_{\psi,\mu\zeta}\|_e)^{\frac{1}{k}} = 0$. Hence $C_{\psi,\phi}$ is essentially normaloid. □

Theorem 3.9. Let ψ be a holomorphic function on $\mathbb{C}$ and ϕ be a analytic self map on $\mathbb{C}$ such that $\phi(\zeta) = \mu\zeta + \nu$ with $|\mu| < 1$. If $C_{\psi,\phi}$ be a bounded operator on the Fock space $\mathcal{F}^2$ such that C_ϕ is normaloid then $C_{\psi,\phi}$ is normaloid.

Proof. By Theorem 3.3, we have $\phi(0) = 0$. This implies $\phi(\zeta) = \mu\zeta$.
From Lemma 3.2, $C^*_{\psi,\phi}K_0 = \overline{\psi(0)}K_{\phi(0)} = \overline{\psi(0)}K_0$ and $\|(C_{\psi,\phi} - \psi(0))K_0\| = \|(C_{\psi,\phi} - \psi(0))^*K_0\|$, we have $(C_{\psi,\phi} - \psi(0))K_0 = 0$.
Since $K_0 = 1$, we have $\psi(\zeta) = \psi(0)$, which is a constant and denote $\psi(0) = u$.
Thus $C_{\psi,\phi} = uC_{\mu\zeta}$ with $|\mu| < 1$.
Following arguments as in Theorem 3.6, we get $\|C_{\psi,\phi}\| = \|uC_{\mu\zeta}\| = |u|$ and $r_\sigma(C_{\psi,\phi}) = r_\sigma(uC_{\mu\zeta}) = |u|$. Hence $C_{\psi,\phi}$ is normaloid. □

Theorem 3.10. Let ϕ be an analytical function on $\mathbb{C}$ such that $\phi(\zeta) = \mu\zeta + \nu$ with $|\mu| < 1$ and $\psi(\zeta) = k_c(\zeta)$. Suppose $\phi(p) = p$ then $C_{\psi,\phi}$ is normaloid if and only if $C_{\psi,\phi}$ is spectraloid.

Proof. It is known that every normaloid is spectraloid. On the other hand, assume that $C_{\psi,\phi}$ is spectraloid. by definition, $r_\sigma(C_{\psi,\phi}) = r_w(C_{\psi,\phi})$. Since $C_{\psi,\phi}$ is unitarily equivalent to C_{Ψ_p,Φ_p} where Ψ_p, Φ_p defined as in Theorem 3.6., by ([7], Proposition 11) and using the fact that unitarily equivalent operators have same numerical range, we get $r_w(C_{\psi,\phi}) = |\Psi_p(0)|$. Also from the proof of Theorem 3.6, we have $\Psi_p(\zeta) = \Psi_p(0)$, a constant. By ([1], Theorem 4), we have $\|C_{\psi,\phi}\| = \|C_{\Psi_p,\Phi_p}\| = |\Psi_p(0)|$. Thus $r_w(C_{\psi,\phi}) = \|C_{\psi,\phi}\|$. Hence $C_{\psi,\phi}$ is normaloid. □

In the next theorem, we will find the conditions for which the $C^2_{\psi,\phi}$ is normaloid. Moreover, this result can be extended to any power of natural number. In order to prove the next result we need following proposition.

Proposition 3.11. Let ϕ be holomorphic on $\mathbb{C}$ such that $\phi(\zeta) = \mu\zeta + \nu$ with $|\mu| < 1$ and $\psi(\zeta) = k_c(\zeta)$. Then $C^2_{\psi,\phi}$ is unitarily equivalent to $C_{\Psi'_p,\Phi'_p}$, where $\Phi'_p(\zeta) = \mu^2\zeta + \mu^2 p + \mu\nu + \nu - p$ and $\Psi'_p(\zeta) = e^{\zeta(\overline{c}(\mu+1)+\overline{p}(\mu^2-1))+\overline{c}(\mu p+\nu+p)-|c|^2}$

Proof. By ([11], Proposition 2.2), C_{k_p,ϕ_p} is unitary and its inverse is $C_{k_{-p},\phi_{-p}}$. Taking $\Psi'_p(\zeta) = k_{-p}(\zeta).\psi \circ \phi_{-p}(\zeta).\psi \circ \phi \circ \phi_{-p}(\zeta).k_p \circ \phi \circ \phi \circ \phi_{-p}(\zeta)$ and $\Phi'_p(\zeta) = \phi_p \circ \phi \circ \phi \circ \phi_{-p}(\zeta)$, and with the fact that reproducing kernels are dense in the Fock space $\mathcal{F}^2$, we have

$$C_{k_{-p},\phi_{-p}} C_{\psi,\phi} C_{\psi,\phi} C_{k_p,\phi_p} K_w(\zeta) = C_{\Psi'_p,\Phi'_p} K_w(\zeta) \tag{3.4}$$

This implies $C_{\psi,\phi}$ is unitarily equivalent to $C_{\Psi'_p,\Phi'_p}$ where $\Phi'_p(\zeta) = \mu^2\zeta + \mu^2 p + \mu\nu + \nu - p$ and $\Psi'_p(\zeta) = e^{\zeta(\overline{c}(\mu+1)+\overline{p}(\mu^2-1))+\overline{c}(\mu p+\nu+p)-|c|^2}$. □

Theorem 3.12. Let ϕ be an analytical function on $\mathbb{C}$ such that $\phi(\zeta) = \mu\zeta + \nu$ with $|\mu| < 1$ and $\psi(\zeta) = k_c(\zeta)$. For some $p \in \mathbb{C}$, if $\phi^2(p) = p$ then $C^2_{\psi,\phi}$ is normaloid.

Proof. By Proposition 3.11, we have $C_{\psi,\phi}$ is unitarily equivaent to $C_{\Psi'_p,\Phi'_p}$. Taking $\phi^2(p) = p$ for some $p \in \mathbb{C}$, we get

$$\begin{aligned} \Phi'_p(\zeta) &= \phi_p \circ \phi \circ \phi \circ \phi_{-p}(\zeta) \\ &= \phi_p(\phi^2(\zeta + p) \\ &= \phi_p(\mu^2\zeta + \mu^2 p + \mu\nu + \nu) \\ &= \mu^2\zeta + \mu^2 p + \mu\nu + \nu - p \\ &= \mu^2\zeta + \phi^2(p) - p \\ &= \mu^2\zeta \end{aligned} \tag{3.5}$$

$$\begin{aligned} \Psi'_p(\zeta) &= k_{-p}(\zeta).\psi \circ \phi_{-p}(\zeta).\psi \circ \phi \circ \phi_{-p}(\zeta).k_p \circ \phi \circ \phi \circ \phi_{-p}(\zeta) \\ &= e^{\zeta(\overline{c}(\mu+1)+\overline{p}(\mu^2-1))+\overline{c}(\mu p+\nu+p)-|c|^2} \end{aligned} \tag{3.6}$$

From $C^*_{\Psi'_p,\Phi'_p}K_0 = \overline{\Psi'_p(0)}K_{\Phi'_p(0)}$ and $\|(C_{\Psi'_p,\Phi'_p} - \Psi'_p(0))K_0\| = \|(C_{\Psi'_p,\Phi'_p} - \Psi'_p(0))^*K_0\|$, we have $(C_{\Psi'_p,\Phi'_p} - \Psi'_p)K_0 = C_{\Psi'_p,\Phi'_p}K_0 - \Psi'_pK_0 = 0$. Therefore $\Psi'_p(\zeta) = \Psi'_p(0)$ which is a constant. Denote $\Psi'_p(\zeta) = \Psi'_p(0) = s$. Thus $C_{\psi,\phi}$ is unitarily equivalent to $sC_{\Phi'_p}$ where $\Phi'_p(\zeta) = \mu^2\zeta$ with $|\mu| < 1$.

By ([7], Proposition 2.2) and following argument as in Theorem 3.6., we have $r_w(C_{\Psi'_p,\Phi'_p} = |s|$ and by ([1], Theorem 4), we get $\|C_{\Psi'_p,\Phi'_p}\| = |s|$. Thus $C_{\Psi'_p,\Phi'_p}$ is normaloid. Since norm and numerical range of unitarily equivalent operators are equal, we get the desire result. □

Corollary 3.13. Let ϕ be an analytical function on $\mathbb{C}$ such that $\phi(\zeta) = \mu\zeta + \nu$ with $|\mu| < 1$ and $\psi(\zeta) = k_c(\zeta)$. For some $wwwp \in \mathbb{C}$ and any natural number n, if $\phi^n(p) = p$ then $C^n_{\psi,\phi}$ is normaloid.

Proof. Result can be proved by following similar argument as in Theorem 3.12. □

References

[1] B. J. Carswell, B. D. MacCluer and A. Schuster, *Composition Operators on the Fock Space*, Acta. Sci. Math. (Szeged)., **(69)**(2003), 871–887.

[2] C. C. Cowen and B. D. MacCluer, *Composition Operators on Spaces of Analytical Functions*, Studies in Advanced Mathematics, CRC Press.

[3] J. Dai, *The Norm of Composition Operators on the Fock Space*, Complex Variables and Elliptic Equations, **(64)(9)**(2019), 1–9.

[4] K. Guo and K. Izuchi, *Composition Opertors on Fock Type Spaces*, Acta Sci. Math.(Szeged)., **(74)(3-4)**(2008), 807–828.

[5] K. E. Gustafson and K. M. Rao, *Numerical Range*, Springer-Verlag New York, 1997.

[6] T. Le, *Normal and Isometric Weighted Composition Operators on the Fock Space*, Bull. Lond. Math. Soc., **(46)(4)**(2014), 847–856.

[7] V. Matache, *Numerical Range of Composition Operators*, Linear. Algebra. Appl., **(331)**(2001), 61–74.

[8] T. Mengestie and W. Seyoum, *Spectra of Composition Operators on Fock-type Spaces*, Questiones Mathematicae, (2019), 1–16.

[9] J. H. Shapiro, *Composition Operators and Classical Tunction Theory*, Universitext: Tracts in Mathematics, Springer - Verlag, New York, 1993.

[10] D. Thompson, *Normaloid Weighted Composition Operators on $\mathscr{H}^2$*, J. Math. Anal. Appl., **(467)(2)**(2018), 1153–1162.

[11] L. Zhao, *Unitary Weighted Composition Operator on the Fock Space of $\mathbb{C}^n$*, Complex. Anal. Oper. Theory., **(8)(2)**(2014), 581–590.

[12] K. Zhu, *Analysis on Fock Spaces*, Graduate Text in Mathematics, Springer, New York.